9 Online Appendix for "Severity reduction and private market distortion effects of voluntary and mandatory public annuity plans"

In this Online Appendix, we address several issues not covered in the main text.

9.1 Appendix C: Symmetric information in a model with two health states

It is more connivent to use a two-state model to obtain the results under the assumption of symmetric information. The model is similar to the one in the main text, except that a retiree's survival probability (θ) only takes two possible values: θ_l and θ_h, where

$$0 < \theta_l < \theta_h < 1. \tag{C1}$$

First, consider the private annuity market before the PA plan is introduced. When the retirees and annuity providers have symmetric information about the retirees' survival probabilities, it is easy to show that it is optimal for the annuity providers to offer two exclusive annuities, one for each type.

Similar to (6), it can be shown that the first-order condition for a retiree's annuitization choice is given by

$$u'\left(w - \widehat{\nu}^*\left(\theta_i\right)\right) = \frac{\theta_i}{1+\rho}\widehat{V}u'\left(\widehat{V}\widehat{\nu}^*\left(\theta_i\right)\right), \tag{C2}$$

where $i = l, h$, and we use $\widehat{\nu}^*\left(\theta_i\right)$ to represent the optimal annuitization choice a type-i retiree in this model.

As in the main model, we assume that the equilibrium annuity payout ($\widehat{V}_i$) for each annuity is determined by the zero-profit condition. Assume that there are n_i type-i retirees ($i = l, h$) in the economy. It can be shown that for each annuity, annuity providers' revenue in Period 1 is given by $n_i\widehat{\nu}^*\left(\theta_i\right)$, and their expected payment in Period 2 is $\theta_i n_i \widehat{V}_i \widehat{\nu}^*\left(\theta_i\right)$. Under the zero-profit condition, the equilibrium value of the payout term $\widehat{V}_i$, denoted by $\widehat{V}_i^*$, is determined according to

$$\widehat{V}_i^* = \frac{(1+r)\,n_i\widehat{\nu}^*\left(\theta_i\right)}{\theta_i n_i \widehat{\nu}^*\left(\theta_i\right)} = \frac{(1+r)}{\theta_i}, \tag{C3}$$

which is the actuarially fair level. There is no adverse selection in each annuity market. Moreover, $\widehat{V}_h^* < \widehat{V}_l^*$.

In this symmetric information environment, the equilibrium payouts of the two exclusive private annuities are at the actuarially fair levels.[34] It is easy to conclude that there is no strong reason for the government to offer public annuities.

To illustrate the above idea, we now describe, briefly, that even if the government offers, the action does not affect the equilibrium annuity payouts.

9.1.1 VPAc plan

Similar to the private annuity market, the government is assumed to offer two exclusive public annuities, one for each type of retirees. Denote the equilibrium PA payout for type-i retirees as G_i^*. First, focus on the case that the ceiling is set at a non-binding level higher than $\widehat{\nu}^*(\theta_i)$. It can be shown that (i) if $G_i^* < \widehat{V}_i^*$, then no type-i retiree buys the PA; and (ii) if $G_i^* > \widehat{V}_i^*$, then all type-i retirees will buy the PA instead. In the latter case, the PA payout is higher that the actuarially fair level; thus, the government will have a budget deficit, which is inconsistent with the zero-profit condition. Therefore, the only equilibrium is

$$G_i^* = \widehat{V}_i^*. \tag{C4}$$

The above results hold whether the ceiling level is higher or lower than $\widehat{\nu}^*(\theta_i)$. If it is set at a level lower than $\widehat{\nu}^*(\theta_i)$, then type-$i$ retirees can satisfy their residual demand by buying the private annuity, which offers the same payout level as the PA. The equilibrium private annuity payouts remain unchanged.

Offering the VPAc plan in the presence of symmetric information does not lead to any change in the equilibrium payout of each of the two private

[34] Suppose there is only one non-exclusive private annuity that all retirees can buy. According to the zero-profit condition, the equilibrium annuity payout ($\widehat{V}^*$) of this contract is determined according to

$$\widehat{V}^* = \frac{(1+r)\left[n_l\widehat{\nu}^*(\theta_l) + n_h\widehat{\nu}^*(\theta_h)\right]}{\theta_l n_l\widehat{\nu}^*(\theta_l) + \theta_h n_h\widehat{\nu}^*(\theta_h)} = \frac{(1+r)\left[\pi\widehat{\nu}^*(\theta_l) + (1-\pi)\widehat{\nu}^*(\theta_h)\right]}{\theta_l\pi\widehat{\nu}^*(\theta_l) + \theta_h(1-\pi)\widehat{\nu}^*(\theta_h)},$$

where $\pi = \frac{n_l}{n_l+n_h}$ is the proportion of type-l retirees. Since $\widehat{V}^* < \widehat{V}_l^*$, type-$l$ retirees find it profitable to insure among themselves to obtain a higher return. Equivalently, an annuity company has the incentive to provide such a contract for type-l retirees only, which is possible when survival probability information is symmetric. Eventually, only type-h retirees buy this annuity and $\widehat{V}^*$ becomes $\widehat{V}_h^*$. Unless there are restrictions preventing the introduction of exclusive annuities, the above non-exclusive annuity offered to all retirees is likely to collapse. Thus, we do not consider non-exclusive annuities when survival probability information is symmetric.

annuities.[35] As a result, there is no real effect on the retirees' consumption behavior (and their welfare).

9.1.2 MPAf plan

The analysis of the MPAf plan is similar to that of the VPAc plan if the minimum mandated level for type-i retirees is lower than $\widehat{\nu}^*(\theta_i)$.

If the minimum mandated level for type-i retirees is higher than $\widehat{\nu}^*(\theta_i)$, it is straightforward to see that the retirees' optimal annuitization choices before the introduction of the PA plan cannot be attained. In this case, the retirees are worse off under the MPAf plan. Thus, there is no reason supporting the government to offer the MPAf plan.

9.2 Appendix D: Buyers' behavior

(A) We examine the buyers' behavior under the MPAf plan (as described in parts (a) to (d) in Section 4.2), conditional on $G^* > V^*$ implied by Proposition 1.

When an interior solution of γ_θ occurs, $\left.\frac{\partial U}{\partial \gamma_\theta}\right|_{\gamma_\theta^*} = 0$ in (12). At the optimal choices (γ_θ^* and ν_θ^*) and the equilibrium values (G^* and V^*), we have

$$u'\left(w - \gamma_\theta^* - \nu_\theta^*\right) = \frac{\theta}{1+\rho} G^* u'\left(G^*\gamma_\theta^* + V^*\nu_\theta^*\right). \tag{D1}$$

Conditional on $G^* > V^*$, (D1) implies that $\left.\frac{\partial U}{\partial \nu_\theta}\right|_{\nu_\theta^*} < 0$ in (13), and thus $\nu_\theta^* = 0$, a corner solution (at the lowest possible value).

Substituting $\nu_\theta^* = 0$ into (D1), we obtain (23), evaluated at the equilibrium value G^*. It is straightforward to obtain (24) from (23). Substituting $\gamma_\theta^* = f$ into (23), we obtain θ_{fo} in (29). Substituting $\gamma_\theta^* = m$ into (23), we obtain θ_{om} in (25). This proves part (b).

[35] As in the private annuity market, there is no incentive for the government to offer a non-exclusive PA plan that any retiree can buy, as shown in the following. Suppose the government offers a non-exclusive PA plan. Denote the equilibrium PA payout as G^*. It can be shown that (i) if $G^* < \widehat{V}_h^* < \widehat{V}_l^*$, then no retiree buy the PA; (ii) if $\widehat{V}_h^* < G^* < \widehat{V}_l^*$, then only type-$h$ retirees will buy the PA and the zero-profit condition does not hold because $G^* > \frac{(1+r)}{\theta_h}$; and (iii) if $G^* \geq \widehat{V}_l^*$, then all retirees may buy the PA and the zero-profit condition does not hold. Therefore, the only equilibrium is

$$G^* = \widehat{V}_h^*.$$

In this case, type-h retirees are indifferent in buying the PA or the private annuity, and type-l retirees only buy the private annuity.

Since (24) holds for $\theta \geq \theta_{fo}$ and $\gamma_\theta^* = f$ (the minimum mandated level) when $\theta = \theta_{fo}$, part (a) follows.

An interior solution of ν_θ suggests that $\left.\frac{\partial U}{\partial \nu_\theta}\right|_{\nu_\theta^*} = 0$ in (13), which implies

$$u'\left(w - \gamma_\theta^* - \nu_\theta^*\right) = \frac{\theta}{1+\rho} V^* u'\left(G^* \gamma_\theta^* + V^* \nu_\theta^*\right). \tag{D2}$$

Conditional on $G^* > V^*$, (D2) implies that $\left.\frac{\partial U}{\partial \gamma_\theta}\right|_{\gamma_\theta^*} > 0$ in (12), and thus $\gamma_\theta^* = m$, a corner solution (at the highest possible value).

Substituting $\gamma_\theta^* = m$ into (D2), we obtain (15) evaluated at the equilibrium values G^* and V^*. It is straightforward to obtain (16) from (15). Substituting $\nu_\theta^* = 0$ into (15), we obtain θ_{mb} in (17). This proves part (d).

We prove part (c) by combining the following components: (i) $\theta_{mb} > \theta_{om}$, based on (17), (25) and $G^* > V^*$; and (ii) when $\theta_{om} < \theta < \theta_{mb}$, $\left.\frac{\partial U}{\partial \gamma_\theta}\right|_{\gamma_\theta^*} > 0$ in (12) and thus $\gamma_\theta^* = m$ (the maximum level allowed), and $\left.\frac{\partial U}{\partial \nu_\theta}\right|_{\nu_\theta^*} < 0$ in (13) and thus $\nu_\theta^* = 0$.

(B) We examine the buyers' behavior under the VPAc plan (as described in parts (a) to (c) in Section 4.1), conditional on $G^* > V^*$.[36]

When an interior solution of γ_θ occurs, $\left.\frac{\partial U}{\partial \gamma_\theta}\right|_{\gamma_\theta^*} = 0$ in (12). At the optimal choices (γ_θ^* and ν_θ^*) and the equilibrium values (G^* and V^*), we have (D1). Conditional on $G^* > V^*$, (D1) implies that $\left.\frac{\partial U}{\partial \nu_\theta}\right|_{\nu_\theta^*} < 0$ in (13), and thus $\nu_\theta^* = 0$, a corner solution (at the lowest possible value).

Substituting $\nu_\theta^* = 0$ into (D1), we obtain (23), evaluated at the equilibrium value G^*. It is straightforward to obtain (24) from (23). Substituting $\gamma_\theta^* = m$ into (23), we obtain θ_{om} in (25). This proves part (a).

An interior solution of ν_θ suggests that $\left.\frac{\partial U}{\partial \nu_\theta}\right|_{\nu_\theta^*} = 0$ in (13), which implies (D2). Conditional on $G^* > V^*$, (D2) implies that $\left.\frac{\partial U}{\partial \gamma_\theta}\right|_{\gamma_\theta^*} > 0$ in (12), and thus $\gamma_\theta^* = m$, a corner solution (at the highest possible value).

Substituting $\gamma_\theta^* = m$ into (D2), we obtain (15) evaluated at the equilibrium values G^* and V^*. It is straightforward to obtain (16) from (15). Substituting $\nu_\theta^* = 0$ into (15), we obtain θ_{mb} in (17). This proves part (c).

We prove part (b) by combining the following components: (i) $\theta_{mb} > \theta_{om}$, based on (17), (25) and $G^* > V^*$; and (ii) when $\theta_{om} < \theta < \theta_{mb}$, $\left.\frac{\partial U}{\partial \gamma_\theta}\right|_{\gamma_\theta^*} > 0$ in

[36] Buyers' behavior under the VPAc plan are very similar to those under the MPAf plan. The only major difference is that there is no threshold level θ_{fo} under the VPAc plan.

(12) and thus $\gamma_\theta^* = m$ (the maximum level allowed), and $\left.\frac{\partial U}{\partial \nu_\theta}\right|_{\nu_\theta^*} < 0$ in (13) and thus $\nu_\theta^* = 0$. ■

9.3 Appendix E: Results related to Proposition 1

First, we will provide a direct proof of $G^* > V^*$. Second, we will present the proof of Proposition 1 for the VPAc plan.

9.3.1 A direct proof of $G^* > V^*$

Using (18) and (30) for the MPAf plan (or (26) for the VPAc plan), we obtain[37]

$$\begin{aligned}\frac{1+r}{V^*} - \frac{1+r}{G^*} &= \frac{\int_{\theta_{mb}}^{\overline{\theta}} \theta \nu_\theta^* dF(\theta)}{\int_{\theta_{mb}}^{\overline{\theta}} \nu_\theta^* dF(\theta)} - \frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta \gamma_\theta^* dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}} \gamma_\theta^* dF(\theta)} \\ &> \frac{\int_{\theta_{mb}}^{\overline{\theta}} \theta dF(\theta)}{\int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)} - \frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta \gamma_\theta^* dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}} \gamma_\theta^* dF(\theta)}. \qquad \text{(E1)}\end{aligned}$$

Furthermore, we obtain[38]

$$\frac{\int_{\theta_{mb}}^{\overline{\theta}} \theta dF(\theta)}{\int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)} - \frac{\int_{\underline{\theta}}^{\overline{\theta}} \theta \gamma_\theta^* dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}} \gamma_\theta^* dF(\theta)}$$

$$= \frac{\left[\int_{\underline{\theta}}^{\theta_{mb}} \gamma_\theta^* dF(\theta) + \int_{\theta_{mb}}^{\overline{\theta}} m dF(\theta)\right] \int_{\theta_{mb}}^{\overline{\theta}} \theta dF(\theta) - \left[\int_{\underline{\theta}}^{\theta_{mb}} \theta\gamma_\theta^* dF(\theta) + \int_{\theta_{mb}}^{\overline{\theta}} \theta m dF(\theta)\right] \int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)}{\left[\int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)\right]\left[\int_{\underline{\theta}}^{\overline{\theta}} \gamma_\theta^* dF(\theta)\right]}$$

[37] Define $H(\theta) = \frac{F(\theta)-F(\theta_{mb})}{F(\overline{\theta})-F(\theta_{mb})}$ for $\theta \in \left[\theta_{mb}, \overline{\theta}\right]$, which is a cumulative density function because $H(\theta_{mb}) = 0$ and $H(\overline{\theta}) = 1$. Applying the covariance formula, it can be shown that $\frac{\int_{\theta_{mb}}^{\overline{\theta}} \theta\nu_\theta^* dF(\theta)}{\int_{\theta_{mb}}^{\overline{\theta}} \nu_\theta^* dF(\theta)} = \frac{\int_{\theta_{mb}}^{\overline{\theta}} \theta\nu_\theta^* dH(\theta)}{\int_{\theta_{mb}}^{\overline{\theta}} \nu_\theta^* dH(\theta)} = \frac{cov(\theta,\nu_\theta^*;H(\theta))+[E(\theta;H(\theta))][E(\nu_\theta^*;H(\theta))]}{E\left(\nu_\theta^*;H(\theta)\right)} = \frac{cov(\theta,\nu_\theta^*;H(\theta))}{E\left(\nu_\theta^*;H(\theta)\right)} + E\left(\theta;H\left(\theta\right)\right)$, where $E\left(\theta;H\left(\theta\right)\right)$, $E\left(\nu_\theta^*;H\left(\theta\right)\right)$ and $cov\left(\theta,\nu_\theta^*;H\left(\theta\right)\right)$ are defined with respect to $H(\theta)$ for $\theta \in \left[\theta_{mb}, \overline{\theta}\right]$. Since $\frac{\int_{\theta_{mb}}^{\overline{\theta}} \theta dF(\theta)}{\int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)} = \frac{\int_{\theta_{mb}}^{\overline{\theta}} \theta dH(\theta)}{\int_{\theta_{mb}}^{\overline{\theta}} dH(\theta)} = E\left(\theta;H\left(\theta\right)\right)$ and the covariance of θ and ν_θ^* is positive when $\theta \in \left[\theta_{mb}, \overline{\theta}\right]$, we obtain $\frac{\int_{\theta_{mb}}^{\overline{\theta}} \theta\nu_\theta^* dF(\theta)}{\int_{\theta_{mb}}^{\overline{\theta}} \nu_\theta^* dF(\theta)} > \frac{\int_{\theta_{mb}}^{\overline{\theta}} \theta dF(\theta)}{\int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)}$.

[38] The equality in line 2 appears because $\gamma_\theta^* = m$ when $\theta > \theta_{mb}$. The inequality in line 4 appears because $\int_{\theta_{mb}}^{\overline{\theta}} \theta dF(\theta) > \theta_{mb} \int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)$ when $\theta > \theta_{mb}$ and $\int_{\underline{\theta}}^{\theta_{mb}} \theta\gamma_\theta^* dF(\theta) < \theta_{mb} \int_{\underline{\theta}}^{\theta_{mb}} \gamma_\theta^* dF(\theta)$ when $\theta < \theta_{mb}$.

$$= \frac{\left[\int_{\underline{\theta}}^{\theta_{mb}} \gamma_\theta^* dF(\theta)\right]\left[\int_{\theta_{mb}}^{\overline{\theta}} \theta dF(\theta)\right] - \left[\int_{\underline{\theta}}^{\theta_{mb}} \theta\gamma_\theta^* dF(\theta)\right]\left[\int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)\right]}{\left[\int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)\right]\left[\int_{\underline{\theta}}^{\overline{\theta}} \gamma_\theta^* dF(\theta)\right]}$$

$$> \frac{\left[\int_{\underline{\theta}}^{\theta_{mb}} \gamma_\theta^* dF(\theta)\right]\theta_{mb}\left[\int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)\right] - \theta_{mb}\left[\int_{\underline{\theta}}^{\theta_{mb}} \gamma_\theta^* dF(\theta)\right]\left[\int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)\right]}{\left[\int_{\theta_{mb}}^{\overline{\theta}} dF(\theta)\right]\left[\int_{\underline{\theta}}^{\overline{\theta}} \gamma_\theta^* dF(\theta)\right]}$$

$$= 0. \tag{E2}$$

Combining (E1) and (E2), we obtain $G^* > V^*$. ■

9.3.2 Proof of Proposition 1 (VPAc plan)

(A) Under the VPAc plan with (20), define the function

$$\gamma_\theta(G, m) = \min\{m, \gamma_\theta^*\}, \tag{E3}$$

where γ_θ^* is the optimal choice of PA according to (23). Based on (E3), define the function

$$J(G, w, m) = \frac{(1+r)\int_{\underline{\theta}}^{\overline{\theta}} \gamma_\theta(G, w, m)\, dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}} \theta\gamma_\theta(G, w, m)\, dF(\theta)}. \tag{E4}$$

This function is useful because it can be seen from (26) and (E4) that the equilibrium payout of the VPAc plan, G^*, is defined by

$$J(G^*, w, m) = \frac{(1+r)\int_{\underline{\theta}}^{\overline{\theta}} \gamma_\theta(G^*, w, m)\, dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}} \theta\gamma_\theta(G^*, w, m)\, dF(\theta)} = G^*. \tag{E5}$$

We study $J(G, w, m)$ as a function of G when $G \in \left[\widehat{V}^*, \frac{1+r}{E(\theta)}\right]$ and look for the intersection of the $J(G, w, m)$ function and the 45-degree line. First, consider $J(G, w, m)$ when $G = \widehat{V}^*$. It can be shown from (6), (23) and (E3) that $\gamma_\theta\left(\widehat{V}^*, w, m\right)$ equals to (i) $\widehat{\nu}_\theta^* \equiv \gamma_\theta\left(\widehat{V}^*, w, 0\right)$ if $\widehat{\nu}_\theta^* \leq m$, or (ii) m if $\widehat{\nu}_\theta^* > m$. As a result, the function $\frac{\gamma_\theta\left(\widehat{V}^*, w, m\right)}{\widehat{\nu}_\theta^*}$ is weakly decreasing for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$, with a strictly decreasing part over some interval.[39] Thus, the Chebyshev's Sum

$$\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\theta}}^{\overline{\theta}} (x-y)\left[\frac{\gamma_x\left(\widehat{V}^*, w, m\right)}{\widehat{\nu}_x^*} - \frac{\gamma_y\left(\widehat{V}^*, w, m\right)}{\widehat{\nu}_y^*}\right]\widehat{\nu}_x^*\widehat{\nu}_y^* dF(x)dF(y)$$

[39]The ratio $\frac{\gamma_\theta\left(\widehat{V}^*, w, m\right)}{\widehat{\nu}_\theta^*}$ (i) equals to 1 for low values of θ, or (ii) is less than 1 and strictly decreasing for high value of θ such that $\widehat{\nu}_\theta^* > m$.

is negative, where x and y are two arbitrary indexes. This leads to

$$J\left(\widehat{V}^{*},w,m\right)=\frac{(1+r)\int_{\underline{\theta}}^{\overline{\theta}}\gamma_{\theta}\left(\widehat{V}^{*},w,m\right)dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}}\theta\gamma_{\theta}\left(\widehat{V}^{*},w,m\right)dF(\theta)}>\frac{(1+r)\int_{\underline{\theta}}^{\overline{\theta}}\widehat{\nu}_{\theta}^{*}dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}}\theta\widehat{\nu}_{\theta}^{*}dF(\theta)}=\widehat{V}^{*}. \tag{E6}$$

Second, consider $J(G,w,m)$ when $G=\frac{1+r}{E(\theta)}$. It is easy to show that

$$J\left(\frac{1+r}{E(\theta)},w,m\right)<\frac{1+r}{E(\theta)}, \tag{E7}$$

because when adverse selection is present, the PA payout based on the market behavior (the LHS term) is always lower than the (hypothetical) payout based on the average survival probability of the population (the RHS term).

Combining (E5), (E6), (E7) and the continuity of the $J(G,w,m)$ function, we conclude that there exists an equilibrium value of G^{*} which is larger than $\widehat{V}^{*}$. This proves part (a) of Proposition 1.

(B) The proof of part (b) of Proposition 1 for the VPAc plan is exactly the same as that in Appendix A. ■

9.4 Appendix F: Proof of Proposition 2

The three PA plans considered in Proposition 2 differ in the floor level (f) but are subject to the same ceiling (m). Thus, we prove this proposition by considering two discrete values of the floor: f_a and f_b, where $0\leq f_a<f_b\leq m$. For the PA plan with $f=f_a$, denote the two equilibrium payouts as G_a^{*} and V_a^{*}. For the PA plan with $f=f_b$, denote the two equilibrium payouts as G_b^{*} and V_b^{*}.

We start with the PA plan where $f=f_a$. We consider a new PA plan with $f=f_b$ and see how the two equilibrium payouts change. Specifically, we compare G_b^{*} with G_a^{*} and compare V_b^{*} with V_a^{*}.

(A) Recall the PA choice function $\gamma_{\theta}(G,w,m,f)$ in (A1) and the PA payout function $J(G,w,m,f)$ in (A2). Based on these functions, the equilibrium PA payouts are defined by $J(G_a^{*},w,m,f_a)\equiv G_a^{*}$ and

$$J\left(G_b^{*},w,m,f_b\right)=\frac{(1+r)\int_{\underline{\theta}}^{\overline{\theta}}\gamma_{\theta}\left(G_b^{*},w,m,f_b\right)dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}}\theta\gamma_{\theta}\left(G_b^{*},w,m,f_b\right)dF(\theta)}\equiv G_b^{*}. \tag{F1}$$

We study $J(G,w,m,f_b)$ as a function of G when $G\in\left[G_a^{*},\frac{1+r}{E(\theta)}\right]$ and look for the intersection of the $J(G,w,m,f_b)$ function and the 45-degree line.

First, consider $J(G, w, m, f_b)$ when $G = G_a^*$. Comparing $\gamma_\theta(G_a^*, w, m, f_b)$ with $\gamma_\theta(G_a^*, w, m, f_a)$, we can see from (A1) that (i) the ratio $\frac{\gamma_\theta(G_a^*;w,m;f_b)}{\gamma_\theta(G_a^*,w,m,f_a)} = \frac{f_b}{f_a} > 1$ when θ is low such that $\gamma_\theta(G_a^*, w, m, f_b) = f_b$ and $\gamma_\theta(G_a^*, w, m, f_a) = f_a$; (ii) the ratio $\frac{\gamma_\theta(G_a^*;w,m;f_b)}{\gamma_\theta(G_a^*,w,m,f_a)}$ is decreasing in θ when θ is in the intermediate range such that $\gamma_\theta(G_a^*, w, m, f_b) = f_b$ but $f_a < \gamma_\theta(G_a^*, w, m, f_a) \leq f_b$; and (iii) the ratio $\frac{\gamma_\theta(G_a^*;w,m;f_b)}{\gamma_\theta(G_a^*,w,m,f_a)} = 1$ when θ is high such that both $\gamma_\theta(G_a^*, w, m, f_b) > f_b$ and $\gamma_\theta(G_a^*, w, m, f_a) > f_b$. As a result, this ratio is weakly decreasing in θ. Thus, the Chebyshev's Sum

$$\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} (x - y) \left[\frac{\gamma_x (G_a^*, w, m, f_b)}{\gamma_x (G_a^*, w, m, f_a)} - \frac{\gamma_y (G_a^*, w, m, f_b)}{\gamma_y (G_a^*, w, m, f_a)} \right]$$

$$[\gamma_x (G_a^*, w, m, f_a)] \left[\gamma_y (G_a^*, w, m, f_a)\right] dF(x)dF(y) \tag{F2}$$

is negative, where x and y are two arbitrary indexes. This leads to

$$J(G_a^*, w, m, f_b) = \frac{(1+r)\int_{\underline{\theta}}^{\overline{\theta}} \gamma_\theta (G_a^*, w, m, f_b) \, dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}} \theta\gamma_\theta (G_a^*, w, m, f_b) \, dF(\theta)}$$

$$> \frac{(1+r)\int_{\underline{\theta}}^{\overline{\theta}} \gamma_\theta (G_a^*, w, m, f_a) \, dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}} \theta\gamma_\theta (G_a^*, w, m, f_a) \, dF(\theta)} \equiv G_a^*. \tag{F3}$$

Second, consider $J(G, w, m, f_b)$ when $G = \frac{1+r}{E(\theta)}$. It is easy to show that

$$J\left(\frac{1+r}{E(\theta)}, w, m, f_b\right) < \frac{1+r}{E(\theta)}, \tag{F4}$$

because of the presence of adverse selection.

Combining (F1), (F3), (F4) and the continuity of the $J(G, w, m, f_b)$ function over G, we conclude that there exists an equilibrium value G_b^* such that $G_b^* \equiv J(G_b^*, w, m, f_b) > G_a^*$. This proves $G_{VPAc}^* < G_{MPAf}^* < G_{MPA}^*$ of (34).

(B) Based on the optimal private annuity purchase (ν_θ^*) in (15), define the function

$$\nu_\theta (V, G, m) = \max\{0, \nu_\theta^*\}. \tag{F5}$$

where other parameters are unchanged. Based on (F5), define the function

$$K(V, G, w) = \frac{(1+r)\int_{\underline{\theta}}^{\overline{\theta}} \nu_\theta (V, G, w) \, dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}} \theta\nu_\theta (V, G, w) \, dF(\theta)}, \tag{F6}$$

which is useful because it can be seen from (18) and (F6) that the equilibrium payout of the private annuity (V_a^*) in the presence of the PA plan (with G_a^*) is defined by

$$K\left(V_a^*, G_a^*, w\right) = \frac{(1+r)\int_{\underline{\theta}}^{\overline{\theta}} \nu_\theta\left(V_a^*, G_a^*, w\right) dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}} \theta\nu_\theta\left(V_a^*, G_a^*, w\right) dF(\theta)} = V_a^*, \tag{F7}$$

and similarly for

$$K\left(V_b^*, G_b^*, w\right) = \frac{(1+r)\int_{\underline{\theta}}^{\overline{\theta}} \nu_\theta\left(V_b^*, G_b^*, w\right) dF(\theta)}{\int_{\underline{\theta}}^{\overline{\theta}} \theta\nu_\theta\left(V_b^*, G_b^*, w\right) dF(\theta)} = V_b^*. \tag{F8}$$

We study $K(V, G_b^*, w)$ as a function of V when $V \in \left[\frac{1+r}{\overline{\theta}}, V_a^*\right]$ and look for the intersection of the $K(V, G_b^*, w)$ function and the 45-degree line. First, consider $K\left(V, G_b^*, w\right)$ when $V = \frac{1+r}{\overline{\theta}}$. As in part (B) of the proof for Proposition 1, we can easily show that

$$K\left(\frac{1+r}{\overline{\theta}}, G_b^*, w\right) > \frac{1+r}{\overline{\theta}}. \tag{F9}$$

because of the problem of adverse selection.

Second, consider $K\left(V, G_b^*, w\right)$ when $V = V_a^*$. Our objective is to compare $\nu_\theta\left(V_a^*, G_b^*, w\right)$ with $\nu_\theta\left(V_a^*, G_a^*, w\right)$. The comparison relies on an intermediate term: $\nu_\theta\left(V_a^*, G_a^*, w^h\right)$, as follows.

For a retiree whose survival probability θ is higher than $\frac{(1+\rho)u'(w-m)}{V_a^* u'\left(G_b^* m\right)}$, which is the threshold in (17) with $V^* = V_a^*$ and $G^* = G_b^*$, we consider the hypothetical environment in which she is given extra resources up to the new wealth level of w^h but is allowed to buy PA (with the payout G_a^*) and private annuity (with the payout V_a^*). We want to find the level of w^h such that the individual's optimal choices of $c_{1\theta}$ and $c_{2\theta}$ will be the same (and thus the first-order condition is the same) as those chosen at her original level of wealth (w) when both PA (with the payout G_b^*) and private annuity (with the payout V_a^*) are available. We can use (10) and (11) to show that the optimal annuity choices are related by

$$\nu_\theta\left(V_a^*, G_a^*, w^h\right) = \nu_\theta\left(V_a^*, G_b^*, w\right) + \frac{G_b^* - G_a^*}{V_a^*} m, \tag{F10}$$

and the actual and hypothetical levels of wealth are related by

$$w^h = w + \frac{G_b^* - G_a^*}{V_a^*} m, \tag{F11}$$

where $\nu_\theta\left(V_a^*, G_b^*, w\right)$ and $\nu_\theta\left(V_a^*, G_a^*, w^h\right)$ are determined according to (F5).

We now examine $\frac{\nu_\theta\left(V_a^*, G_a^*, w^h\right)}{\nu_\theta\left(V_a^*, G_a^*, w\right)}$, where $\nu_\theta\left(V_a^*, G_a^*, w^h\right)$ and $\nu_\theta\left(V_a^*, G_a^*, w\right)$ are the optimal choices in two similar environments which have the same annuity payouts (V_a^* and G_a^*) but different wealth levels. When the utility function $U\left(c_{1\theta}, c_{2\theta}; \theta\right)$ in (1) is homothetic, condition (2) holds, leading to the result that the wealth elasticity of consumption at either period is unity. Together with (10) and (11) with $\gamma_\theta = m$, the share of "equivalent annuity" purchase to "total wealth" is constant, where the level of total wealth is captured by $w + \frac{G-V}{V}m$ instead of w and the amount of equivalent annuity is given by $\nu_\theta\left(V, G, w\right) + \frac{G}{V}m$ instead of $\nu_\theta\left(V, G, w\right)$.[40] Putting $t = \frac{w^h + \frac{G_a^* - V_a^*}{V_a^*}m}{w + \frac{G_a^* - V_a^*}{V_a^*}m} = \frac{\left(w + \frac{G_b^* - G_a^*}{V_a^*}m\right) + \frac{G_a^* - V_a^*}{V_a^*}m}{w + \frac{G_a^* - V_a^*}{V_a^*}m}$ in (2) where w^h is defined in (F11), it can be shown that

$$\frac{\nu_\theta\left(V_a^*, G_a^*, w^h\right) + \frac{G_a^*}{V_a^*}m}{\nu_\theta\left(V_a^*, G_a^*, w\right) + \frac{G_a^*}{V_a^*}m} = t = \frac{w + \frac{G_b^* - V_a^*}{V_a^*}m}{w + \frac{G_a^* - V_a^*}{V_a^*}m} = \frac{V_a^*\left(w - m\right) + G_b^* m}{V_a^*\left(w - m\right) + G_a^* m} \qquad \text{(F12)}$$

is a constant (and is larger than 1).

Combining (F10) and (F12), it can be shown that[41]

$$\begin{aligned}
\frac{\nu_\theta\left(V_a^*, G_b^*, w\right)}{\nu_\theta\left(V_a^*, G_a^*, w\right)} &= \frac{\nu_\theta\left(V_a^*, G_b^*, w\right)}{\frac{1}{t}\left[\nu_\theta\left(V_a^*, G_a^*, w^h\right) + \frac{G_a^*}{V_a^*}m\right] - \frac{G_a^*}{V_a^*}m} \\
&= \frac{\nu_\theta\left(V_a^*, G_b^*, w\right)}{\frac{1}{t}\left[\nu_\theta\left(V_a^*, G_b^*, w\right) + \frac{G_b^*}{V_a^*}m\right] - \frac{G_a^*}{V_a^*}m} \\
&= \frac{V_a^*\left(w - m\right) + G_b^* m}{\left[V_a^*\left(w - m\right) + G_a^* m\right] + \frac{\left(G_b^* - G_a^*\right)m(w-m)}{\nu_\theta\left(V_a^*, G_b^*, w\right)}}
\end{aligned} \qquad \text{(F13)}$$

is strictly increasing in θ for $\theta > \theta_{mb}\left(V_a^*, G_b^*\right) = \frac{(1+\rho)u'(w-m)}{V_a^* u'\left(G_b^* m\right)}$, because $\nu_\theta\left(V_a^*, G_b^*, w\right)$ is increasing in θ for interior solutions. Together with $\frac{\nu_\theta\left(V_a^*, G_b^*, w\right)}{\nu_\theta\left(V_a^*, G_a^*, w\right)} = 0$ for

[40] Note $c_{1\theta}$ in (10) can be written as $c_{1\theta} = (w + \frac{G-V}{V}m) - (\nu_\theta + \frac{G}{V}m)$ while $c_{2\theta}$ in (11) being written as $c_{2\theta} = V(\nu_\theta + \frac{G}{V}m)$. Thus, $w + \frac{G-V}{V}m$ and $\nu_\theta + \frac{G}{V}m$ can be interpreted as total wealth and equivalent annuity, respectively.

[41] In deriving (F13), we use (F12) in the first equality, and use (F10) in the second equality. To obtain the third equality, we use straightforward algebra to show that $\frac{1}{t}\frac{G_b^*}{V_a^*}m - \frac{G_a^*}{V_a^*}m = \frac{(G_b^* - G_a^*)m(w-m)}{V_a^*(w-m) + G_b^* m} > 0$.

$\theta \in [\theta_{mb}(V_a^*, G_a^*), \theta_{mb}(V_a^*, G_b^*)]$ where $\theta_{mb}(V_a^*, G_a^*) = \frac{(1+\rho)u'(w-m)}{V_a^* u'(G_a^* m)}$, we conclude that the Chebyshev's Sum[42]

$$\int_{\theta_{mb}(V_a^*,G_a^*)}^{\overline{\theta}} \int_{\theta_{mb}(V_a^*,G_a^*)}^{\overline{\theta}} (x-y)\left[\frac{\nu_x(V_a^*, G_b^*, w)}{\nu_x(V_a^*, G_a^*, w)} - \frac{\nu_y(V_a^*, G_b^*, w)}{\nu_y(V_a^*, G_a^*, w)}\right]$$

$$[\nu_x(V_a^*, G_a^*, w)]\,[\nu_y(V_a^*, G_a^*, w)]\, dF(x)dF(y)$$

is positive, leading to[43]

$$K(V_a^*, G_b^*, w) = \frac{(1+r)\int_{\theta_{mb}(V_a^*,G_a^*)}^{\overline{\theta}} \nu_\theta(V_a^*, G_b^*, w)\, dF(\theta)}{\int_{\theta_{mb}(V_a^*,G_a^*)}^{\overline{\theta}} \theta\nu_\theta(V_a^*, G_b^*, w)\, dF(\theta)}$$

$$< \frac{(1+r)\int_{\theta_{mb}(V_a^*,G_a^*)}^{\overline{\theta}} \nu_\theta(V_a^*, G_a^*, w)\, dF(\theta)}{\int_{\theta_{mb}(V_a^*,G_a^*)}^{\overline{\theta}} \theta\nu_\theta(V_a^*, G_a^*, w)\, dF(\theta)} \equiv V_a^*. \tag{F14}$$

Combining (F8), (F9), (F14) and the continuity of the $K(V, G_b^*, w)$ function over V, we conclude that there exists an equilibrium value V_b^* such that $V_b^* \equiv K(V_b^*, G_b^*; w) < V_a^*$. This proves $V_{MPA}^* < V_{MPAf}^* < V_{VPAc}^*$.

Combining the above results with Proposition 1, we obtain (34). ■

9.5 Appendix G: Proof of Lemma 2

We consider optimal annuity choices (γ_θ^*, ν_θ^* and $\widehat{\nu}_\theta^*$) as functions of θ. (See Figure A2.) Choose the point (n, ν_n^*) on the ν_θ^* curve such that

$$\nu_n^* = \frac{G^* - \widehat{V}^*}{\widehat{V}^* - V^*} m > 0. \tag{G1}$$

Since ν_n^* is an interior solution, (15) with $G = G^*$ and $V = V^*$ implies that $u'(w - m - \nu_n^*) = \frac{n}{1+\rho} V^* u'(G^* m + V^* \nu_n^*)$, which is equivalent to

$$u'\left(w - \frac{G^* - V^*}{\widehat{V}^* - V^*} m\right) = \frac{n}{1+\rho} V^* u'\left(\widehat{V}^* \frac{G^* - V^*}{\widehat{V}^* - V^*} m\right). \tag{G2}$$

[42] Note that $\nu_\theta(V_a^*, G_a^*, w) = 0$ for $\theta \in [\underline{\theta}, \theta_{mb}(V_a^*, G_a^*)]$ and $\nu_\theta(V_a^*, G_b^*, w) = 0$ for $\theta \in [\underline{\theta}, \theta_{mb}(V_a^*, G_b^*)]$. Thus, we need to adjust the lower limit of integration in this Chebyshev's Sum.

[43] The results that $\nu_\theta(V_a^*, G_a^*, w) = 0$ and $\nu_\theta(V_a^*, G_b^*, w) = 0$ for $\theta \in [\underline{\theta}, \theta_{mb}(V_a^*, G_a^*)]$ have been used in the derivation.

Combining (21), (22) and (G2), we obtain $n \leq \overline{\theta}$. Based on (16) and (G1), we conclude that when $\theta \in [n, \overline{\theta}]$,

$$\nu_\theta^* \geq \nu_n^* = \frac{G^* - \widehat{V}^*}{\widehat{V}^* - V^*} m. \tag{G3}$$

Besides the point (n, ν_n^*), we find a related point $(n, \widehat{\nu}_n)$, where

$$\widehat{\nu}_n = \nu_n^* + m = \frac{G^* - V^*}{\widehat{V}^* - V^*} m. \tag{G4}$$

The points (n, ν_n^*) and $(n, \widehat{\nu}_n)$ are linked by the following relationship. Using (3), (4), (10) and (11), it is easy to see that the consumption levels (c_{1n}^* and c_{2n}^*) corresponding to the optimal choice ν_n^* after the PA plan is introduced are identical to those corresponding to $\widehat{\nu}_n$ (which is not necessarily the optimal choice) before the PA plan is introduced. As a result, the corresponding utility levels are the same. Denote this level as $U_n^* = U(c_{1n}^*, c_{2n}^*; n)$.

Combining (15), (32) and (G4) and using the definitions of c_{1n}^* and c_{2n}^*, we obtain

$$u'(w - \widehat{\nu}_n) = u'(c_{1n}^*) = \frac{n}{1+\rho} V^* u'(c_{2n}^*) < \frac{n}{1+\rho} \widehat{V}^* u'\left(\widehat{V}^* \widehat{\nu}_n\right),$$

which implies that

$$\widehat{\nu}_n < \widehat{\nu}_n^*, \tag{G5}$$

after using (6) with $\widehat{V} = \widehat{V}^*$. Since $\widehat{\nu}_n$ is different from $\widehat{\nu}_n^*$ according to (G5), we conclude that $\widehat{\nu}_n$ is feasible but not optimal. Therefore,

$$U_n^* = U(c_{1n}^*, c_{2n}^*; n) < U(\widehat{c}_{1n}^*, \widehat{c}_{2n}^*; n) = \widehat{U}_n^*. \tag{G6}$$

Next, we show by contradiction that

$$c_{2\theta}^* < \widehat{c}_{2\theta}^* \tag{G7}$$

for $\theta \in [n, \overline{\theta}]$. Suppose it were true that $c_{2\theta}^* \geq \widehat{c}_{2\theta}^*$. Based on (6) and (15), we have $u'(\widehat{c}_{1\theta}^*) = \frac{\theta}{1+\rho} \widehat{V}^* u'(\widehat{c}_{2\theta}^*)$ and $u'(c_{1\theta}^*) = \frac{\theta}{1+\rho} V^* u'(c_{2\theta}^*)$. Given (32) and the supposed inequality $c_{2\theta}^* \geq \widehat{c}_{2\theta}^*$, these two first-order conditions imply that $u'(c_{1\theta}^*) < u'(\widehat{c}_{1\theta}^*)$, and thus $c_{1\theta}^* > \widehat{c}_{1\theta}^*$. Using (3) and (10), we obtain

$$\widehat{\nu}_\theta^* > m + \nu_\theta^*, \tag{G8}$$

Using (4) and (11), the supposed inequality $c_{2\theta}^* \geq \widehat{c}_{2\theta}^*$ implies that

$$G^* m + V^* \nu_\theta^* \geq \widehat{V}^* \widehat{\nu}_\theta^*. \tag{G9}$$

Substituting (G8) into (G9) and simplifying, we obtain $\nu_\theta^* < \frac{G^* - \widehat{V}^*}{\widehat{V}^* - V^*} m$, which, however, contradicts (G3). Therefore, we conclude that (G7) is correct.

Lemma 1 and (G7) imply that $\frac{\partial \left(U_\theta^* - \widehat{U}_\theta^*\right)}{\partial \theta} < 0$ when $\theta \in \left[n, \overline{\theta}\right]$. Together with (G6), we conclude that $U_\theta^* - \widehat{U}_\theta^* < 0$, including when $\theta = \overline{\theta}$. This proves Lemma 2. ■

9.6 Appendix H: Proof of Lemma A1

We prove by contradiction. Suppose that $c_{2\theta}^*$ and $\widehat{c}_{2\theta}^*$ intersect at least two times in Region C. Select arbitrarily two levels of θ, denoted by j and k with $k > j$, that the intersection occurs. Thus,

$$c_{2\theta}^* = \widehat{c}_{2\theta}^*; \ \theta = j, k \ \& \ k > j. \tag{H1}$$

Since the annuitization choice $(\widehat{\nu}_\theta^*)$ is increasing in θ according to (7) and $\widehat{c}_{2\theta}^*$ is related to $\widehat{\nu}_\theta^*$ according to (4), we have

$$\widehat{c}_{2j}^* \neq \widehat{c}_{2k}^*. \tag{H2}$$

Consider either of the two intersecting points, say, the point at $\theta = j$. The assumption of homothetic and time-separable utility function in (1) and (2) implies that the equality $\frac{u'(t\widehat{c}_{1j}^*)}{u'(t\widehat{c}_{2j}^*)} = \frac{u'(\widehat{c}_{1j}^*)}{u'(\widehat{c}_{2j}^*)}$ holds for any positive value of t before the PA plan is introduced. Similarly, $\frac{u'(tc_{1j}^*)}{u'(tc_{2j}^*)} = \frac{u'(c_{1j}^*)}{u'(c_{2j}^*)}$ after the PA plan is introduced. The two equalities, together with $c_{2j}^* = \widehat{c}_{2j}^*$ according to (H1), imply that

$$\frac{u'(t\widehat{c}_{1j}^*)}{u'(tc_{1j}^*)} = \frac{u'(\widehat{c}_{1j}^*)}{u'(c_{1j}^*)}. \tag{H3}$$

The annuitization choices ($\widehat{\nu}_\theta^*$ and ν_θ^*) are interior solutions in Region C before and after the PA plan is introduced, respectively. Thus, the first-order conditions (6) and (15) hold for both j and k. The two conditions before the introduction of the PA plan lead to $\frac{u'(\widehat{c}_{1j}^*)}{u'(\widehat{c}_{1k}^*)} = \frac{ju'(\widehat{c}_{2j}^*)}{ku'(\widehat{c}_{2k}^*)}$, and the two conditions after the introduction of the PA plan lead to $\frac{u'(c_{1j}^*)}{u'(c_{1k}^*)} = \frac{ju'(c_{2j}^*)}{ku'(c_{2k}^*)}$. Combining these two equalities with (H1), we have

$$\frac{u'(\widehat{c}_{1j}^*)}{u'(c_{1j}^*)} = \frac{u'(\widehat{c}_{1k}^*)}{u'(c_{1k}^*)}. \tag{H4}$$

Choose the particular value of t (to be called t_0) such that

$$t_0 \widehat{c}_{1j}^* = \widehat{c}_{1k}^*. \tag{H5}$$

Substituting (H5) into (H3), and using (H4), we obtain $\frac{u'(\widehat{c}_{1k}^*)}{u'(t_0 c_{1j}^*)} = \frac{u'(\widehat{c}_{1k}^*)}{u'(c_{1k}^*)}$, which implies that

$$t_0 c_{1j}^* = c_{1k}^*. \tag{H6}$$

Combining (H5) and (H6) leads to

$$\frac{\widehat{c}_{1j}^*}{c_{1j}^*} = \frac{\widehat{c}_{1k}^*}{c_{1k}^*}. \tag{H7}$$

The various budget constraints (3), (4), (10) and (11) hold for both $\theta = j$ and $\theta = k$, where γ_θ^* equals to m in Region C. Using (H7) and performing algebraic manipulations, we obtain

$$\frac{V^*w + (G^* - V^*)m - c_{2j}^*}{\widehat{V}^*w - \widehat{c}_{2j}^*} = \frac{V^*c_{1j}^*}{\widehat{V}^*\widehat{c}_{1j}^*} = \frac{V^*c_{1k}^*}{\widehat{V}^*\widehat{c}_{1k}^*} = \frac{V^*w + (G^* - V^*)m - c_{2k}^*}{\widehat{V}^*w - \widehat{c}_{2k}^*},$$

which, with the use of the two equalities in (H1), can further be reduced to

$$\left[(\widehat{V}^* - V^*)w - (G^* - V^*)m\right]\left(\widehat{c}_{2j}^* - \widehat{c}_{2k}^*\right) = 0. \tag{H8}$$

Since the term $\left[(\widehat{V}^* - V^*)w - (G^* - V^*)m\right]$ in (H8) is non-zero,[44] we obtain $\widehat{c}_{2j}^* = \widehat{c}_{2k}^*$, which contradicts (H2). Thus, it is impossible for $c_{2\theta}^*$ and $\widehat{c}_{2\theta}^*$ to intersect two or more times in Region C. ■

9.7 Appendix I: Proof of Proposition 4

For the MPAf plan, it is convenient to label $\theta \in [\underline{\theta}, \theta_{fo})$ as Region A1, $\theta \in [\theta_{fo}, \theta_{om})$ as Region A2, $\theta \in [\theta_{om}, \theta_{mb})$ as Region B, and $\theta \in [\theta_{mb}, \overline{\theta}]$ as Region C. (See Figure A3.)

[Insert Figure A3 here.]

It is easy to see that Lemmas 1 and A1 are also applicable to the MPAf plan.[45]

[44] One can show that there exists a value of $\theta \in [\underline{\theta}, \overline{\theta}]$ such that $\widehat{\nu}_\theta^* = \frac{G^* - V^*}{\widehat{V}^* - V^*}m$ satisfies the first-order condition (6). Thus, $\frac{G^* - V^*}{\widehat{V}^* - V^*}m$ must be smaller than the retiree's wealth w, leading to the result that $\left[(\widehat{V}^* - V^*)w - (G^* - V^*)m\right]$ is positive.

[45] The proof of Lemma A1 is also applicable to the MPAf plan. The proof of Lemma 1 for the MPAf plan, which is very similar to that in Appendix B, is omitted.

9.7.1 Proof of Lemma 3

The following three results can be shown. First, the function $U_{\underline{\theta}}\left(\underline{f}\right)$ first increases in $\underline{f}$ and then decreases in $\underline{f}$ after reaching the unique maximum. Second, there are two roots to (40). Moreover, $U_{\underline{\theta}}\left(\widehat{v}_{\underline{\theta}}^{*}\right) > \widehat{U}_{\underline{\theta}}^{*}$ because $G^{*} > \widehat{V}^{*}$ according to Proposition 1(a). Thus, the smaller root is eliminated because it is smaller than $\widehat{v}_{\underline{\theta}}^{*}$ and (28) is violated. Third, $U_{\underline{\theta}}\left(\underline{f}\right)$ is decreasing in the neighborhood of the larger root. As a result, (41) is satisfied.

Combining all the above results, we conclude that (41) holds if condition (39) is satisfied. On the other hand, if (42) holds, then $U_{\underline{\theta}}^{*} \geq \widehat{U}_{\underline{\theta}}^{*}$ and (41) does not hold. ■

9.7.2 Proof of Proposition 4

First, we focus on Region A2. The choices $\widehat{c}_{1\theta}^{*}$ and $\widehat{c}_{2\theta}^{*}$ are feasible under the MPAf plan because of (31), but they are not optimal since they do not satisfy the first-order condition (23). Moreover, the combination of $c_{1\theta}^{*} \leq \widehat{c}_{1\theta}^{*}$ and $c_{2\theta}^{*} \leq \widehat{c}_{2\theta}^{*}$ is inconsistent with the two lifetime budget constraints (with and without the PA plan), and the combination of $c_{1\theta}^{*} > \widehat{c}_{1\theta}^{*}$ and $c_{2\theta}^{*} \leq \widehat{c}_{2\theta}^{*}$ is inconsistent with the two relevant first-order conditions (6) and (23). Combining the above results, we conclude that $c_{2\theta}^{*} > \widehat{c}_{2\theta}^{*}$ and $U_{\theta}^{*} > \widehat{U}_{\theta}^{*}$ in Region A2. This is Result 1.

In Region A1, $c_{2\theta}^{*}$ is constant but $\widehat{c}_{2\theta}^{*} = \widehat{V}^{*}\widehat{\nu}_{\theta}^{*}$ is increasing in θ. Since $c_{2\theta}^{*} > \widehat{c}_{2\theta}^{*}$ when $\theta = \theta_{fo}$, we conclude that $c_{2\theta}^{*} > \widehat{c}_{2\theta}^{*}$ for all retirees in Region A1. This is Result 2.

Applying Lemma 1, we conclude that $\frac{\partial\left(U_{\theta}^{*}-\widehat{U}_{\theta}^{*}\right)}{\partial\theta} > 0$ in Regions A1 and A2; thus, $U_{\theta}^{*} - \widehat{U}_{\theta}^{*}$ is increasing in these two regions. Combining it with condition (41) and $U_{\theta}^{*} > \widehat{U}_{\theta}^{*}$ in Region A2, we conclude that $U_{\theta}^{*} - \widehat{U}_{\theta}^{*}$ starts from a negative value when $\theta = \underline{\theta}$, and it increases and crosses the horizontal axis *once* to become positive in Region A1. This is Result 3.

Next, it is easy to see that $c_{2\theta}^{*}$ is constant and $\widehat{c}_{2\theta}^{*}$ is increasing in Region B. Together with Result 1, we conclude that there are two possible cases in Region B: (a) $\widehat{c}_{2\theta}^{*}$ is increasing and passes the constant level of $c_{2\theta}^{*}$ *once*, and as a result, $U_{\theta}^{*} - \widehat{U}_{\theta}^{*}$ is first increasing and then decreasing after reaching a maximum; or (b) $\widehat{c}_{2\theta}^{*}$ is increasing but still below $c_{2\theta}^{*}$, and as a result, $U_{\theta}^{*} - \widehat{U}_{\theta}^{*}$ is increasing (and is always positive). Call them Case (a) and Case (b), respectively.

Regarding Region C, we show that the combination of $c_{1\overline{\theta}}^{*} < \widehat{c}_{1\overline{\theta}}^{*}$ and $c_{2\overline{\theta}}^{*} \geq \widehat{c}_{2\overline{\theta}}^{*}$ is inconsistent with (32) and the first-order conditions (6) and (15). On the other hand, the combination of $c_{1\overline{\theta}}^{*} \geq \widehat{c}_{1\overline{\theta}}^{*}$ and $c_{2\overline{\theta}}^{*} \geq \widehat{c}_{2\overline{\theta}}^{*}$ is inconsistent with (38). Therefore, we conclude that $c^*_{2\bar{\theta}} < \widehat{c}^*_{2\bar{\theta}}$. This is Result 4.

Combining Results 1 to 4 and Lemma A1, we end up with two possibilities. First, with Case (a), we conclude that $c^*_{2\theta}$ and $\widehat{c}^*_{2\theta}$ do not intersect in Region C, since intersecting once in Region C will lead to an inconsistency with Result 4. Therefore, there are two thresholds, with the lower threshold θ_L^{MPAf} in Region A1 and the higher threshold θ_H^{MPAf} in either Region B or C. (Some key features are shown in Figure A3, which is based on Case (a).)

On the other hand, with Case (b), we conclude that $c^*_{2\theta}$ and $\widehat{c}^*_{2\theta}$ intersect once in Region C, since no intersection will lead to an ever-increasing (and positive) $U^*_\theta - \widehat{U}^*_\theta$ in Region C, which is inconsistent with (38). Therefore, there are two thresholds, with the lower threshold θ_L^{MPAf} in Region A1 and the higher threshold θ_H^{MPAf} in Region C. ■

9.8 Appendix J: Utility effects when health and wealth are correlated

In the following, we provide a brief computational analysis in a model with wealth heterogeneity to examine the effects of the two observed PA plans on retirees' welfare. To make the analysis closely related to the main sections, we make as minimal changes as possible. We add wealth heterogeneity to the model by interpreting w in (3) as randomly distributed, and allow for the well-known feature that retirees' health and wealth are positively correlated (Meer et al., 2003; Michaud and Van Soest, 2008). We represent the two sources of randomness (survival probability and wealth) by a bivariate normal distribution, with a non-negative correlation coefficient, as follows:

$$\begin{pmatrix} \theta \\ w \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_\theta \\ \mu_w \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \psi\sigma_\theta\sigma_w \\ \psi\sigma_\theta\sigma_w & \sigma_w^2 \end{pmatrix} \right), \tag{J1}$$

where μ_θ and μ_w are the expected values of survival probability and wealth, respectively; σ_θ^2 and σ_w^2 are their variances; and ψ $(0 \leq \psi < 1)$ is the correlation coefficient.

In the following computational analysis, we use the CRRA utility function: $u(c_{i\theta}) = \frac{(c_{i\theta})^{1-\phi}-1}{1-\phi}$, where $\phi = 2$. Other parameter values are assumed to be: $\underline{\theta} = 0.1$, $\overline{\theta} = 0.9$, $\underline{w} = 2$, $\overline{w} = 8$, $\mu_\theta = 0.45$, $\sigma_\theta = 0.13$, $\mu_w = 5$, $\sigma_w = 0.9$, $\psi = 0.1$, $r = 0.3$, $\rho = 0.28$, $m = 1.2$ and $f = 1.0$.[46]

[46] Strictly speaking, we use a truncated bivariate normal distribution, since the values of survival probability and wealth outside the specified upper and lower limits are truncated. These limits are imposed to eliminate variables with inappropriate values (such as negative survival probability). Based on the bivariate normal distribution with the above parameter values, less than 1% of the distribution is truncated. Note also that in our two-period

We first consider the MPAf plan. In this environment, the annuity choices are given by

$$\widehat{\nu}^*_{\theta w} = \frac{\left(\frac{\theta \widehat{V}^*}{1+\rho}\right)^{\frac{1}{\phi}}}{\widehat{V}^* + \left(\frac{\theta \widehat{V}^*}{1+\rho}\right)^{\frac{1}{\phi}}} w, \tag{J2}$$

$$\gamma^*_{\theta w} = \begin{cases} f & \text{if } \frac{\left(\frac{\theta G^*}{1+\rho}\right)^{\frac{1}{\phi}}}{G^* + \left(\frac{\theta G^*}{1+\rho}\right)^{\frac{1}{\phi}}} w \leq f \\ \frac{\left(\frac{\theta G^*}{1+\rho}\right)^{\frac{1}{\phi}}}{G^* + \left(\frac{\theta G^*}{1+\rho}\right)^{\frac{1}{\phi}}} w & \text{if } f < \frac{\left(\frac{\theta G^*}{1+\rho}\right)^{\frac{1}{\phi}}}{G^* + \left(\frac{\theta G^*}{1+\rho}\right)^{\frac{1}{\phi}}} w < m \\ m & \text{if } \frac{\left(\frac{\theta G^*}{1+\rho}\right)^{\frac{1}{\phi}}}{G^* + \left(\frac{\theta G^*}{1+\rho}\right)^{\frac{1}{\phi}}} w \geq m \end{cases}, \tag{J3}$$

and

$$\nu^*_{\theta w} = \max \left\{ 0, \frac{\left(\frac{\theta V^*}{1+\rho}\right)^{\frac{1}{\phi}}}{V^* + \left(\frac{\theta V^*}{1+\rho}\right)^{\frac{1}{\phi}}} \left[w + \left(\frac{G^*}{\widehat{V}^*} - 1 \right) m \right] - \frac{G^*}{\widehat{V}^*} m \right\}, \tag{J4}$$

where we now use the two-dimensional subscript θw to refer to the survival probability (θ) and wealth (w) of a retiree. Moreover, the equilibrium annuity payouts are given by

$$\widehat{V}^* = \frac{(1+r) \int_{\underline{w}}^{\overline{w}} \int_{\underline{\theta}}^{\overline{\theta}} \widehat{\nu}^*_{\theta w} h(\theta, w) d\theta dw}{\int_{\underline{w}}^{\overline{w}} \int_{\underline{\theta}}^{\overline{\theta}} \theta \widehat{\nu}^*_{\theta w} h(\theta, w) d\theta dw}, \tag{J5}$$

$$G^* = \frac{(1+r) \int_{\underline{w}}^{\overline{w}} \int_{\underline{\theta}}^{\overline{\theta}} \gamma^*_{\theta w} h(\theta, w) d\theta dw}{\int_{\underline{w}}^{\overline{w}} \int_{\underline{\theta}}^{\overline{\theta}} \theta \gamma^*_{\theta w} h(\theta, w) d\theta dw}, \tag{J6}$$

and

$$V^* = \frac{(1+r) \int_{\underline{w}}^{\overline{w}} \int_{\underline{\theta}}^{\overline{\theta}} \nu^*_{\theta w} h(\theta, w) d\theta dw}{\int_{\underline{w}}^{\overline{w}} \int_{\underline{\theta}}^{\overline{\theta}} \theta \nu^*_{\theta w} h(\theta, w) d\theta dw}, \tag{J7}$$

where $h(\theta, w)$ is the joint probability density function of the vector of survival probability and wealth, $\begin{pmatrix} \theta \\ w \end{pmatrix}$, given in (J1). Based on the above information, the equilibrium payouts are $\widehat{V}^* = 2.780$, $G^* = 2.854$ and $V^* = 2.383$.

model, a period corresponds to 15 calendar years. The value of $r = 0.3$ corresponds roughly to an annual interest rate of 1.8%.

In Proposition 4 where the wealth level is constant, we have identified two thresholds (θ_L^{MPAf} and θ_H^{MPAf}) when (39) holds, based on which all annuity buyers are separated into three different groups. With the additional feature of wealth heterogeneity, these two thresholds are also present, but now each of them becomes a curve (on the plane of survival probability versus wealth) rather than a point, as observed in Panel A of Figure A4.[47] Moreover, each of the two threshold curves are downward sloping.[48]

[Insert Figure A4 here.]

We see in Panel A of Figure A4 that the two threshold curves $\theta_L^{MPAf}(w)$ and $\theta_H^{MPAf}(w)$ separate the whole population into three areas. The area below the curve $\theta_L^{MPAf}(w)$ represents the buyers who lose under the MPAf plan due to the mandatory force of the plan. These buyers have a low level of survival probability or wealth. The area above the curve $\theta_H^{MPAf}(w)$ represents the buyers who lose because of the private market distortion effect. These buyers have a high level of survival probability or wealth. Only the middle area between the two curves represents the buyers who benefit from the plan because of the severity reduction effect. The numerical analysis shows that the pattern of three-group classification in Proposition 4 is preserved in a more complicated environment with health and wealth heterogeneity.

We label annuity buyers who benefit from the MPAf plan as the winners. In the case of homogeneous wealth (as in the previous section), we have obtained the all-or-none pattern when the percentage of winners is plotted against the survival probability. According to Proposition 4, the percentage of winner is zero when $\theta < \theta_L^{MPAf}$, jumps to 100% at θ_L^{MPAf} and remaining there when $\theta_L^{MPAf} \leq \theta \leq \theta_H^{MPAf}$, but drops to zero after $\theta > \theta_H^{MPAf}$. Now consider the retirees with the same survival probability (say, $\theta = 0.5$) but

[47]When the ceiling and the floor parameters (m and f) of the MPAf plan are appropriately chosen such that conditions similar to (21) and (39) hold (which leads to $U^*_{\underline{\theta w}} - \widehat{U}^*_{\underline{\theta w}} < 0$ and $U^*_{\overline{\theta w}} - \widehat{U}^*_{\overline{\theta w}} < 0$), the two threshold curves always appear.

[48]Each of the two threshold curves is defined by the equation of identical utility level before and after the PA is introduced: $u(\widehat{c}_{1\theta w}) + \frac{\theta}{1+\rho}u(\widehat{c}_{2\theta w}) = u(c_{1\theta w}) + \frac{\theta}{1+\rho}u(c_{2\theta w})$. Differentiating totally this equation with respect to θ and w, and using the budget constraints and the envelope theorem, we obtain $\frac{d\theta}{dw} = -\frac{u'(c_{1\theta w}) - u'(\widehat{c}_{1\theta w})}{\frac{1}{1+\rho}[u(c_{2\theta w}) - u(\widehat{c}_{2\theta w})]}$. At a given level of wealth, when $\theta = \theta_L^{MPAf}$, we show that $c_{1\theta w} < \widehat{c}_{1\theta w}$ and $c_{2\theta w} > \widehat{c}_{2\theta w}$ because a retiree with this threshold level of survival probability over-purchases the PA (and thus consumes less in Period 1) under the MPAf plan. Thus, $\frac{d\theta}{dw} < 0$ and the threshold curve $\theta_L^{MPAf}(w)$ is downward sloping. Similarly, at a given wealth level, when $\theta = \theta_H^{MPAf}$, we show that $c_{1\theta w} > \widehat{c}_{1\theta w}$ and $c_{2\theta w} < \widehat{c}_{2\theta w}$ under the MPAf plan. Thus, the threshold curve $\theta_H^{MPAf}(w)$ is also downward sloping. (Note that the above results hold generally, and not only for the CRRA utility function used in our computational analysis.)

different wealth levels. Since the two threshold curves are downward sloping, they generally intersect a particular horizontal line (such as $\theta = 0.5$) at some points higher than $\underline{w}$ but lower than $\overline{w}$, leading to the result that there are some but not all winners. Therefore, the original all-or-none pattern for winners disappears when wealth is heterogeneous.

Panels B and C in Figure A4 display the percentage of winners against health and wealth, respectively. We observe that for the buyers with low survival probability or wealth, only a small percentage of them gain from the MPAf plan. They are mainly affected by the distortion caused by the mandatory floor. Similarly, only a small percentage of buyers with high survival probability or wealth gain from the MPAf plan, because they are significantly affected by the private market distortion effect. On the other hand, the majority of buyers with average survival probability or wealth benefit from the MPAf plan due to the severity reduction effect.

Now consider the VPAc plan when the parameter f equals to zero. We keep the values of the other parameters unchanged and obtain the results for the VPAc plan shown in Panels D, E and F in Figure A4. When compared with the MPAf plan, there are two major differences. First, the lower threshold curve $\theta_L^{MPAf}(w)$ in Panel A disappears under the VPAc plan (see Panel D). Second, all buyers with low survival probability (Panel E) or wealth (Panel F) are winners under the VPAc plan. Except for these differences, the other patterns, as well as the interpretation, are similar to those under the MPAf plan.

To summarize, there are two key results in the above computational analysis (as well as in unreported computations with other parameter values). First, the pattern of two-group classification in Proposition 3 and that of three-group classification in Proposition 4 are preserved in a more complicated environment with health and wealth heterogeneity. Second, while the all-or-none pattern in homogeneous wealth environment disappears in the more general environment of health and wealth heterogeneity,[49] we still observe the essential features of Propositions 3 and 4 that (a) a large proportion of good health retirees lose and a large proportion of average health retirees benefit under either PA plan, and (b) a large proportion of poor health re-

[49]The reason of the all-or-none pattern in Propositions 3 and 4 is that we only have one source of heterogeneity (in health). In this case, retirees with the same survival probability is identical in all other aspects. As a result, they either all benefit or all lose under a particular PA plan. When there are other sources of heterogeneity (such as wealth heterogeneity), retirees with a particular survival probability have different levels of wealth. Since the utility effects also depend on wealth, it is not surprising to find that among retirees with the same survival probability, some of them benefit and some lose when a particular PA plan is introduced.

tirees benefit under the VPAc plan but lose under the MPAf plan. Moreover, in the presence of health and wealth heterogeneity, there is a parallel set of results in terms of wealth: (a) a large proportion of rich retirees lose and a large proportion of middle income retirees benefit under either PA plan, and (b) a large proportion of low income retirees benefit under the VPAc plan but lose under the MPAf plan.

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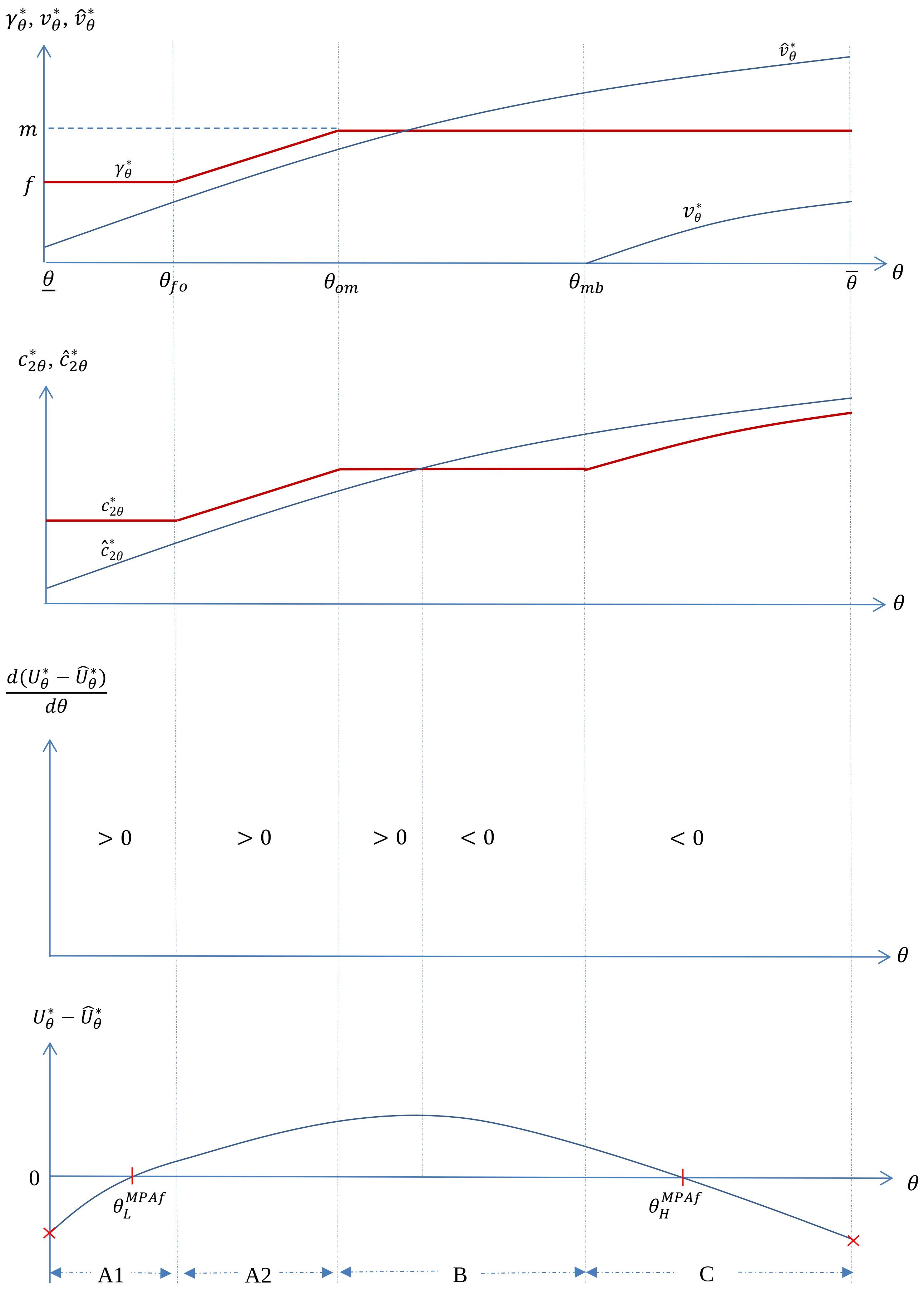

Figure A3: Utility comparison under the MPAf plan

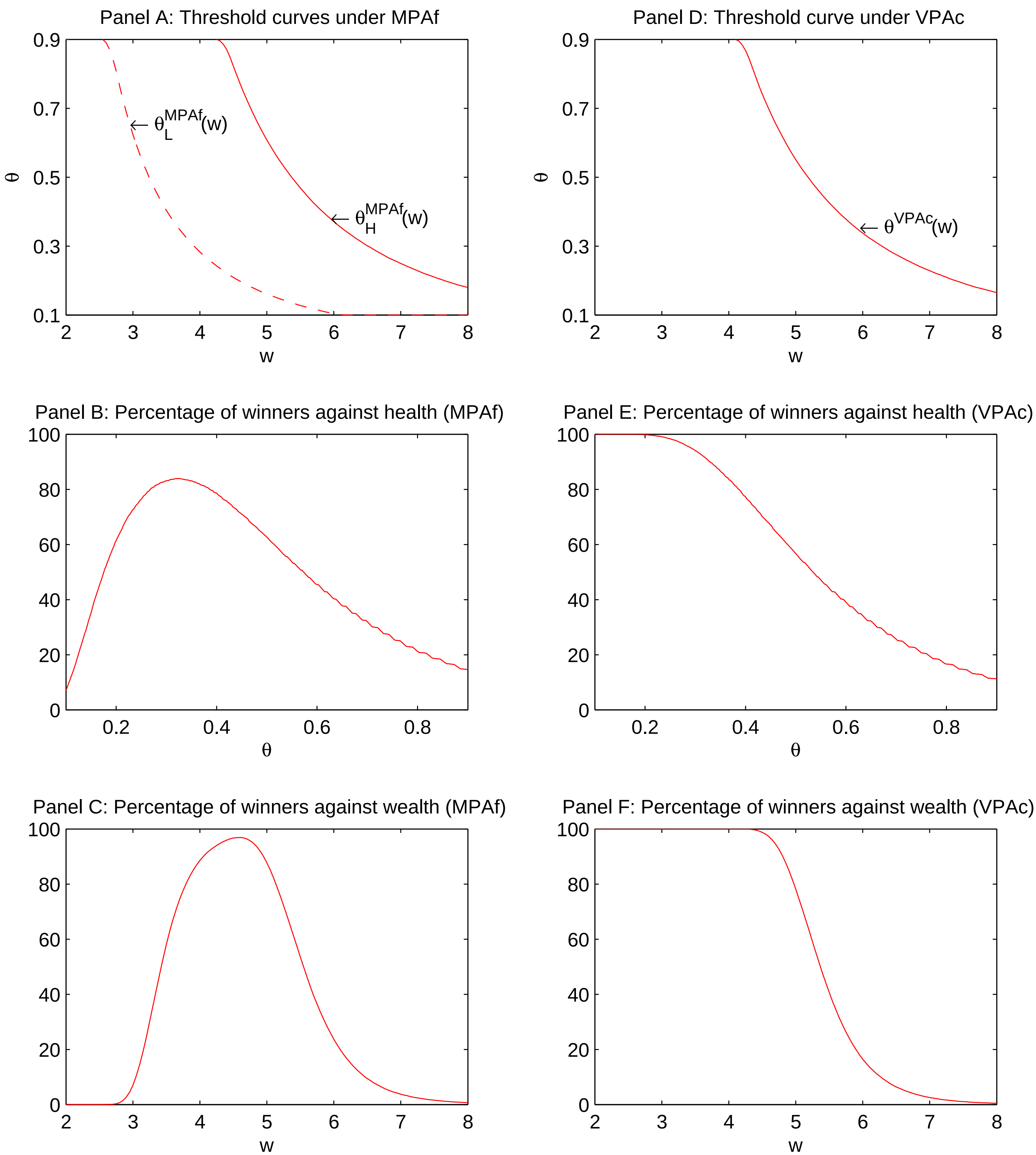

Figure A4: With health and wealth heterogeneity