Appendix for the paper of Vicious Circle Principle and Logic Programs with Aggregates

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1 Appendix

In this appendix, given an $\mathscr{A}log$ program Π, a set A of literals and a rule $r \in \Pi$, we use $\alpha(r,A)$ to denote the rule obtained from r in the aggregate reduct of Π with respect to A. $\alpha(r,A)$ is *nil*, called an *empty rule*, if r is discarded in the aggregate reduct. We use $\alpha(\Pi,A)$ to denote the aggregate reduct of Π, i.e., $\{\alpha(r,A) : r \in \Pi \text{ and } \alpha(r,A) \neq nil\}$.

Proposition 1 (*Rule Satisfaction and Supportedness*)
Let A be an answer set of a ground $\mathscr{A}log$ program Π. Then

1. A satisfies every rule r of Π.
2. If $p \in A$ then there is a rule r from Π such that the body of r is satisfied by A and p is the only atom in the head of r which is true in A. (It is often said that rule r supports atom p.)

Proof: Let

(1) A be an answer set of Π.

We first prove A satisfies every rule r of Π. Let r be a rule of Π such that

(2) A satisfies the body of r.

Statement (2) implies that every aggregate atom, if there is any, of the body of r is satisfied by A. By the definition of the aggregate reduct, there must be a non-empty rule $r' \in \alpha(\Pi,A)$ such that

(3) $r' = \alpha(r,A)$.

By the definition of aggregate reduct, A satisfies the body of r iff it satisfies that of r'. Therefore, (2) and (3) imply that

(4) A satisfies the body of r'.

By the definition of answer set of $\mathscr{A}log$, (1) implies that

(5) A is an answer set of $\alpha(\Pi,A)$.

Since $\alpha(\Pi, A)$ is an ASP program, (3) and (5) imply that

(6) A satisfies r'.

Statements (4) and (6) imply A satisfies the head of r' and thus the head of r because r and and r' have the same head.

Therefore r is satisfied by A, which concludes our proof of the first part of the proposition.

We next prove the second part of the propostion. Consider $p \in A$. (1) implies that A is an answer set of $\alpha(\Pi, A)$. By the supportedness Lemma for ASP programs (**?**), there is a rule $r' \in \alpha(\Pi, A)$ such that

(7) r' supports p.

Let $r \in \Pi$ be a rule such that $r' = \alpha(r, A)$. By the definition of aggregate reduct,

(8) A satisfies the body of r iff A satisfies that of r'.

Since r and r' have the same heads, (7) and (8) imply that rule r of Π supports p in A, which concludes the proof of the second part of the proposition. □

Proposition 2 (*Anti-chain Property*)
Let A_1 be an answer set of an $\mathscr{A}log$ program Π. Then there is no answer set A_2 of Π such that A_1 is a proper subset of A_2.

Proof: Let us assume that there are A_1 and A_2 such that

(1) $A_1 \subseteq A_2$ and

(2) A_1 and A_2 are answer sets of Π

and show that $A_1 = A_2$.

Let R_1 and R_2 be the aggregate reducts of Π with respect to A_1 and A_2 respectively. Let us first show that A_1 satisfies the rules of R_2. Consider

(3) $r_2 \in R_2$.

By the definition of aggregate reduct there is $r \in \Pi$ such that

(4) $r_2 = \alpha(r, A_2)$.

Consider

(5) $r_1 = \alpha(r, A_1)$.

If r contains no aggregate atoms then

(6) $r_1 = r_2$.

By (5) and (6), $r_2 \in R_1$ and hence, by (2) A_1 satisfies r_2.

Assume now that r contains one aggregate term, $f\{X : p(X)\}$, i.e. r is of the form

(7) $h \leftarrow B, C(f\{X : p(X)\})$

where C is some property of the aggregate.

Then r_2 has the form

(8) $h \leftarrow B, P_2$

where

(9) $P_2 = \{p(t) : p(t) \in A_2\}$ and $f(P_2)$ satisfies condition C.

Let

(10) $P_1 = \{p(t) : p(t) \in A_1\}$

and consider two cases:

(11a) $\alpha(r, A_1) = \emptyset$.

In this case $C(f(P_1))$ does not hold. Hence, $P_1 \neq P_2$. Since $A_1 \subseteq A_2$ we have that $P_1 \subset P_2$, the body of rule (8) is not satisfied by A_1, and hence the rule (8) is.

(11b) $\alpha(r, A_1) \neq \emptyset$.

Then r_1 has the form

(12) $h \leftarrow B, P_1$

where

(13) $P_1 = \{p(t) : p(t) \in A_1\}$ and $f(P_1)$ satisfies condition C.

Assume that A_1 satisfies the body, B, P_2, of rule (8). Then

(14) $P_2 \subseteq A_1$

This, together with (9) and (10) implies

(15) $P_2 \subseteq P_1$.

From (1), (9), and (10) we have $P_1 \subseteq P_2$. Hence

(16) $P_1 = P_2$.

This means that A_1 satisfies the body of r_1 and hence it satisfies h and, therefore, r_2.

Similar argument works for rules containing multiple aggregate atoms and, therefore, A_1 satisfies R_2.

Since A_2 is a minimal set satisfying R_2 and A_1 satisfies R_2 and $A_1 \subseteq A_2$ we have that $A_1 = A_2$.

This completes our proof. □

Proposition 3 (*Splitting Set Theorem*)
Let

1. Π_1 and Π_2 be ground programs of $\mathscr{A}log$ such that no atom occurring in Π_1 is unifiable with any atom occurring in the heads of Π_2,
2. S be a set of ground literals containing all head literals of Π_1 but no head literals of Π_2,

Then

(3) A is an answer set of $\Pi_1 \cup \Pi_2$

iff

(4a) $A \cap S$ is an answer set of Π_1 and

(4b) A is an answer set of $(A \cap S) \cup \Pi_2$.

Proof. By the definitions of answer set and aggregate reduct

(3) holds iff

(5) A is an answer set of $\alpha(\Pi_1, A) \cup \alpha(\Pi_2, A)$

It is easy to see that conditions (1), (2), and the definition of α imply that $\alpha(\Pi_1, A)$, $\alpha(\Pi_2, A)$, and S satisfy condition of the splitting set theorem for ASP (**?**). Hence

(5) holds iff

(6a) $A \cap S$ is an answer set of $\alpha(\Pi_1, A)$

and

(6b) A is an answer set of $(A \cap S) \cup \alpha(\Pi_2, A)$.

To complete the proof it suffices to show that

(7) Statements (6a) and (6b) hold iff (4a) and (4b) hold.

By definition of α ,

(8) $(A \cap S) \cup \alpha(\Pi_2, A) = \alpha((A \cap S) \cup \Pi_2, A)$

and hence, by the definition of answer set we have

(9) (6b) iff (4b).

Now notice that from (4b), clause 2 of Proposition **??**, and conditions (1) and (2) of our theorem we have that for any ground instance $p(t)$ of a literal occurring in an aggregate atom of Π_1

(10) $p(t) \in A$ iff $p(t) \in A \cap S$

and, hence

(11) $\alpha(\Pi_1, A) = \alpha(\Pi_1, A \cap S)$.

From (9), (11), and the definition of answer set we have that

(12) (6a) iff (4a)

which completes the proof of our theorem. □

Lemma 1
Checking whether a set M of literals is an answer set of P, a program with aggregates, is in co-NP.

Proof: To prove that M is not an answer set of P, we first check if M is not a model of the aggregate reduct of P, which is in polynomial time. If M is not a model, M is not an answer set of P. Otherwise, we guess a set M' of P, and check if M' is a model of the aggregate reduct of P and $M' \subset M$. This checking is also in polynomial time. Therefore, the problem of checking whether a set M of literals is an answer set of P is in co-NP. □

Proposition 4 (*Complexity*)
The problem of checking if a ground atom a belongs to all answer sets of an $\mathscr{A}log$ program is Π_2^P complete.

Proof: First we show that the cautious reasoning problem is in Π_2^P. We verify that a ground atom a is not a cautious consequence of a program P as follows: Guess a set M of literals and check that (1) M is an answer set for P, and (2) a is not true wrt M. Task (2) is clearly polynomial, while (1) is in co-NP by virtue of Lemma 1. The problem therefore lies in Π_2^P.

Next, cautious reasoning over programs without aggregates is Π_2^P hard by (**?**). Therefore, cautious reasoning over programs with aggregates is Π_2^P hard too.

In summary, cautious reasoning over programs with aggregates is Π_2^P complete. □