Online appendix for the paper

Fuzzy Answer Sets Approximations

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Appendix A Proofs

Theorem 1
Let $U \in \mathcal{I}$, and P be a program. The fixpoint $T_P^U \Uparrow \mathbf{0}$ is reached after a linear number of iterations, measured on the number of atoms appearing in P.

Proof
W.l.o.g. let us assume that all empty rule bodies are replaced by constant $\overline{1}$. Let $L_0 := \mathbf{0}$ and $L_{i+1} := T_P^U(L_i)$, $i \geq 0$. For every $i \geq 0$ and $a \in \mathcal{B}$ such that (i) $L_i(a) < L_{i+1}(a)$, there are a rule r and a literal $b \in B^+(r)$ such that $H(r) = a$ and $L_{i+1}(a) = \langle L_i, U\rangle(r)$. In particular, note that (ii) $L_{i+1}(a) \leq L_i(b)$. In this case we say that a is *inferred* by b.

Let n be the number of propositional atoms in P. We prove that any chain of inferred atoms has length at most $n+1$, which implies that n applications of T_P give the fixpoint $T_P^U \Uparrow \mathbf{0}$. Suppose on the contrary that there are $a_0, \ldots, a_{n+1}$ such that a_0 is a numeric constant and $a_{i+1} \in \mathcal{B}$ is inferred by $a_i \in \mathcal{B}$, $0 \leq i \leq n$. As P contains n propositional atoms, there exist $1 \leq j < k \leq n+1$ such that $a_j = a_k$. Hence, from (i) we have $L_{i+1}(a_{i+1}) > L_i(a_{i+1})$ for $i = 0, \ldots, n$, and thus $L_k(a_k) > L_{k-1}(a_k) \geq L_j(a_k)$ (where the last inequality is due to the monotonicity of T_P). From (ii) we have $L_{i+1}(a_{i+1}) \leq L_i(a_i)$ for $i = 0, \ldots, n$, and thus $L_k(a_k) \leq L_j(a_j) = L_j(a_k)$. Therefore, we have $L_k(a_k) > L_j(a_k)$ and $L_k(a_k) \leq L_j(a_k)$, that is, a contradiction. □

Theorem 2
For FASP programs without numeric constants and crisp sets, Definition 2 coincides with the original notion of unfounded set by Van Gelder et al. (1991).

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Proof
Let L, U be crisp sets, $L \subseteq U \subseteq \mathcal{B}$, and P be an ASP program. According to Van Gelder et al. (1991), a crisp set $Y \subseteq \mathcal{B}$ is an unfouded set for P w.r.t. (L, U) if for each $r \in P$ such that $H(r) \in Y$, (1) $B^+(r) \not\subseteq U$, or (2) $B^-(r) \cap L \neq \emptyset$, or (3) $B^+ \cap Y \neq \emptyset$. Let $X \in \mathcal{I}$ be such that $X(a) = 1$ if $a \in Y$, and $X(a) = 0$ otherwise. We have to show that Y is an unfounded set for P w.r.t. (L, U) (according to Van Gelder et al.) if and only if X is a fuzzy unfounded set for P w.r.t. (L, U) (according to Definition 2).
($\Rightarrow$) Consider a rule $r \in P$ such that $X(H(r)) > 0$. We have $H(r) \in Y$. Any of (1), (2) and (3) implies $\langle U \cap (\mathbf{1} \setminus X), L\rangle(r) = 0$.
($\Leftarrow$) Consider a rule $r \in P$ such that $H(r) \in Y$. We have $X(H(r)) = 1$, and hence $0 = [U \cap (\mathbf{1} \setminus X)](H(r)) \geq \langle U \cap (\mathbf{1} \setminus X), L\rangle(r) = 0$. We prove that if (1) and (3) do not hold, then (2) holds. Falsity of (3) implies $\langle U \cap (\mathbf{1} \setminus X), L\rangle(r) = \langle U, L\rangle(r)$, and falsity of (1) implies $\langle U, L\rangle(r) = L(\mathit{not}\ B^-(r))$. Therefore, there is an element b of $B^-(r)$ such that $L(b) = 1$, and hence $B^-(r) \cap L \neq \emptyset$, i.e., condition (2) is satisfied. □

Theorem 3
Let X_1, X_2 be two fuzzy unfounded sets for P w.r.t. (L, U). Then also $X_1 \cup X_2$ is an unfounded set for P w.r.t. (L, U).

Proof
Let $X = X_1 \cup X_2$ and $r \in P$ such that $X(H(r)) > 0$ holds. We have to show that $[U \cap (\mathbf{1} \setminus X)](H(r)) \geq \langle U \cap (\mathbf{1} \setminus X), L\rangle(r)$. Assume w.l.o.g. that $X(H(r)) = X_1(H(r))$. Since X_1 is unfounded, it follows that

$$[U \cap (\mathbf{1} \setminus X)](H(r)) = [U \cap (\mathbf{1} \setminus X_1)](H(r)) \geq \langle U \cap (\mathbf{1} \setminus X_1), L\rangle(r) \geq \langle U \cap (\mathbf{1} \setminus X), L\rangle(r),$$

which proves the result. □

Theorem 4
M is a fuzzy answer set of a program P if and only if $GUS_P^{M,M} = \mathbf{1} \setminus M$.

Proof
($\Rightarrow$) Since $\mathbf{1} \setminus (\mathbf{1} \setminus M) = M$ and $M \models P$, $\mathbf{1} \setminus M$ is an unfounded set. Moreover, we can prove that for any unfounded set X for P w.r.t. (M, M), $M \cap (\mathbf{1} \setminus X) \models P^M$ holds, from which $X \subseteq \mathbf{1} \setminus M$ follows, since M is a minimal model of P^M. Consider a rule $r \in P$. As the interepretation of $B^-(r)$ is fixed in the reduct P^M, we have to show that $[M \cap (\mathbf{1} \setminus X)](H(r)) \geq \langle M \cap (\mathbf{1} \setminus X), M\rangle(r)$ holds. If $X(H(r)) = 0$, then $[M \cap (\mathbf{1} \setminus X)](H(r)) = M(H(r)) \geq \langle M, M\rangle(r) \geq \langle M \cap (\mathbf{1} \setminus X), M\rangle$. Otherwise, it follows that $[M \cap (\mathbf{1} \setminus X)](H(r)) \geq \langle M \cap (\mathbf{1} \setminus X), M\rangle$ by Definition 2. Thus, $GUS_P^{M,M} = \mathbf{1} \setminus M$.
($\Leftarrow$) Let $GUS_P^{M,M} = \mathbf{1} \setminus M$. We first show that $M \models P$. Let $r \in P$. If $[\mathbf{1} \setminus M](H(r)) = 0$, then $M \models r$ because $M(H(r)) = 1$. If $[\mathbf{1} \setminus M](H(r)) > 0$, then $M \models r$ follows from Definition 2 and the fact $M \cap (\mathbf{1} \setminus (\mathbf{1} \setminus M)) = M$. Hence $M \models P$, which in turn implies $M \models P^M$. We now prove that for any $M' \subseteq M$ such that $M' \models P^M$, $X = \mathbf{1} \setminus M'$ is an unfounded set for P w.r.t. (M, M), from which $X \subseteq GUS_P^{M,M} = \mathbf{1} \setminus M$ and thus $M' = M$. Consider a rule $r \in P$ such that $X(H(r)) > 0$. Since $M' \models P^M$, it holds that $M'(H(r)) \geq \langle M', M\rangle(r)$. From $M' = \mathbf{1} \setminus X$ and $M' \subseteq M$, it then holds that $[M \cap (\mathbf{1} \setminus X)](H(r)) = M'(H(r)) \geq \langle M', M\rangle(r) = \langle M \cap (\mathbf{1} \setminus X), M\rangle(r)$, which shows that X is an unfounded set. □

Theorem 5
Let $L, U \in \mathcal{I}$, $L \subseteq U$, and P be a program. If $\mathbf{1} \setminus R_P^{L,U} \Downarrow \mathbf{1} \subseteq U$, then $R_P^{L,U} \Downarrow \mathbf{1} = GUS_P^{L,U}$.

Proof
($\subseteq$) We show that $X = R_P^{L,U} \Downarrow \mathbf{1}$ is an unfounded set. Consider $r \in P$ such that $X(H(r)) > 0$. We have to show that $[U \cap (\mathbf{1} \setminus X)](H(r)) \geq \langle U \cap (\mathbf{1} \setminus X), L\rangle (r)$. Since $\mathbf{1} \setminus X \subseteq U$, we can equivalently show $[\mathbf{1} \setminus X](H(r)) \geq \langle U \cap (\mathbf{1} \setminus X), L\rangle (r)$. As X is a fixpoint of $R_P^{L,P}$, we have $X(H(r)) \leq 1 - \langle U \cap (\mathbf{1} \setminus X), L\rangle (r)$, which implies that $1 - X(H(r)) \geq \langle U \cap (\mathbf{1} \setminus X), L\rangle (r)$.
($\supseteq$) Let now Y be an unfounded set. We will show that (i) $R_P^{L,U}(Y) = Y$ and (ii) $I \subseteq J$ implies $R_P^{L,U}(I) \subseteq R_P^{L,U}(J)$, from which we derive $Y \subseteq R_P^{L,U} \Downarrow \mathbf{1}$. To show (i), consider $a \in \mathcal{B}$. If $Y(a) = 0$, then also $R_P^{L,U}(Y) = 0$ by definition. Let now $Y(a) > 0$, and suppose that there is some $r \in P$ such that $Y(a) > 1 - \langle U \cap (\mathbf{1} \setminus Y), L\rangle (r)$. This is equivalent to $1 - Y(a) < \langle U \cap (\mathbf{1} \setminus Y), L\rangle (r)$. Since Y is unfounded, it must hold that $[U \cap (\mathbf{1} \setminus Y)](a) \geq \langle U \cap (\mathbf{1} \setminus Y), L\rangle (r)$, and thus we have that

$$[U \cap (\mathbf{1} \setminus Y)](a) \geq \langle U \cap (\mathbf{1} \setminus Y), L\rangle (r) > 1 - Y(a),$$

which is a contradiction with the fact that $[U \cap (\mathbf{1} \setminus Y)](a) \leq 1 - Y(a)$ must hold. To show (ii), we just note that $1 - \langle U \cap (\mathbf{1} \setminus I), L\rangle (r) \leq 1 - \langle U \cap (\mathbf{1} \setminus J), L\rangle (r)$ holds for every $r \in P$. $\square$

Theorem 6
Let $L, U \in \mathcal{I}$ and P be a program. The fixpoint $R_P^{L,U} \Downarrow \mathbf{1}$ can be computed in polynomial time on the size of P.

Proof
It is easy to see that one application of the R_P operator requires linear time in the number of rules P. Moreover, as in Theorem 1, the greatest fixpoint $R_P^{L,U} \Downarrow \mathbf{1}$ is obtained after at most as many applications of this operator as there are atoms in P. In total, this means that this fixpoint can be computed in polynomial time on the size of P. $\square$

Theorem 7
Let P be a program, L, U two interpretations, $(L', U') = W_P(L, U)$, and M an answer set of P. If $L \subseteq M \subseteq U$, then $L' \subseteq M \subseteq U'$.

Proof
By definition, $L'(a) = [T_P^U(L)](a) = \max\{\langle L, U\rangle (r) \mid H(r) = a\}$. By assumption, $L \subseteq M \subseteq U$ and hence $\langle L, U\rangle (r) \leq \langle M, M\rangle (r) = M(B(r))$ for every rule r. Since M is a model of its reduct, we have $L'(a) \leq \max\{M(B(r)) \mid H(r) = a\} \leq M(a)$.

On the other hand, since M is an answer set of P, by Theorems 4 and 5 it follows that $\mathbf{1} \setminus M = GUS_P^{M,M} = R_P^{M,M} \Downarrow \mathbf{1} \supseteq R_P^{L,U} \Downarrow \mathbf{1}$. This implies that $M \subseteq 1 \setminus R_P^{L,U} \Downarrow \mathbf{1} = U'$ $\square$

Theorem 8
Let P be a stratified program. The least fixpoint of W_P coincides with the unique answer set of P and is computable in polynomial time.

Proof

The fact that $W_P \Uparrow (\mathbf{0}, \mathbf{1})$ is the unique answer set of P follows using the same ideas presented in (Lukasiewicz 2006). Each application of W_P requires a computation of T_P and one of $R_P^{L,U} \Downarrow \mathbf{1}$, each of which is polynomial on the number of atoms in P. An increase of $L(a)$ in the i-th iteration of W_P is caused by an increase of $L(b)$ or a decrease of $U(b)$ in the previous iteration, for some $b \in \mathcal{B}$. As in the proof of Theorem 1, this implies that at most linearly many iterations (on the number of atoms appearing in P) can be applied before a fixpoint is reached. □

Theorem 9

Let $L, U \in \mathcal{I}$, and P be a program. If $M \models P$ and $L \subseteq M \subseteq U$, then $S_P^U(L) \subseteq M \subseteq U$.

Proof

Since $M \models P$ and $L \subseteq M$, we have that $[S_P^U(L)](a) \leq M(a)$ for every atom $a \in \mathcal{B}$, and hence $S_P^U(L) \subseteq M$. □

Theorem 10

Let P be a program over the Łukasiewicz t-norm, and $L, U \in \mathcal{I}$. For every atom $a \in \mathcal{B}$ it holds that $[S_P^U(L)](a) = \min\{I(a) \mid I \text{ satisfies } Ł_P \cup \{L(b) \leq I(b) \leq U(b) \mid b \in \mathcal{B}\}\}$.

Proof

Let r be a rule of the form (1), and I an interpretation. $I \models r$ if and only if

$$I(a) \geq I(B(r)) = \max\{I(b_1) + \ldots + I(b_m) - I(b_{m+1}) - \ldots - I(b_n) + 1 - m, 0\}.$$

Since $I(a) \geq 0$, $I \models P$ if and only if I satisfies the system $Ł_P$. Additionally, $L \subseteq I \subseteq U$ if and only if for every $b \in \mathcal{B}$ it holds that $L(b) \leq I(b) \leq U(b)$. Finally, as the feasible region is closed, the optimal can be reached. □

Theorem 11

Let $L, U \in \mathcal{I}$, and P be a program over the Łukasiewicz t-norm. $S_P^U(L)$ is computable in polynomial time w.r.t. the number of rules.

Proof

The computation of the S_P operator requires to solve one linear programming problem for each atom a appearing in P. Linear programming is well-known to be solvable in polynomial time on the number of restrictions. As the size of $Ł_P$ corresponds to the number of rules in P, this yields a polynomial complexity upper bound. □

Theorem 12

The fixpoint of W_P gives the well-founded semantics by Damásio and Pereira (2001).

Proof

The well-founded semantics by Damásio and Pereira (2001) is defined as the fixpoint of the following operator (adapted to our notation):

$$\Omega(L, U) := (T_P^U \Uparrow \mathbf{0}, T_P^L \Uparrow \mathbf{0})$$

Let $U' := T_P^L \Uparrow \mathbf{0}$. The claim immediately follows by Theorem 5 and the following property (proved below):

$$U' \subseteq U \Longrightarrow \mathbf{1} \setminus U' = GUS_P^{L,U}.$$

We first prove that $\mathbf{1} \setminus U'$ is unfounded w.r.t. (L, U). Let r be a rule in P such that $U'(H(r)) > 0$. We have

$$\begin{aligned}
[U \cap (\mathbf{1} \setminus (\mathbf{1} \setminus U')](H(r)) &= [U \cap U'](H(r)) = U'(H(r)) \\
&\geq \max\{\langle U, L\rangle(r') \mid r' \in P, H(r') = H(r)\} \\
&\geq \langle U, L\rangle(r) \\
&\geq \langle U \cap U', L\rangle(r) \\
&= \langle U \cap (\mathbf{1} \setminus (\mathbf{1} \setminus U')), L\rangle(r).
\end{aligned}$$

We complete the proof by proving $X \subseteq \mathbf{1} \setminus U'$, or equivalently $U' \subseteq \mathbf{1} \setminus X$, for every unfounded set X w.r.t. (L, U). To this aim, let U_n be the n-th application of T_P^U to $\mathbf{0}$, $n \geq 0$. We prove $U_n \subseteq \mathbf{1} \setminus X$ by induction on $n \geq 0$.

For $n = 0$, the result holds trivially, since $U_0 = \mathbf{0}$. Suppose now that $U_n \subseteq \mathbf{1} \setminus X$ holds for $n \geq 0$ in order to show $U_{n+1} \subseteq \mathbf{1} \setminus X$. Since T_P is monotonic, we know that $U_n \subseteq U'$. By combining with the induction hypothesis and the original assumption $U' \subseteq U$, we have that $U_n \subseteq U \cap (\mathbf{1} \setminus X)$. Consider now an atom a such that $X(a) > 0$. We have

$$\begin{aligned}
U_{n+1}(a) &= \max\{\langle(U_n, L\rangle(r) \mid r \in P, H(r) = a\} \\
&\leq \max\{\langle U \cap (\mathbf{1} \setminus X), L\rangle(r) \mid r \in P, H(r) = a\} \\
&\leq [U \cap (\mathbf{1} \setminus X)](a) \qquad \text{(A1)} \\
&\leq [\mathbf{1} \setminus X](a),
\end{aligned}$$

where line (A1) follows from the assumption that X is an unfounded set w.r.t. (L, U). Hence, $U_{n+1}(a) \leq 1 - X(a)$ holds for every $a \in \mathcal{B}$, which complete our proof. □

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