Online appendix for the paper

Relational theories with null values and non-Herbrand stable models

published in Theory and Practice of Logic Programming

Vladimir Lifschitz, Karl Pichotta, and Fangkai Yang
Department of Computer Science
University of Texas at Austin
{vl,pichotta,fkyang}@cs.utexas.edu

Lemma 2

A DCA-interpretation I satisfies a second-order sentence F of the signature σ iff the Herbrand interpretation $D(I)^{=}_{\overline{Eq}}$ satisfies $F^{=}_{\overline{Eq}}$.

Proof

The proof is by induction on the size of F; size is understood as follows. About second-order sentences F and G we say that F is *smaller than* G if

- F has fewer second-order quantifiers than G, or
- F has the same number of second-order quantifiers as G, and the total number of first-order quantifiers and propositional connectives in F is less than in G.

The induction hypothesis is that the assertion of the lemma holds for all sentences that are smaller than F. If F is atomic then

$$\begin{array}{lll} I \models F & \text{iff} & F \in D(I) \\ & \text{iff} & F^{=}_{\overline{Eq}} \in D(I)^{=}_{\overline{Eq}} \\ & \text{iff} & D(I)^{=}_{\overline{Eq}} \models F^{=}_{\overline{Eq}}. \end{array}$$

If F is $G \wedge H$ then $F^{=}_{\overline{Eq}}$ is $G^{=}_{\overline{Eq}} \wedge H^{=}_{\overline{Eq}}$. Using the induction hypothesis, we calculate:

$$\begin{array}{lll} I \models F & \text{iff} & I \models G \text{ and } I \models H \\ & \text{iff} & D(I)^{=}_{\overline{Eq}} \models G^{=}_{\overline{Eq}} \text{ and } D(I)^{=}_{\overline{Eq}} \models H^{=}_{\overline{Eq}} \\ & \text{iff} & D(I)^{=}_{\overline{Eq}} \models F^{=}_{\overline{Eq}}. \end{array}$$

For other propositional connectives the reasoning is similar. If F is $\forall x G(x)$ then $F^{=}_{\overline{Eq}}$ is $\forall x \left(G(x)^{=}_{\overline{Eq}}\right)$. Using the induction hypothesis and the fact that I satisfies DCA, we calculate:

$$\begin{array}{lll} I \models F & \text{iff} & \text{for all object constants } a,\ I \models G(a) \\ & \text{iff} & \text{for all object constants } a,\ D(I)^{=}_{\overline{Eq}} \models G(a)^{=}_{\overline{Eq}} \\ & \text{iff} & D(I)^{=}_{\overline{Eq}} \models F^{=}_{\overline{Eq}}. \end{array}$$

For the first-order existential quantifier the reasoning is similar.

It remains to consider the case when F is $\exists v G(v)$, where v is a predicate variable. To simplify notation, we will assume that the arity of v is 1. For any set V of object constants, by exp_V we denote the lambda-expression[1] $\lambda x \bigvee_{a\in V}(x=a)$. Since I is a *DCA*-interpretation, $I \models F$ iff

$$\text{for some } V, \quad I \models G(exp_V)\cdot$$

By the induction hypothesis, this is equivalent to the condition

$$\text{for some } V, \quad D(I)^{=}_{Eq} \models H((exp_V)^{=}_{Eq}), \tag{1}$$

where $H(v)$ stands for $G(v)^{=}_{Eq}$. On the other hand, $F^{=}_{Eq}$ is $\exists v(Sub(v)\wedge H(v))$. The Herbrand interpretation $D(I)^{=}_{Eq}$ satisfies this formula iff

$$\text{for some } V, \quad D(I)^{=}_{Eq} \models Sub(exp_V) \text{ and } D(I)^{=}_{Eq} \models H(exp_V)\cdot \tag{2}$$

We need to show that (2) is equivalent to (1).

Consider first the part

$$D(I)^{=}_{Eq} \models Sub(exp_V) \tag{3}$$

of condition (2), that is,

$$D(I)^{=}_{Eq} \models \forall xy(exp_V(x)\wedge Eq(x,y)\rightarrow exp_V(y))\cdot$$

It is equivalent to

$$D(I)^{=}_{Eq} \models \forall y(\exists x(exp_V(x)\wedge Eq(x,y))\rightarrow exp_V(y))\cdot$$

Interpretation $D(I)^{=}_{Eq}$ satisfies the inverse of this implication, because it satisfies $\forall x Eq(x,x)$. Consequently condition (3) can be equivalently rewritten as

$$D(I)^{=}_{Eq} \models \forall y(\exists x(exp_V(x)\wedge Eq(x,y))\leftrightarrow exp_V(y))\cdot$$

The left-hand side of this equivalence can be rewritten as $\bigvee_{a\in V} Eq(a,y)$. It follows that condition (3) is equivalent to

$$D(I)^{=}_{Eq} \models \forall y\left(\bigvee_{a\in V} Eq(a,y)\leftrightarrow exp_V(y)\right)\cdot$$

Furthermore, $Eq(a,y)$ can be replaced here by $Eq(y,a)$, because $D(I)^{=}_{Eq}$ satisfies $\forall xy(Eq(x,y)\leftrightarrow Eq(y,x))$. Hence (3) is equivalent to

$$D(I)^{=}_{Eq} \models (exp_V)^{=}_{Eq} = exp_V\cdot$$

It follows that (2) is equivalent to the condition

$$\text{for some } V,\ D(I)^{=}_{Eq} \models (exp_V)^{=}_{Eq} = exp_V \text{ and } D(I)^{=}_{Eq} \models H((exp_V)^{=}_{Eq})\cdot \tag{4}$$

It is clear that (4) implies (1).

It remains to check that (1) implies (4). Assume that

$$D(I)^{=}_{Eq} \models H((exp_V)^{=}_{Eq}), \tag{5}$$

and let V' be the set of object constants a such that, for some $b\in V$, $I\models a=b$. We will show that V' can be taken as V in (4). The argument uses two properties of the set V' that are immediate from its definition:

[1] On the use of lambda-expressions in logical formulas, see (**?**, Section 3.1).

(a) $V \subseteq V'$;
(b) if $I \models a = b$ and $a \in V'$ then $b \in V'$.

Consider the first half of (4) with V' as V:

$$D(I)^{=}_{Eq} \models (exp_{V'})^{=}_{Eq} = exp_{V'}.$$

This condition can be restated as follows: for every object constant a,

$$D(I)^{=}_{Eq} \models \bigvee_{b \in V'} Eq(a,b) \quad \text{iff} \quad D(I)^{=}_{Eq} \models \bigvee_{b \in V'}(a = b),$$

or, equivalently,

$$I \models \bigvee_{b \in V'}(a = b) \quad \text{iff} \quad a \in V'.$$

The implication left-to-right follows from property (b) of V'; the implication right-to-left is obvious (take b to be a).

Consider now the second half of (4) with V' as V:

$$D(I)^{=}_{Eq} \models H((exp_{V'})^{=}_{Eq}).$$

To derive it from (5), we only need to check that

$$D(I)^{=}_{Eq} \models (exp_{V'})^{=}_{Eq} = (exp_{V})^{=}_{Eq}.$$

This claim is equivalent to

$$I \models exp_{V'} = exp_{V} \tag{6}$$

and can be restated as follows: for every object constant a,

$$I \models \bigvee_{b \in V'}(a = b) \quad \text{iff} \quad I \models \bigvee_{b \in V}(a = b).$$

The implication left-to-right is immediate from the definition of V'; the implication righ-to-left is immediate from property (a). $\square$