Online appendix for the paper

# *Module Theorem for the General Theory of Stable Models*

published in Theory and Practice of Logic Programming

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## Appendix A  Proofs

### *A.1  Splitting Lemma*

We use the splitting lemma (Ferraris et al. 2009) to prove a few theorems below.

*Splitting Lemma*
Let $F$ be a first-order sentence, and let $\mathbf{p}$, $\mathbf{q}$ be lists of distinct predicate constants. If each strongly connected component of $\mathrm{DG}[F; \mathbf{pq}]$ is a subset of $\mathbf{p}$ or a subset of $\mathbf{q}$ then

$$\mathrm{SM}[F; \mathbf{pq}] \quad \text{is equivalent to} \quad \mathrm{SM}[F; \mathbf{p}] \wedge \mathrm{SM}[F; \mathbf{q}] \ .$$

The statement is slightly more general than the one from (Ferraris et al. 2009) in that $\mathbf{p}$ and $\mathbf{q}$ are not required to be disjoint. The proof of this enhancement follows from the Version 3 of the Splitting Lemma from (Ferraris et al. 2009).

### *A.2  Proof of Lemma ??*

*Lemma ??*
$X$ is a module answer set of $(\Pi, \mathcal{I}, \mathcal{O})$ iff $X$ is an answer set of $\Pi \cup \{\{p\} \leftarrow \mid p \in \mathcal{I}\}$.

*Proof*

$$X \text{ is an answer set of } \Pi \cup \{p \leftarrow \mid p \in (\mathcal{I} \cap X)\}$$

iff

$$X \text{ is an answer set of } \Pi \cup \{p \leftarrow \mathit{not\ not\ } p \mid p \in \mathcal{I}\}$$

iff

$$X \text{ is an answer set of } \Pi \cup \{\{p\} \leftarrow \mid p \in \mathcal{I}\} \ .$$

The equivalence between the first and the second follows from the equivalence between the reducts of each program relative to $X$.

The equivalence between the second and third is because the transformation preserves strong equivalence. ∎

## A.3 Proof of Theorem ??

*Theorem ??*
Let $F$, $G$, $H$ be first-order sentences, and let $\mathbf{p}$, $\mathbf{q}$ be finite lists of distinct predicate constants. If

(a) each strongly connected component of $\mathrm{DG}[F \wedge G \wedge H;\ \mathbf{pq}]$ is a subset of $\mathbf{p}$ or a subset of $\mathbf{q}$,

(b) $F$ is negative on $\mathbf{q}$, and

(c) $G$ is negative on $\mathbf{p}$

then

$$\mathrm{SM}[F \wedge G \wedge H;\ \mathbf{pq}] \quad \text{is equivalent to} \quad \mathrm{SM}[F \wedge H;\ \mathbf{p}] \wedge \mathrm{SM}[G \wedge H;\ \mathbf{q}]\ .$$

*Proof*
By the Splitting Lemma above, $\mathrm{SM}[F \wedge G \wedge H;\ \mathbf{pq}]$ is equivalent to

$$\mathrm{SM}[F \wedge G \wedge H;\ \mathbf{p}] \wedge \mathrm{SM}[F \wedge G \wedge H;\ \mathbf{q}]\ .$$

Since $G$ is negative on $\mathbf{p}$, the first conjunctive term can be rewritten as

$$\mathrm{SM}[F \wedge H;\ \mathbf{p}] \wedge G\ . \tag{A1}$$

Similarly, the second conjunctive term can be rewritten as

$$\mathrm{SM}[G \wedge H;\ \mathbf{q}] \wedge F\ . \tag{A2}$$

It remains to observe that the second conjunctive term of each of the formulas (A1) and (A2) is entailed by the first conjunctive term of the other. ∎

## A.4 Proof of Proposition ??

*Proposition ??*
For any first-order modules $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, the following properties hold:

- $\mathbf{F}_1 \sqcup \mathbf{F}_2$ is defined iff $\mathbf{F}_2 \sqcup \mathbf{F}_1$ is defined.
- $\mathrm{SM}[\mathbf{F}_1 \sqcup \mathbf{F}_2]$ is equivalent to $\mathrm{SM}[\mathbf{F}_2 \sqcup \mathbf{F}_1]$.
- $(\mathbf{F}_1 \sqcup \mathbf{F}_2) \sqcup \mathbf{F}_3$ is defined iff $\mathbf{F}_1 \sqcup (\mathbf{F}_2 \sqcup \mathbf{F}_3)$ is defined.
- $\mathrm{SM}[(\mathbf{F}_1 \sqcup \mathbf{F}_2) \sqcup \mathbf{F}_3]$ is equivalent to $\mathrm{SM}[\mathbf{F}_1 \sqcup (\mathbf{F}_2 \sqcup \mathbf{F}_3)]$.

*Proof*

Claims (a) and (b) follow immediately from the definitions.

We prove Claim (c). Let $\mathbf{F}_i = (F_i, \mathcal{I}_i, \mathcal{O}_i)$ for each $i \in \{1, 2, 3\}$ and without loss of generality assume that each $F_i$ is a conjunction of the form $F_{i,1} \wedge \cdots \wedge F_{i,k_i}$.

*From left to right:* Assume that $(\mathbf{F}_1 \sqcup \mathbf{F}_2) \sqcup \mathbf{F}_3$ is defined. Since $\mathbf{F}_1$ and $\mathbf{F}_2$ are joinable,

(i) $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$;

(ii) each conjunctive term of $F_1$ is negative on $\mathcal{O}_2$, or is one of the conjunctive terms of $F_2$;

(iii) each conjunctive term of $F_2$ is negative on $\mathcal{O}_1$, or is one of the conjunctive terms of $F_1$;

(iv) each strongly connected component of $\mathrm{DG}[F_1 \wedge F_2; \mathcal{O}_1\mathcal{O}_2]$ is a subset of $\mathcal{O}_1$ or a subset of $\mathcal{O}_2$.

Also, since $(\mathbf{F}_1 \sqcup \mathbf{F}_2)$ and $\mathbf{F}_3$ are joinable,

(v) $(\mathcal{O}_1 \cup \mathcal{O}_2) \cap \mathcal{O}_3 = \emptyset$;

(vi) each conjunctive term of $F_1 \wedge F_2$ is negative on $\mathcal{O}_3$, or is one of the conjunctive terms of $F_3$;

(vii) each conjunctive term of $F_3$ is negative on $\mathcal{O}_1 \cup \mathcal{O}_2$, or is one of the conjunctive terms of $F_1 \wedge F_2$;

(viii) each strongly connected component of $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$ is a subset of $\mathcal{O}_1 \cup \mathcal{O}_2$ or a subset of $\mathcal{O}_3$.

We first prove that $\mathbf{F}_2 \sqcup \mathbf{F}_3$ is defined.

(ix) From (v), it follows that $\mathcal{O}_2 \cap \mathcal{O}_3 = \emptyset$.

(x) From (vi), it follows that each conjunctive term of $F_2$ is negative on $\mathcal{O}_3$ or is one of the conjunctive terms of $F_3$.

(xi) We prove that each conjunctive term of $F_3$ is negative on $\mathcal{O}_2$ or is one of the conjunctive terms of $F_2$.

Consider any conjunctive term $C$ of $F_3$. By (vii), $C$ is negative on $\mathcal{O}_1 \cup \mathcal{O}_2$, or is one of the conjunctive terms of $F_1 \wedge F_2$.

— Case 1: $C$ is negative on $\mathcal{O}_1 \cup \mathcal{O}_2$. Clearly, it is negative on $\mathcal{O}_2$ as well.

— Case 2: $C$ is one of the conjunctive terms of $F_1 \wedge F_2$. If $C$ is one of the conjunctive terms of $F_2$, the claim trivially follows. If $C$ is one of the conjunctive terms of $F_1$, by (ii), it is either negative on $\mathcal{O}_2$ or is one of the conjunctive terms of $F_2$. In either case, the claim follows.

(xii) We first prove that each strongly connected component of $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$ is contained in only one of $\mathcal{O}_1$, $\mathcal{O}_2$ or $\mathcal{O}_3$, from which the fact that each strongly connected component of $\mathrm{DG}[F_2 \wedge F_3; \mathcal{O}_2\mathcal{O}_3]$ is contained in $\mathcal{O}_2$ or $\mathcal{O}_3$ follows, as $\mathrm{DG}[F_2 \wedge F_3; \mathcal{O}_2\mathcal{O}_3]$ is a subgraph of $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$. By (i) and (v), $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ are pairwise disjoint. Consider any strongly connected component $S$ of $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$. By (viii) $S$ is a subset of $\mathcal{O}_1 \cup O_2$ or a subset of $\mathcal{O}_3$. Assume that $S$ is a subset of $\mathcal{O}_1 \cup O_2$. Clearly, $S$ is also a strongly connected component of $\mathrm{DG}[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2]$. In view

of (vii), $DG[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2]$ is the same as $DG[F_1 \wedge F_2; \mathcal{O}_1\mathcal{O}_2]$, so that $S$ is a strongly connected component of $DG[F_1 \wedge F_2; \mathcal{O}_1\mathcal{O}_2]$ as well. By (iv) $S$ is contained in $\mathcal{O}_1$ or $\mathcal{O}_2$.

We now prove that $\mathbf{F_1} \sqcup (\mathbf{F_2} \sqcup \mathbf{F_3})$ is defined.

- From (i) and (v), it follows that $\mathcal{O}_1 \cap (\mathcal{O}_2 \cup O_3) = \emptyset$;
- From (ii) and (vi), it follows that each conjunctive term of $F_1$ is negative on $\mathcal{O}_2 \cup \mathcal{O}_3$ or is one of the conjunctive terms of $F_2 \wedge F_3$;
- From (iii) and (vii), it follows that each conjunctive term of $F_2 \wedge F_3$ is negative on $\mathcal{O}_1$ or is one of the conjunctive terms of $F_1$;
- From the claim proven in (viii), it follows that each strongly connected component of $DG[F_1 \wedge F_2 \wedge F_3; \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$ is contained in $\mathcal{O}_1$ or $\mathcal{O}_2 \cup \mathcal{O}_3$.

*From right to left:* Assume that $\mathbf{F_1} \sqcup (\mathbf{F_2} \sqcup \mathbf{F_3})$ is defined. By Claim (a), $(\mathbf{F_2} \sqcup \mathbf{F_3}) \sqcup \mathbf{F_1}$ is defined, and then $(\mathbf{F_3} \sqcup \mathbf{F_2}) \sqcup \mathbf{F_1}$ is defined. By the first part of Claim (c) that was proven, $\mathbf{F_3} \sqcup (\mathbf{F_2} \sqcup \mathbf{F_1})$ is defined, and then by applying Claim (a) twice, we have that $(\mathbf{F_1} \sqcup \mathbf{F_2}) \sqcup \mathbf{F_3}$ is defined.

We now prove Claim (d). Using Theorem **??** and Claim (c),

$$
\begin{aligned}
SM[(\mathbf{F_1} \sqcup \mathbf{F_2}) \sqcup \mathbf{F_3}] &\Leftrightarrow SM[\mathbf{F_1} \sqcup \mathbf{F_2}] \wedge SM[\mathbf{F_3}] \\
&\Leftrightarrow SM[\mathbf{F_1}] \wedge SM[\mathbf{F_2}] \wedge SM[\mathbf{F_3}] \\
&\Leftrightarrow SM[\mathbf{F_1}] \wedge SM[\mathbf{F_2} \sqcup \mathbf{F_3}] \\
&\Leftrightarrow SM[\mathbf{F_1} \sqcup (\mathbf{F_2} \sqcup \mathbf{F_3})] \ .
\end{aligned}
$$

■

### A.5 Proof of Theorem ??

*Theorem* **??**
Let $\mathbf{F}_1 = (F_1, \mathcal{I}_1, \mathcal{O}_1)$ and $\mathbf{F}_2 = (F_2, \mathcal{I}_2, \mathcal{O}_2)$ be first-order modules of a signature $\sigma$ that are joinable, and, for $i = 0, 1$, let $\mathbf{c}_i$ be a subset of $\sigma$ that contains $\mathbf{c}(F_i) \cup \mathcal{O}_i$, and let $I_i$ be a $\mathbf{c}_i$-partial interpretation of $\sigma$. If $I_1$ and $I_2$ are compatible with each other, then

$$I_1 \cup I_2 \models SM[\mathbf{F}_1 \sqcup \mathbf{F}_2] \quad \text{iff} \quad I_1 \models SM[\mathbf{F}_1] \ \text{and} \ I_2 \models SM[\mathbf{F}_2] \ .$$

*Proof*
Let us identify $\mathbf{F}_1$ with $(F_1' \wedge H, \mathcal{I}_1, \mathcal{O}_1)$ and $\mathbf{F}_2$ with $(F_2' \wedge H, \mathcal{I}_2, \mathcal{O}_2)$ as in the definition of join (Definition **??**).

By definition $SM[\mathbf{F}_1 \sqcup \mathbf{F}_2]$ is $SM[F_1' \wedge F_2' \wedge H; \ \mathcal{O}_1 \cup \mathcal{O}_2]$. By Theorem **??**,

$$I_1 \cup I_2 \models SM[F_1' \wedge F_2' \wedge H; \ \mathcal{O}_1 \cup \mathcal{O}_2] \ \text{iff}$$
$$I_1 \cup I_2 \models SM[F_1' \wedge H; \ \mathcal{O}_1] \ \text{and} \ I_1 \cup I_2 \models SM[F_2' \wedge H; \ \mathcal{O}_2]$$

Clearly, $I_1 \cup I_2$ is compatible with $I_1$. Since $\mathbf{c}_1$ contains $\mathbf{c}(F_1' \wedge H) \cup \mathcal{O}_1$, it follows that $I_1 \cup I_2 \models \mathrm{SM}[F_1' \wedge H;\ \mathcal{O}_1]$ iff $I_1 \models \mathrm{SM}[F_1' \wedge H;\ \mathcal{O}_1]$. Similarly, $I_1 \cup I_2 \models \mathrm{SM}[F_2' \wedge H;\ \mathcal{O}_2]$ iff $I_2 \models \mathrm{SM}[F_2' \wedge H;\ \mathcal{O}_2]$. Consequently, the claim follows. ∎

## *A.6 Proof of Proposition* ??

*Lemma 1*

Let $\langle B, P[t], Q[t] \rangle$ be an incremental first-order theory, and let $\mathbf{P}_i$ and $\mathbf{R}_k$ be as in Proposition ??. It holds that

$$Out(\mathbf{P}_i) = pr(B \wedge P[1] \wedge \cdots \wedge P[i]),$$
$$Out(\mathbf{R}_k) = pr(B \wedge P[1] \wedge \cdots \wedge P[k] \wedge Q[k]).$$

*Proof*

We show the first clause by induction. The second clause is similar.

- Base case: $\mathbf{P}_0 = FM(B, \emptyset) = (B^\omega,\ \emptyset,\ pr(B))$.
- Inductive step: Assume that $Out(\mathbf{P}_{i-1}) = pr(B \wedge P[1] \wedge \cdots \wedge P[i-1])$. The module $FM(P[i], Out(\mathbf{P}_{i-1}))$ is

$$(P[i]^\omega,\ Out(\mathbf{P}_{i-1}),\ pr(P[i]) \backslash Out(\mathbf{P}_{i-1})) \ .$$

Thus

$$Out(\mathbf{P}_i) = Out(\mathbf{P}_{i-1}) \cup \big(pr(P[i]) \backslash Out(\mathbf{P}_{i-1})\big) = Out(\mathbf{P}_{i-1}) \cup pr(P[i])$$

and by the I.H., this is then $pr(B \wedge P[1] \wedge \cdots \wedge P[i])$.

∎

*Lemma 2*

Given any two first-order formulas $F_1, F_2$ and disjoint sets of predicate constants $\mathbf{p}_1, \mathbf{p}_2$ such that $pr(F_1) \subseteq \mathbf{p}_1$, and $F_2$ is negative on $\mathbf{p}_1$. Every strongly connected component of $\mathrm{DG}[F_1 \wedge F_2; \mathbf{p}_1 \mathbf{p}_2]$ is contained in $\mathbf{p}_1$ or $\mathbf{p}_2$.

*Proof*

Since $F_2$ is negative on $\mathbf{p}_1$, we have that $head(F_2) \cap \mathbf{p}_1 = \emptyset$. Thus every outgoing edge in the dependency graph from a predicate constant in $\mathbf{p}_1$ must be obtained from $F_1$. Since $pr(F_1) \subseteq \mathbf{p}_1$, such outgoing edge always leads to a vertex in $\mathbf{p}_1$. Consequently, every strongly connected component of $\mathrm{DG}[F_1 \wedge F_2; \mathbf{p}_1 \mathbf{p}_2]$ containing a predicate constant from $head(F_1)$ is contained in $\mathbf{p}_1$, so the claim follows. ∎

*Proposition* ??

If an incremental first-order theory $\langle B, P[t], Q[t] \rangle$ is acyclic, then the following modules are defined for all $k \geq 0$.

$$\begin{aligned}
\mathbf{P}_0 &= FM(B, \emptyset), \\
\mathbf{P}_i &= \mathbf{P}_{i-1} \sqcup FM(P[i], Out(\mathbf{P}_{i-1})), && (1 \leq i \leq k) \\
\mathbf{R}_k &= \mathbf{P}_k \sqcup FM(Q[k], Out(\mathbf{P}_k)) \ .
\end{aligned}$$

6

*Proof*

We first prove by induction that $\mathbf{P}_i$ is defined.

**Base case:** It is clear that $\mathbf{P}_0 = FM(B, \emptyset)$ is defined.

**Inductive step:** Assume that $\mathbf{P}_{i-1} = (F_{i-1}, \mathcal{I}_{i-1}, \mathcal{O}_{i-1})$ is defined for any $i > 0$. Also,

$$FM(P[i], \mathcal{O}_{i-1}) = (P[i]^\omega, \mathcal{O}_{i-1}, pr(P[i]) \backslash \mathcal{O}_{i-1})$$

is trivially defined. To show that they are joinable, we will check the following:

(i) $head(F_{i-1}) \cap (pr(P[i]) \backslash \mathcal{O}_{i-1}) = \emptyset$;
(ii) $head(P[i]^\omega) \cap \mathcal{O}_{i-1} = \emptyset$;
(iii) every strongly connected component of

$$\mathrm{DG}[F_{i-1} \wedge P[i]^\omega; \ \mathcal{O}_{i-1} \cup (pr(P[i]) \backslash \mathcal{O}_{i-1})]$$

is a subset of $\mathcal{O}_{i-1}$ or $pr(P[i]) \backslash \mathcal{O}_{i-1}$.

Note that

$$pr(F_{i-1}) \subseteq pr(B \wedge P[1] \wedge \ldots P[i-1]) \tag{A3}$$

and

$$head(P[i]^\omega) \subseteq head(P[i]) . \tag{A4}$$

*Proof of Claim (i):* By Lemma 1, $\mathcal{O}_{i-1}$ is $pr(B \wedge P[1] \wedge \cdots \wedge P[i-1])$, and Claim (i) trivially follows in view of (A3) and the fact that $head(F_{i-1}) \subseteq pr(F_{i-1})$.

*Proof of Claim (ii):* Since the theory is acyclic,

$$head(P[i]) \cap pr(B \wedge P[1] \wedge \cdots \wedge P[i-1]) = \emptyset ,$$

and from (A4) and Lemma 1, we have that

$$head(P[i]^\omega) \cap \mathcal{O}_{i-1} = \emptyset . \tag{A5}$$

*Proof of Claim (iii):* The claim follows from (A5) and Lemma 2.

We next show that $\mathbf{R}_k$ is defined. By our previous result, $\mathbf{P}_k = (F_k, \mathcal{I}_k, \mathcal{O}_k)$ is defined. It also holds that

$$FM(Q[k], \mathcal{O}_k) = (Q[k]^\omega, \mathcal{O}_k, pr(Q[k]) \backslash \mathcal{O}_k)$$

is defined trivially. The rest of the reasoning is similar to the previous one.

### A.7 Proof of Proposition ??

*Proposition ??*
Let $\langle B, P[t], Q[t] \rangle$ be an acyclic incremental first-order theory and let $\mathbf{R}_k$ be the

module as defined in the statement of Proposition **??**. For any nonnegative integer $k$,

$$I_B \cup I_{P[1]} \cup \cdots \cup I_{P[k]} \cup I_{Q[k]} \models \mathrm{SM}[\mathbf{R}_k]$$

$$\text{iff} \quad I_B \models \mathrm{SM}[FM(B, \emptyset)]$$
$$\text{and} \quad I_{P[1]} \models \mathrm{SM}[FM(P[1], Out(\mathbf{P}_0))]$$
$$\text{and} \quad \ldots$$
$$\text{and} \quad I_{P[k]} \models \mathrm{SM}[FM(P[k], Out(\mathbf{P}_{k-1}))]$$
$$\text{and} \quad I_{Q[k]} \models \mathrm{SM}[FM(Q[k], Out(\mathbf{P}_k))] \ .$$

where $I_B$ ($I_{P[1]}, \ldots, I_{P[k]}, I_{Q[k]}$, respectively) is a $\mathbf{c}(B)$-partial interpretation ($\mathbf{c}(P[1])$, $\ldots, \mathbf{c}(P[k]), \mathbf{c}(Q[k])$-partial interpretation, respectively) such that $I_B, I_{P[1]}, \ldots, I_{P[k]}, I_{Q[k]}$ are pairwise compatible.

*Proof*
Via repeated applications of Theorem **??** on $\mathbf{R}_k$ as indicated by Proposition **??**.
∎

### A.8 Proof of Proposition **??**

*Lemma 3*
Let $\langle B, P[t], Q[t] \rangle$ be an acyclic incremental first-order theory, let $k$ be a nonnegative integer, let $H_k = B \wedge P[1] \wedge \cdots \wedge P[k]$, and let $R_k$ be the $k$-expansion of the incremental theory. It holds that $I_B \cup I_{P[1]} \cup \cdots \cup I_{P[k]} \cup I_{Q[k]} \models \mathrm{SM}[R_k]$ iff

$$I_B \models \mathrm{SM}[B; \ pr(B)]$$
$$\text{and} \quad I_{P[1]} \models \mathrm{SM}[P[1]; \ pr(P[1]) \setminus pr(H_0)]$$
$$\text{and} \quad \ldots \tag{A6}$$
$$\text{and} \quad I_{P[k]} \models \mathrm{SM}[P[k]; \ pr(P[k]) \setminus pr(H_{k-1})]$$
$$\text{and} \quad I_{Q[k]} \models \mathrm{SM}[Q[k]; \ pr(Q[k]) \setminus pr(H_k)]$$

where $I_B$ ($I_{P[1]}, \ldots, I_{P[k]}, I_{Q[k]}$, respectively) is a $\mathbf{c}(B)$-partial interpretation ($\mathbf{c}(P[1])$, $\ldots, \mathbf{c}(P[k]), \mathbf{c}(Q[k])$-partial interpretation, respectively) such that $I_B, I_{P[1]}, \ldots, I_{P[k]}, I_{Q[k]}$ are pairwise compatible.

*Proof*
Formula $H_k$ is trivially negative on $pr(Q[k]) \setminus pr(H_k)$, and since the theory is acyclic, $Q[k]$ is negative on $pr(H_k)$. Also, by Lemma 2, every strongly connected component of $\mathrm{DG}[H_k \wedge Q[k]; \ pr(H_k) \cup pr(Q[k])]$ is a subset of $pr(H_k)$ or $pr(Q[k]) \setminus pr(H_k)$. By Theorem **??**, it then holds that

$$I_{H_k} \cup I_{Q[k]} \models \mathrm{SM}[R_k] \quad \text{iff} \quad I_{H_k} \models \mathrm{SM}[H_k] \text{ and } I_{Q[k]} \models \mathrm{SM}[Q[k]; \ pr(Q[k]) \setminus pr(H_k)]$$

where $I_{H_k}$ is a $\mathbf{c}(H_k)$-partial interpretation that is compatible with $I_{Q[k]}$.

Next we check by induction that $I_{H_k} \models \text{SM}[H_k]$ is equivalent to

$$
\begin{aligned}
&I_B \models \text{SM}[B] \\
&\text{and} \quad I_{P[1]} \models \text{SM}[P[1];\ pr(P[1]) \setminus pr(H_0)] \\
&\text{and} \quad \dots \\
&\text{and} \quad I_{P[k]} \models \text{SM}[P[k];\ pr(P[k]) \setminus pr(H_{k-1})] \ .
\end{aligned}
\tag{A7}
$$

**Base case:** when $k = 0$, $H_k = B$. Trivial.

**Inductive step:** Let the property hold for $H_{k-1}$. By definition, $H_k = H_{k-1} \wedge P[k]$. $H_{k-1}$ is trivially negative on $pr(P[k]) \setminus pr(H_{k-1})$ and since the theory is acyclic, $P[k]$ is negative on $pr(H_{k-1})$. Also, by Lemma 2, every strongly connected component of $\text{DG}[H_k; pr(H_k)]$ is a subset of $pr(H_{k-1})$ or $pr(P[k]) \setminus pr(H_{k-1})$. By Theorem **??**, it then holds that

$$
I_{H_k} \models \text{SM}[H_k] \quad \text{iff} \quad I_{H_{k-1}} \models \text{SM}[H_{k-1}] \text{ and } I_{P[k]} \models \text{SM}[P[k];\ pr(P[k]) \setminus pr(H_{k-1})].
$$

The property then holds by the I.H. ∎

*Lemma 4*
For any first-order formula $F$, $\text{SM}[FM(F, \mathcal{I})]$ is equivalent to $\text{SM}[F; pr(F) \setminus \mathcal{I}]$.

*Proof*
We introduce a notion that helps us prove. By $Simpl(F)$ we denote the least fixpoint of the sequence $F_0, F_1, \dots$: formula $F_0$ is defined as $F$, and $F_{i+1}$ is defined as $F_i|_{head(F_i)}$.

Formula $\text{SM}[FM(F, \mathcal{I})]$ is $\text{SM}[(F^\omega, \mathcal{I}, pr(F) \setminus \mathcal{I})]$, which in turn is defined as $\text{SM}[F^\omega;\ pr(F) \setminus \mathcal{I}]$. By Theorem 2 from (Ferraris et al. 2011), this is equivalent to $\text{SM}[F^\omega \wedge Choice(\mathcal{I});\ pr(F)]$. From the definition of $Simpl$, the latter is equivalent to $\text{SM}[Simpl(F \wedge Choice(\mathcal{I}));\ pr(F)]$, and, furthermore, by Theorem 4 from (Ferraris et al. 2011), is equivalent to $\text{SM}[F \wedge Choice(\mathcal{I});\ pr(F)]$. ∎

*Proposition **??***
Let $\langle B, P[t], Q[t] \rangle$ be an acyclic incremental theory, let $k$ be a nonnegative integer, let $R_k$ be the $k$-expansion of the incremental theory, and let $\mathbf{R}_k$ be the module as defined in Proposition **??**. For any $\mathbf{c}$-partial interpretation $I$ such that $\mathbf{c} \supseteq \mathbf{c}(R_k)$, we have that

$$
I \models \text{SM}[R_k] \quad \text{iff} \quad I \models \text{SM}[\mathbf{R}_k].
$$

*Proof*
Without loss of generality, let $I = I_B \cup I_{P[1]} \cup \cdots \cup I_{P[k]} \cup I_{Q[k]}$. By Lemma 3,

$I \models \mathrm{SM}[R_k]$ is equivalent to (A6), and by Lemma 1, this is further equivalent to

$$I_B \models \mathrm{SM}[B; \; pr(B)]$$
$$\text{and} \quad I_{P[1]} \models \mathrm{SM}[P[1]; \; pr(P[1]) \setminus Out(\mathbf{P}_0)]$$
$$\text{and} \quad \dots$$
$$\text{and} \quad I_{P[k]} \models \mathrm{SM}[P[k]; \; pr(P[k]) \setminus Out(\mathbf{P}_{k-1})]$$
$$\text{and} \quad I_{Q[k]} \models \mathrm{SM}[Q[k]; \; pr(Q[k]) \setminus Out(\mathbf{P}_k)] \; .$$

We check the following:

- $I_B \models \mathrm{SM}[B; \; pr(B)]$ iff $I_B \models \mathrm{SM}[FM(B, \emptyset)]$;
- $I_{P[i]} \models \mathrm{SM}[P[i]; \; pr(P[i]) \setminus Out(\mathbf{P}_{i-1})]$ iff $I_{P[i]} \models \mathrm{SM}[FM(P[i], Out(\mathbf{P}_{i-1}))]$;
- $I_{Q[k]} \models \mathrm{SM}[Q[k]; \; pr(Q[k]) \setminus Out(\mathbf{P}_k)]$ iff $I_{Q[k]} \models \mathrm{SM}[FM(Q[k], Out(\mathbf{P}_k))]$.

The first clause is clear. The last two clauses follow from Lemma 4.
Therefore, by Proposition **??**,

$$I_B \cup I_{P[1]} \cup \cdots \cup I_{P[k]} \cup I_{Q[k]} \models \mathrm{SM}[\mathbf{R}_k] \; .$$

■

## References

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