Simulating Dynamic Systems Using Linear Time Calculus Theories - Online Appendix

Bart Bogaerts, Joachim Jansen, Maurice Bruynooghe,

Broes De Cat, Joost Vennekens, and Marc Denecker

Department of Computer Science, KU Leuven

(*e-mail:* {firstname.lastname}@cs.kuleuven.be)

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Appendix A Weak Progression

In this section, we show how a weaker variant of progression can be defined using three-valued logic. We will restrict our attention to function-free vocabularies (i.e.,vocabularies containing only constant and predicate symbols) here to simplify the presentation. However, all definitions can be extended to the general case.

A.1 Three-valued logic

We briefly summarise some concepts from three-valued logic. A truth-value is one of the following: $\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$ (true, false and unknown). We define $\mathbf{f}^{-1} = \mathbf{t}, \mathbf{t}^{-1} = \mathbf{f}$ and $\mathbf{u}^{-1} = \mathbf{u}$. We define two orders on truth values: the precision order $\leq_p$ is given by $\mathbf{u} \leq_p \mathbf{t}$ and $\mathbf{u} \leq_p \mathbf{f}$. And the truth order $\leq$ is given by $\mathbf{f} \leq \mathbf{u} \leq \mathbf{t}$.

Definition Appendix A.1

A *partial set* $\mathcal{P}$ over D is a function from D to $\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$.

The precision order is extended to partial sets over D: $\mathcal{P} \leq_p \mathcal{P}'$ if for all $d \in D$: $\mathcal{P}(d) \leq_p \mathcal{P}'(d)$.

A partial Σ-structure J consists of 1) a *domain*, D^J: a set of elements, and 2) a mapping associating a value to each symbol in Σ. For predicate symbols P of arity n, this is a partial set P^J over $(D^J)^n$. For constants, this is a value in D^J.

We assume that a (partial) structure also interprets variable symbols and denote $J[x : d]$ for the structure equal to J except interpreting x by d.

If J and J' are two partial structures with the same interpretation for constants, J is less precise than J' ($J \leq_p J'$) if for all symbols σ, $\sigma^J \leq_p \sigma^{J'}$. A partial structure J is two-valued if interpretations of its symbols map nothing to $\mathbf{u}$. A two-valued partial structure is exactly a structure.

Definition Appendix A.2
Given a partial structure J, the Kleene valuation (Kl_J) is defined inductively based on the Kleene truth tables (Kleene 1938):

- $Kl_J(P(\bar{t})) = P^J(\bar{t}^J)$,
- $Kl_J(\neg\varphi) = (Kl_J(\varphi))^{-1}$
- $Kl_J(\varphi \wedge \psi) = \min_{\leq}(Kl_J(\varphi), Kl_J(\psi))$
- $Kl_J(\varphi \vee \psi) = \max_{\leq}(Kl_J(\varphi), Kl_J(\psi))$
- $Kl_J(\forall x : \varphi) = \min_{\leq}\left\{Kl_{J[x:d]}(\varphi) \mid d \in D^J\right\}$
- $Kl_J(\exists x : \varphi) = \max_{\leq}\left\{Kl_{J[x:d]}(\varphi) \mid d \in D^J\right\}$

The Kleene valuation is extended to definitions and theories. For definitions, intuitively, the value of Δ is true if all its defined atoms are two-valued and have the correct (defined) interpretation, its value is false if some defined atom is interpreted incorrectly, and is unknown otherwise. The exact definition can be found in (Denecker and Ternovska 2008). In this text, we will only use the following property.

Proposition Appendix A.3
If all defined atoms in a non-empty definition Δ are interpreted as $\mathbf{u}$ in J, then $Kl_J(\Delta) = \mathbf{u}$.

We use $Kl_J(\mathcal{T})$ to denote the Kleene value of a theory $\mathcal{T}$ over a structure J. $Kl_J(\mathcal{T}) = \mathbf{t}$ if all of $\mathcal{T}$'s definitions and sentences have value $\mathbf{t}$ in structure J. $Kl_J(\mathcal{T}) = \mathbf{f}$ if one of $\mathcal{T}$'s definitions or sentences has value $\mathbf{f}$ in structure J. $Kl_J(\mathcal{T}) = \mathbf{u}$ otherwise.

We summarise some well-known properties about the Kleene-valuation.

Proposition Appendix A.4
If J is a two-valued partial structure (i.e., a structure), then $Kl_J(\mathcal{T})$ is $\mathbf{t}$ if and only if $J \models \mathcal{T}$ and $Kl_J(\mathcal{T})$ is $\mathbf{f}$ otherwise.

Proposition Appendix A.5
If J and J' are partial structures with $J \leq_p J'$, then for every theory $\mathcal{T}$, $Kl_J(\mathcal{T}) \leq_p Kl_{J'}(\mathcal{T})$.

A.2 Weakly $\mathcal{T}$-Compatible Chains and Weak Progression

For this paper, we are only interested in a special kind of partial structures: partial structures that have complete information on an initial segment of time points and that have no information about other time points. Using the identification of a structure with an ∞-chain, a k-chain corresponds to such a partial structure. If $(J_j)_{j=0}^k$ is a k-chain, we associate to J the partial structure J equal to the J_j on static symbols and such that for dynamic symbols σ

$$\sigma(d_1 \dots d_{n-1}, j)^J = \begin{cases} \mathbf{t} \text{ if } j \leq k \text{ and } (d_1 \dots d_{n-1}) \in \sigma_{curr}^{J_j} \\ \mathbf{f} \text{ if } j \leq k \text{ and } (d_1 \dots d_{n-1}) \notin \sigma_{curr}^{J_j} \\ \mathbf{u} \text{ otherwise} \end{cases}$$

We identify the k-chain and the corresponding partial structure.

Definition Appendix A.6 (*Weakly $\mathcal{T}$-compatible, weak $\mathcal{T}$-successor*)
A k-chain J is *weakly $\mathcal{T}$-compatible* with a Σ-theory $\mathcal{T}$ if $Kl_J(\mathcal{T}) \neq \mathbf{f}$.

A Σ_{ss}-structure S' is a *weak $\mathcal{T}$-successor* of a k-chain J if J::S is weakly $\mathcal{T}$-compatible.

Proposition Appendix A.7
Every $\mathcal{T}$-compatible k-chain J is also weakly $\mathcal{T}$-compatible.

Proof
If J is $\mathcal{T}$-compatible, then there is a model J' of $\mathcal{T}$ that is more precise than J. Since $J' \models \mathcal{T}$, $Kl_{J'}(\mathcal{T}) = \mathbf{t}$ by Proposition Appendix A.4. Now, Proposition Appendix A.5 guarantees that $Kl_J(\mathcal{T})$ is less precise than $\mathbf{t}$, hence it must be either $\mathbf{t}$ or $\mathbf{u}$ and we conclude that J is indeed weakly $\mathcal{T}$-compatible. $\square$

The reverse of Proposition Appendix A.7 does not hold as the following (simple) example shows.

Example Appendix A.8
Let $\mathcal{T}$ be the following first-order theory:

$$\begin{array}{l} P(\mathcal{I}). \\ \forall t : Q(\mathcal{S}(t)) \Leftrightarrow P(t). \\ \forall t : \neg Q(t). \end{array}$$

It is clear that $\mathcal{T}$ has no models, as the second constraint requires Q to be true at time 1, while the last constraint requires Q to be false at all time points. Hence, there are no $\mathcal{T}$-compatible chains.

However, the 0-chain J such that J_0 interprets P by $\mathbf{t}$ and Q by $\mathbf{f}$ is weakly $\mathcal{T}$-compatible. The Kleene-valuation of $\mathcal{T}$ in J is $\mathbf{u}$.

The above example shows that it is possible that a weakly $\mathcal{T}$-compatible chain cannot be extended. Such a situation is often called a deadlock.

Definition Appendix A.9 (*Deadlock*)
A weakly $\mathcal{T}$-compatible chain J is in a *deadlock* if there are no weakly $\mathcal{T}$-compatible extensions of J.

Definition Appendix A.10 (*Weak Progression inference*)
The *weak progression inference* is an inference that takes as input a theory $\mathcal{T}$ and a weakly $\mathcal{T}$-compatible k-chain J and returns all weak $\mathcal{T}$-successors of J.

Definition Appendix A.11 (*Weak Markov property*)
A theory $\mathcal{T}$ satisfies the *weak Markov property* if for every weakly $\mathcal{T}$-compatible k-chain J, and every weakly $\mathcal{T}$-compatible k'-chain J' ending in the same state, i.e., such that $J_k = J'_{k'}$, the weak $\mathcal{T}$-successors of J are exactly the weak $\mathcal{T}$-successors of J'.

The weak Markov property essentially says the same as the Markov property, namely that the successors of a given chain only depend on the last state, i.e., that the system has no history.

Appendix B Proofs

Proposition 3.6
Let Σ be a linear-time vocabulary and Σ_{ss} the corresponding single state vocabulary. Then the mappings $\pi_k^{ss}(\cdot)$ induce a one-to-one correspondence between Σ-structures J and sequences $(J_k)_{k=0}^{\infty}$ of Σ_{ss}-structures sharing the same interpretation of static symbols.

Proof
It is clear that given a structure J, $(\pi_k^{ss}(J))_{k=0}^{\infty}$ is indeed such a sequence.

Now, for the other direction, suppose J_k is a sequence of Σ_{ss}-structures sharing the same interpretation of static symbols. Let J denote the Σ-structure with the same interpretation of static symbols and such that, for dynamic predicates σ,

$$\sigma^J = \{(d_1, \ldots, d_{n-1}, k) \mid (d_1, \ldots, d_{n-1}) \in \sigma_{curr}^{J_k}\}.$$

Then J is indeed a structure such that $\pi_k^{ss}(J) = J_k$, as desired. $\square$

Theorem 3.18
Let $\mathcal{T}$ be an LTC-theory and J a Σ-structure. Then J is a model of $\mathcal{T}$ if and only if $\pi_0^{ss}(J) \models \mathcal{T}_0$ and for every $k \in \mathbb{N}$, $\pi_k^{bs}(J) \models \mathcal{T}_t$.

Proof
By the first condition of Definition 3.13, the FO part of the theory only consists of static, initial, single-state, and bistate sentences. Now, a structure J satisfies a static sentence if and only if each of its projections satisfy this sentence. A structure J satisfies an initial sentence, if and only if its initial time-point satisfies the projection of this sentences, etc. Hence, for the FO part, the result easily follows.

Furthermore, Definition 3.13 guarantees that all definitions in $\mathcal{T}$ are stratified over time. Now, it follows immediately from Theorem 4.5 in (Vennekens et al. 2006) that we can split stratified definitions in one definition for each stratification level. Thus, what we obtain is one definition for each point in time, defining the state at $\mathcal{S}(t)$ in terms of the state in t. This definition corresponds exactly to the definition in $\mathcal{T}_t$, as desired. $\square$

Theorem 3.19
Let $\mathcal{T}$ be an LTC-theory and J a k-chain. Then, J is weakly $\mathcal{T}$-compatible if and only if $\pi_0^{ss}(J) \models \mathcal{T}_0$ and for every $j < k$, $\pi_j^{bs}(J) \models \mathcal{T}_t$.

Proof
One direction is clear: if J is weakly $\mathcal{T}$-compatible, then $\pi_0^{ss}(J) \models \mathcal{T}_0$ and for every $j < k$, $\pi_j^{bs}(J) \models \mathcal{T}_t$.

For the other direction, suppose $\pi_0^{ss}(J) \models \mathcal{T}_0$ and for every $j < k$, $\pi_j^{bs}(J) \models \mathcal{T}_t$. We will show that J is weakly $\mathcal{T}$-compatible. In order to show this, we will show that $Kl_J(\mathcal{T}) \neq \mathbf{f}$, or said differently, that for every sentence $\varphi \in \mathcal{T}$, $Kl_J(\varphi) \neq \mathbf{f}$ and that for the definition Δ in $\mathcal{T}$, $Kl_J(\Delta) \neq \mathbf{f}$.

First, let φ be any sentence in $\mathcal{T}$. If φ is an initial, or a static sentence, then $J \models \varphi$ because $\pi_0^{ss}(J) \models \mathcal{T}_0$, thus $Kl_J(\varphi) = \mathbf{t}$ for such sentences. If φ is a universal single-state sentence $\forall t : \varphi'(t)$, we assume that $Kl_J(\varphi) = \mathbf{f}$, and will show that this leads to a contradiction. In this case, using the definition of the Kleene valuation, at least for one i, $Kl_J(\varphi[i/t]) = \mathbf{f}$, or said differently, at least for one i, $J_i \not\models te(\varphi)$. Now, this i should definitely be greater than k, since $\mathcal{T}_t$ contains the constraint $te(\varphi)$. However, since

J_i is completely unknown on dynamic predicates, we see that $J_i \leq_p J_0$. Hence, using Proposition Appendix A.5, we find that also $J_0 \not\models te(\varphi)$, which is in contradiction with the assumption that $\pi_0^{ss}(J) \models \mathcal{T}_0$. For bistate sentences, a similar argument holds. Thus we can conclude that indeed for every sentence φ in $\mathcal{T}$, $Kl_J(\varphi) \neq \mathbf{f}$.

Now, let Δ be the definition of $\mathcal{T}$. We should show that $Kl_J(\Delta) \neq \mathbf{f}$. As Δ is stratified over time, by Theorem 4.5 in (Vennekens et al. 2006), we can split Δ in definitions $(\Delta_i)_{i\in\mathbb{N}}$ for each time point. The definitions Δ_i with $i \leq k$ are satisfied in J because those are the definitions in the theory $\mathcal{T}_t$. For definitions Δ_i with $i > k$, these definitions define only dynamic atoms with dynamic arguments greater than k. Furthermore, these dynamic atoms are completely unknown in J. Proposition Appendix A.3 then yields that $Kl_J(\Delta) = \mathbf{u} \neq \mathbf{f}$.

Thus, we also find that $Kl_J(\mathcal{T}) = \mathbf{u} \neq \mathbf{f}$, i.e., J is indeed weakly $\mathcal{T}$-compatible. □

Corollary 3.20
LTC-theories satisfy the Markov property and the weak Markov property.

Proof
We first prove that LTC theories satisfy the Markov property. Let J and J' be a k-chain and a k'-chain respectively with $J_k = J'_{k'}$. Suppose S is a $\mathcal{T}$-successor of J'. We show that S is also a $\mathcal{T}$-successor of J. Since $J'{::}S$ is $\mathcal{T}$-compatible, there exists a model K' of $\mathcal{T}$ such that $K'_i = J'_i$ for $i \leq k'$ and $K_{k'+1} = S$. Now let K be the structure such that

$$K_j = \begin{cases} J_j & \text{for } j \leq k, \\ K'_{k'+j-k} & \text{otherwise.} \end{cases}$$

We claim that K is a model of $\mathcal{T}$ more precise than $J{::}S$. The fact that it is more precise than $J{::}S$ follows from the fact that $K_{k+1} = K'_{k'+(k+1)-k} = K'_{k'+1}$, which equals S, by construction of K. In order to prove our claim, we show that for every j, (K_j, K_{j+1}), satisfies $\mathcal{T}_t$. For $j \leq k$, this follows from the fact that J is $\mathcal{T}$-compatible; for $j > k$, from the fact that K' is a model of $\mathcal{T}$. Now using Theorem 3.18, we see that K is a model of $\mathcal{T}$, which shows that $J{::}S$ is indeed $\mathcal{T}$-compatible.

We now prove that LTC theories satisfy the weak Markov property. This follows immediately from Theorem 3.19: $J{::}S$ is weakly $\mathcal{T}$-compatible if and only if J is weakly $\mathcal{T}$-compatible and $(J_k, S) \models \mathcal{T}_t$. □

Theorem 4.2
Let $\mathcal{T}$ be an LTC-theory and φ a universal single-state sentence. Then $\mathcal{T} \models \varphi$ if $\mathcal{T}_0 \models te(\varphi)$, and $(\mathcal{T}_t \wedge te(\varphi)) \models te(\varphi[\mathcal{S}(t)/t])$, where t is the unique time-variable in φ.

Proof
This theorem is in fact a reformulation of the principle of proofs by induction. The condition $\mathcal{T}_0 \models te(\varphi)$ expresses that the invariants holds at time 0, i.e., this is the base case. The condition $(\mathcal{T}_t \wedge te(\varphi)) \models te(\varphi[\mathcal{S}(t)/t])$ expresses that whenever the invariants holds at t, it also holds at $\mathcal{S}(t)$. □

Theorem 4.3
Let J be a Σ_s-structure and let φ be a universal single-state sentence with time variable t. Then φ is satisfied in all Σ-structures expanding J if $\mathcal{T}_0 \wedge \neg te(\varphi)$ has no models expanding J, and $\mathcal{T}_t \wedge te(\varphi) \wedge \neg te(\varphi[\mathcal{S}(t)/t])$ has no models expanding J.

Proof

This theorem is also a reformulation of the principle of proofs by induction, analogue to Theorem 4.2. □

Theorem 4.4

Let $\mathcal{T}$ be an LTC-theory and φ a universal bistate sentence. Then $\mathcal{T} \models \varphi$ if and only if $\mathcal{T}_t \models te(\varphi)$.

Proof

One direction, is clear: if $\mathcal{T} \models \varphi$, it follows immediately that $\mathcal{T}_t \models te(\varphi)$.

For the other direction, suppose $\mathcal{T}_t \models te(\varphi)$. We should show that $\mathcal{T} \models \varphi$. Therefore, let J be a model of $\mathcal{T}$. By Theorem 3.18, for every k, $J_k \models \mathcal{T}_t$. Thus, using our assumption, for every k, also $J_k \models te(\varphi)$. But φ is itself an LTC-theory, and $\varphi_i = \mathbf{t}$ and $\varphi_t = te(\varphi)$. Thus, using Theorem 3.18 again, we find that $J \models \varphi$, as desired. □

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