Appendix A Proof of Lemma 2

We will make use of certain facts established in (Ésik and Rondogiannis 2014).

Suppose that L is a basic model. For each $x \in L$ and $\alpha < \kappa$, we define $x|_\alpha = \bigsqcup_\alpha \{x\}$. It was shown in (Ésik and Rondogiannis 2014) that $x =_\alpha x|_\alpha$ and $x|_\alpha =_\alpha x|_\beta$, $x|_\alpha \leq x|_\beta$ for all $\alpha < \beta < \kappa$. Moreover, $x = \bigvee_{\alpha<\kappa} x|_\alpha$. Also, for all $x, y \in L$ and $\alpha < \kappa$, it holds $x =_\alpha y$ iff $x|_\alpha =_\alpha y|_\alpha$ iff $x|_\alpha = y|_\alpha$, and $x \sqsubseteq_\alpha y$ iff $x|_\alpha \sqsubseteq_\alpha y|_\alpha$. And if $x \sqsubseteq_\alpha y$, then $x|_\alpha \leq y|_\alpha$. It is also not difficult to prove that for all $x \in L$ and $\alpha, \beta < \kappa$, $(x|_\alpha)|_\beta = x|_{\min\{\alpha,\beta\}}$. More generally, whenever $X \subseteq (z]_\alpha$ and $\beta \leq \alpha < \kappa$, it holds $(\bigsqcup_\alpha X)|_\beta = \bigsqcup_\beta X$. And if $\alpha < \beta$, then $(\bigsqcup_\alpha X)|_\beta = \bigsqcup_\alpha X$. Finally, we will make use of the following two results from (Ésik and Rondogiannis 2014):

Proposition 1
Let A, B be basic models and let $\alpha < \kappa$. If $f_j : A \to B$ is an α-monotonic function for each $j \in J$, then so is $f = \bigvee_{j\in J} f_j$ defined by $f(x) = \bigvee_{j\in J} f_j(x)$.

Lemma 2
Let Z be an arbitrary set and L be a basic model. Then, $Z \to L$ is a basic model with the pointwise definition of the order of relations $\leq$ and $\sqsubseteq_\alpha$ for all $\alpha < \kappa$.

Suppose that A, B are basic models. By Lemma 2 the set $A \to B$ is also a model, where the relations $\leq$ and $\sqsubseteq_\alpha$, $\alpha < \kappa$, are defined in a pointwise way (see (Ésik and Rondogiannis 2014, Subsection 5.3) for details). It follows that for any set F of functions $A \to B$, $\bigvee F$ can be computed pointwisely. Also, when $F \subseteq (f]_\alpha$ for some $f : A \to B$, $\bigsqcup_\alpha F$ for $\alpha < \kappa$ can be computed pointwisely.

We want to show that whenever $f : A \to B$, $\beta < \kappa$ and $F \subseteq (f]_\beta$ is a set of functions such that $F \subseteq [A \stackrel{m}{\to} B]$, then $\bigsqcup_\beta F \in [A \stackrel{m}{\to} B]$. We will make use of a lemma.

Lemma 3
Let L be a basic model. For all $x, y \in L$ and $\alpha, \beta < \kappa$ with $\alpha \neq \beta$, $x|_\beta \sqsubseteq_\alpha y|_\beta$ iff either $\beta < \alpha$ and $x|_\beta = y|_\beta$ (or equivalently, $x =_\beta y$), or $\beta > \alpha$ and $x|_\alpha \sqsubseteq_\alpha y|_\alpha$.

Proof
Let $x|_\beta \sqsubseteq_\alpha y|_\beta$. If $\beta < \alpha$ then $x|_\beta = (x|_\beta)|_\beta = (y|_\beta)|_\beta = y|_\beta$. If $\beta > \alpha$ then $x|_\alpha = (x|_\beta)|_\alpha \sqsubseteq_\alpha (y|_\beta)|_\alpha = y|_\alpha$.

Suppose now that $\beta < \alpha$ and $x|_\beta = y|_\beta$. Then $(x|_\beta)|_\alpha = x|_\beta = y|_\beta = (y|_\beta)|_\alpha$ and thus $x|_\beta =_\alpha y|_\beta$. Finally, let $\beta > \alpha$ and $x|_\alpha \sqsubseteq_\alpha y|_\alpha$. Then $(x|_\beta)|_\alpha = x|_\alpha \sqsubseteq_\alpha y|_\alpha = (y|_\beta)|_\alpha$ and thus $x|_\beta \sqsubseteq_\alpha y|_\beta$. $\square$

Remark 1
Under the above assumptions, if $\beta < \alpha$, then $x|_\beta \sqsubseteq_\alpha y|_\beta$ iff $x|_\beta =_\alpha y|_\beta$ iff $x|_\beta = y|_\beta$.

Corollary 1
For all $X, Y \subseteq L$ and $\alpha \neq \beta$, $\bigsqcup_\beta X \sqsubseteq_\alpha \bigsqcup_\beta Y$ iff $\beta < \alpha$ and $\bigsqcup_\beta X = \bigsqcup_\beta Y$, or $\beta > \alpha$ and $\bigsqcup_\alpha X \sqsubseteq_\alpha \bigsqcup_\alpha Y$.

Proof
Let $x = \bigsqcup_\beta X$ and $y = \bigsqcup_\beta Y$. Then $x = \bigsqcup_\beta X = \bigsqcup_\beta\{\bigsqcup_\beta X\} = x|_\beta$ and $y = y|_\beta$. Let $\beta < \alpha$. Then $x \sqsubseteq_\alpha y$ iff $x = y$. Let $\beta > \alpha$. Then $x \sqsubseteq_\alpha y$ iff $x|_\alpha \sqsubseteq_\alpha y|_\alpha$. But $x|_\alpha = \bigsqcup_\alpha\{\bigsqcup_\beta X\} = \bigsqcup_\alpha X$ and similarly for Y. $\square$

Lemma 4
Let A and B be basic models. Suppose that $f : A \to B$ and $F \subseteq (f]_\beta$ (where $\beta < \kappa$) is a set of functions in $[A \xrightarrow{m} B]$. Then $\bigsqcup_\beta F$ is also α-monotonic for all $\alpha < \kappa$.

Proof
Suppose that $\alpha, \beta < \kappa$ and $x \sqsubseteq_\alpha y$ in A. Then $(\bigsqcup_\beta F)(x) = \bigsqcup_\beta\{f(x) : f \in F\}$ and $(\bigsqcup_\beta F)(y) = \bigsqcup_\beta\{f(y) : f \in F\}$. We have that $f(x) \sqsubseteq_\alpha f(y)$ for all $f \in F$. Thus, if $\alpha = \beta$, then clearly $(\bigsqcup_\beta F)(x) \sqsubseteq_\alpha (\bigsqcup_\beta F)(y)$.

Suppose that $\beta < \alpha$. Then $\bigsqcup_\beta\{f(x) : f \in F\} = \bigsqcup_\beta\{f(y) : f \in F\}$ since $f(x) =_\beta f(y)$ for all $f \in F$. Thus, by Corollary 1, $(\bigsqcup_\beta F)(x) \sqsubseteq_\alpha (\bigsqcup_\beta F)(y)$.

Suppose that $\beta > \alpha$. Then $(\bigsqcup_\beta F)(x) \sqsubseteq_\alpha (\bigsqcup_\beta F)(y)$ follows by Corollary 1 from $\bigsqcup_\alpha\{f(x) : f \in F\} \sqsubseteq_\alpha \bigsqcup_\alpha\{f(y) : f \in F\}$. $\square$

We equip $[A \xrightarrow{m} B]$ with the order relations $\leq$ and $\sqsubseteq_\alpha$ inherited from $A \to B$. We have the following lemma:

Lemma 5
If A and B are basic models, then so is $[A \xrightarrow{m} B]$ with the pointwise definition of the order of relations $\leq$ and $\sqsubseteq_\alpha$ for all $\alpha < \kappa$.

Proof
It is proved in (Ésik and Rondogiannis 2014) that the set of functions $A \to B$ is a basic model with the pointwise definition of the relations $\leq$ and $\sqsubseteq_\alpha$, so that for all $f, g : A \to B$ and $\alpha < \kappa$, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in A$ and $f \sqsubseteq_\alpha g$ iff $f(x) \sqsubseteq_\alpha g(x)$ for all $x \in A$. It follows that for any $F \subseteq B^A$ and $\alpha < \kappa$, $\bigvee F$ and $\bigsqcup_\alpha F$ can also be computed pointwise: $(\bigvee F)(x) = \bigvee\{f(x) : x \in A\}$ and $(\bigsqcup_\alpha F)(x) = \bigsqcup_\alpha\{f(x) : f \in F\}$. By Proposition 1 and Lemma 4, for all $F \subseteq B^A$, if F is a set of functions α-monotonic for all α, then $\bigvee F$ and $\bigsqcup_\beta F$ are also α-monotonic for all α. Since the relations $\leq$ and $\sqsubseteq_\alpha$, $\alpha < \kappa$ on $[A \xrightarrow{m} B]$ are the restrictions of the corresponding relations on B^A, in view of Proposition 1 and Lemma 4, $[A \xrightarrow{m} B]$ also satisfies the axioms in Definition 1, so that $[A \xrightarrow{m} B]$ is a basic model. $\square$

The following lemma is shown in (Ésik and Rondogiannis 2014, Subsection 5.2) and will be used in the proof of the basis case of the next lemma:

Lemma 6
$(V, \leq)$ is a complete lattice and a basic model.

Lemma 2
Let D be a nonempty set and π be a predicate type. Then, $(\llbracket\pi\rrbracket_D, \leq_\pi)$ is a complete lattice and a basic model.

Proof
Let π be a predicate type. We prove that $[\![\pi]\!]_D$ is a basic model by induction on the structure of π. When $\pi = o$, $[\![\pi]\!]_D = V$, a basic model. Suppose that π is of the sort $\iota \to \pi'$. Then $[\![\pi]\!]_D = D \to [\![\pi']\!]_D$, which is a basic model, since $[\![\pi']\!]_D$ is a model by the induction hypothesis. Finally, let π be of the sort $\pi_1 \to \pi_2$. By the induction hypothesis, $[\![\pi_i]\!]_D$ is a model for $i = 1, 2$. Thus, by Lemma 5, $[\![\pi]\!]_D = [[\![\pi_1]\!]_D \overset{m}{\to} [\![\pi_2]\!]_D]$ is also a basic model. □

Remark 2
Let $\mathcal{C}$ denote the category of all basic models and α-monotonic functions. The above results show that $\mathcal{C}$ is cartesian closed, since for all basic models A, B, the evaluation function $\mathsf{eval} : (A \times B) \times A \to B$ is α-monotonic (in both arguments) for all $\alpha < \kappa$.

Indeed, suppose that $f, g \in [A \overset{m}{\to} B]$ and $x, y \in A$ with $f \sqsubseteq_\alpha g$ and $x \sqsubseteq_\alpha y$. Then $\mathsf{eval}(f, x) = f(x) \sqsubseteq_\alpha g(x) = \mathsf{eval}(g, x)$ by the pointwise definition of $f \sqsubseteq_\alpha g$. Also, $\mathsf{eval}(f, x) = f(x) \sqsubseteq_\alpha f(y) = \mathsf{eval}(f, y)$ since f is α-monotonic.

Since $\mathcal{C}$ is cartesian closed, for all $f \in [B \times A \overset{m}{\to} C]$ there is a unique $\Lambda f \in [B \overset{m}{\to} [A \overset{m}{\to} C]]$ in with $f(y, x) = \mathsf{eval}(\Lambda f(y), x)$ for all $x \in A$ and $y \in B$.

Appendix B Proofs of Lemmas 3, 4 and 5

Lemma 3
Let $\mathsf{E} : \rho$ be an expression and let D be a nonempty set. Moreover, let s be a state over D and let I be an interpretation over D. Then, $[\![\mathsf{E}]\!]_s(I) \in [\![\rho]\!]_D$.

Proof
If $\rho = \iota$ then the claim is clear. Let E be of a predicate type π. We prove simultaneously the following auxiliary statement. Let $\alpha < \kappa$, $\mathsf{V} : \pi$, $x, y \in [\![\pi']\!]_D$. If $x \sqsubseteq_\alpha y$ then $[\![\mathsf{E}]\!]_{s[\mathsf{V}/x]}(I) \sqsubseteq_\alpha [\![\mathsf{E}]\!]_{s[\mathsf{V}/y]}(I)$. The proof is by structural induction on E. We will cover only the nontrivial cases.

Case $(\mathsf{E}_1\ \mathsf{E}_2)$*:* The main statement follows directly by the induction hypothesis of E_1 and E_2. There are two cases. Suppose that $E_1 : \pi_1 \to \pi$ and $E_2 : \pi_1$. Then $[\![\mathsf{E}_1]\!]_s(I) \in [\![\pi_1 \to \pi]\!]_D = [[\![\pi_1]\!]_D \overset{m}{\to} [\![\pi]\!]_D]$ and $[\![\mathsf{E}_2]\!]_s(I) \in [\![\pi_1]\!]_D$ by the induction hypothesis. Thus, $[\![\mathsf{E}_1]\!]_s(I)\ ([\![\mathsf{E}]\!]_s(I)) \in [\![\pi]\!]_D$. Suppose now that $E_1 : \iota \to \pi$ and $E_2 : \iota$. Then $[\![\mathsf{E}_1]\!]_s(I) \in [\![\iota \to \pi]\!]_D = D \to [\![\pi]\!]_D$ by the induction hypothesis and $[\![\mathsf{E}_2]\!]_s(I) \in [\![\iota]\!]_D = D$. It follows again that $[\![\mathsf{E}_1]\!]_s(I)\ ([\![\mathsf{E}]\!]_s(I)) \in [\![\pi]\!]_D$.

Auxiliary statement: Let $x, y \in [\![\pi']\!]_D$ and assume $x \sqsubseteq_\alpha y$. We have by definition $[\![(\mathsf{E}_1\ \mathsf{E}_2)]\!]_{s[\mathsf{V}/x]}(I) = [\![\mathsf{E}_1]\!]_{s[\mathsf{V}/x]}(I)\ ([\![\mathsf{E}_2]\!]_{s[\mathsf{V}/x]}(I))$, and similarly for $[\![(\mathsf{E}_1\ \mathsf{E}_2)]\!]_{s[\mathsf{V}/y]}(I)$. We have $E_1 : \pi_1 \to \pi$ and $E_2 : \pi_1$ or $E_1 : \iota \to \pi$ and $E_2 : \iota$. In the first case, by induction hypothesis $[\![\mathsf{E}_1]\!]_{s[\mathsf{V}/x]}(I) \in [\![\pi_1 \to \pi]\!]_D$, and thus is α-monotonic. Also, $[\![\mathsf{E}_1]\!]_{s[\mathsf{V}/x]}(I) \sqsubseteq_\alpha [\![\mathsf{E}_1]\!]_{s[\mathsf{V}/y]}(I)$ and $[\![\mathsf{E}_2]\!]_{s[\mathsf{V}/x]}(I) \sqsubseteq_\alpha [\![\mathsf{E}_2]\!]_{s[\mathsf{V}/y]}(I)$ by the induction hypothesis. It follows that

$$[\![\mathsf{E}_1]\!]_{s[\mathsf{V}/x]}(I)\,([\![\mathsf{E}_2]\!]_{s[\mathsf{V}/x]}(I)) \sqsubseteq_\alpha [\![\mathsf{E}_1]\!]_{s[\mathsf{V}/x]}(I)\,([\![\mathsf{E}_2]\!]_{s[\mathsf{V}/y]}(I)) \sqsubseteq_\alpha [\![\mathsf{E}_1]\!]_{s[\mathsf{V}/y]}(I)\,([\![\mathsf{E}_2]\!]_{s[\mathsf{V}/y]}(I)).$$

The second case is similar. We have $[\![\mathsf{E}_1]\!]_{s[\mathsf{V}/x]}(I) \sqsubseteq_\alpha [\![\mathsf{E}_1]\!]_{s[\mathsf{V}/y]}(I)$ by the induction hypothesis, moreover, $[\![\mathsf{E}_2]\!]_{s[\mathsf{V}/x]}(I) = [\![\mathsf{E}_2]\!]_{s[\mathsf{V}/y]}(I)$. Therefore, $[\![\mathsf{E}_1]\!]_{s[\mathsf{V}/x]}(I)\ ([\![\mathsf{E}_2]\!]_{s[\mathsf{V}/x]}(I)) \sqsubseteq_\alpha [\![\mathsf{E}_1]\!]_{s[\mathsf{V}/y]}(I)\ ([\![\mathsf{E}_2]\!]_{s[\mathsf{V}/y]}(I))$.

Case $(\lambda \mathsf{V}.\mathsf{E})$: Assume $V : \rho_1$ and $\mathsf{E} : \pi_2$. We will show that $[\![\lambda \mathsf{V}.\mathsf{E}]\!]_s(I) \in [\![\rho_1 \to \pi_2]\!]_D$. If $\rho_1 = \iota$ then the result follows easily from the induction hypothesis of the first statement. Assume $\rho_1 = \pi_1$. We show that $[\![\lambda \mathsf{V}.\mathsf{E}]\!]_s(I) \in [\![\pi_1 \to \pi_2]\!]_D$, that is, $\lambda d.[\![\mathsf{E}]\!]_{s[\mathsf{V}/d]}(I)$ is α-monotonic for all $\alpha < \kappa$. That follows directly by the induction hypothesis of the auxiliary statement.

Auxiliary statement: It suffices to show that $[\![(\lambda \mathsf{U}.\mathsf{E})]\!]_{s[\mathsf{V}/x]}(I) \sqsubseteq_\alpha [\![(\lambda \mathsf{U}.\mathsf{E})]\!]_{s[\mathsf{V}/y]}(I)$ and equivalently for every d, $[\![\mathsf{E}]\!]_{s[\mathsf{V}/x][\mathsf{U}/d]}(I) \sqsubseteq_\alpha [\![\mathsf{E}]\!]_{s[\mathsf{V}/y][\mathsf{U}/d]}(I)$ which follows from induction hypothesis. $\square$

Lemma 4
Let P be a program. Then, $\mathcal{I}_\mathsf{P}$ is a complete lattice and a basic model.

Proof
From Lemma 2 we have that for all predicate types π, $[\![\pi]\!]_{U_\mathsf{P}}$ is a complete lattice and a basic model. It follows, by Lemma 2, that for all predicate types π, $\mathcal{P}_\pi \to [\![\pi]\!]_{U_\mathsf{P}}$ is also a complete lattice and a model, where $\mathcal{P}_\pi$ is the set of predicate constants of type π. Then, $\mathcal{I}_\mathsf{P}$ is $\prod_\pi \mathcal{P}_\pi \to [\![\pi]\!]_{U_\mathsf{P}}$ which is also a basic model (proved in (Ésik and Rondogiannis 2014)). $\square$

Lemma 5 (α-Monotonicity of Semantics)
Let P be a program and let $\mathsf{E} : \pi$ be an expression. Let I, J be Herbrand interpretations and s be a Herbrand state of P. For all $\alpha < \kappa$, if $I \sqsubseteq_\alpha J$ then $[\![\mathsf{E}]\!]_s(I) \sqsubseteq_\alpha [\![\mathsf{E}]\!]_s(J)$.

Proof
The proof is by structural induction on E.

Induction Base: The cases $\mathsf{V}, \mathsf{false}, \mathsf{true}$ are straightforward since their meanings do not depend on I. Let $I \sqsubseteq_\alpha J$. If E is a predicate constant p then we have $I(\mathsf{p}) \sqsubseteq_\alpha J(\mathsf{p})$.

Induction Step: Assume that the statement holds for expressions E_1 and E_2 and let $I \sqsubseteq_\alpha J$.

Case $(\mathsf{E}_1\ \mathsf{E}_2)$: It holds $[\![(\mathsf{E}_1\ \mathsf{E}_2)]\!]_s(I) = [\![\mathsf{E}_1]\!]_s(I)([\![\mathsf{E}_2]\!]_s(I))$. By induction hypothesis we have $[\![\mathsf{E}_1]\!]_s(I) \sqsubseteq_\alpha [\![\mathsf{E}_1]\!]_s(J)$ and therefore $[\![\mathsf{E}_1]\!]_s(I)([\![\mathsf{E}_2]\!]_s(I)) \sqsubseteq_\alpha [\![\mathsf{E}_1]\!]_s(J)([\![\mathsf{E}_2]\!]_s(I))$. We perform a case analysis on the type of E_2. If E_2 is of type ι and since I, J are Herbrand interpretations, it is clear that $[\![\mathsf{E}_2]\!]_s(I) = [\![\mathsf{E}_2]\!]_s(J)$ and therefore $[\![\mathsf{E}_1]\!]_s(I)([\![\mathsf{E}_2]\!]_s(I)) \sqsubseteq_\alpha [\![\mathsf{E}_1]\!]_s(J)([\![\mathsf{E}_2]\!]_s(J))$. By definition of application we get $[\![(\mathsf{E}_1\ \mathsf{E}_2)]\!]_s(I) \sqsubseteq_\alpha [\![(\mathsf{E}_1\ \mathsf{E}_2)]\!]_s(J)$. If E_2 is of type π then by induction hypothesis we have $[\![\mathsf{E}_2]\!]_s(I) \sqsubseteq_\alpha [\![\mathsf{E}_2]\!]_s(J)$ and since $[\![\mathsf{E}_1]\!]_s(J)$ is α-monotonic we get that $[\![\mathsf{E}_1]\!]_s(J)([\![\mathsf{E}_2]\!]_s(I)) \sqsubseteq_\alpha [\![\mathsf{E}_1]\!]_s(J)([\![\mathsf{E}_2]\!]_s(J))$. By transitivity of $\sqsubseteq_\alpha$ and by the definition of application we conclude that $[\![(\mathsf{E}_1\ \mathsf{E}_2)]\!]_s(I) \sqsubseteq_\alpha [\![(\mathsf{E}_1\ \mathsf{E}_2)]\!]_s(J)$.

Case $(\lambda \mathsf{V}.\mathsf{E}_1)$: It holds by definition that $[\![(\lambda \mathsf{V}.\mathsf{E}_1)]\!]_s(I) = \lambda d.[\![\mathsf{E}_1]\!]_{s[\mathsf{V}/d]}(I)$. It suffices to show that $\lambda d.[\![\mathsf{E}_1]\!]_{s[\mathsf{V}/d]}(I) \sqsubseteq_\alpha \lambda d.[\![\mathsf{E}_1]\!]_{s[\mathsf{V}/d]}(J)$ and equivalently that for every d, $[\![\mathsf{E}_1]\!]_{s[\mathsf{V}/d]}(I) \sqsubseteq_\alpha [\![\mathsf{E}_1]\!]_{s[\mathsf{V}/d]}(J)$ which holds by induction hypothesis.

Case $(\mathsf{E}_1 \bigvee_\pi \mathsf{E}_2)$: It holds $[\![(\mathsf{E}_1 \bigvee_\pi \mathsf{E}_2)]\!]_s(I) = \bigvee\{[\![\mathsf{E}_1]\!]_s(I), [\![\mathsf{E}_2]\!]_s(I)\}$. It suffices to show that $\bigvee\{[\![\mathsf{E}_1]\!]_s(I), [\![\mathsf{E}_2]\!]_s(I)\} \sqsubseteq_\alpha \bigvee\{[\![\mathsf{E}_1]\!]_s(J), [\![\mathsf{E}_2]\!]_s(J)\}$ which holds by induction hypothesis and Axiom 4.

Case $(\mathsf{E}_1 \bigwedge_\pi \mathsf{E}_2)$: It holds $[\![(\mathsf{E}_1 \bigwedge_\pi \mathsf{E}_2)]\!]_s(I) = \bigwedge\{[\![\mathsf{E}_1]\!]_s(I), [\![\mathsf{E}_2]\!]_s(I)\}$. Let $\pi = \rho_1 \to \cdots \to$

$\rho_n \to o$, it suffices to show for all $d_i \in [\![\rho_i]\!]_{U_\mathsf{P}}$, $\bigwedge\{[\![\mathsf{E}_1]\!]_s(I)\ d_1 \cdots d_n, [\![\mathsf{E}_2]\!]_s(I)\ d_1 \cdots d_n\} \sqsubseteq_\alpha \bigwedge\{[\![\mathsf{E}_1]\!]_s(J)\ d_1 \cdots d_n, [\![\mathsf{E}_2]\!]_s(J)\ d_1 \cdots d_n\}$. We define $x_i = [\![\mathsf{E}_i]\!]_s(I)\ d_1 \cdots d_n$ and $y_i = [\![\mathsf{E}_i]\!]_s(J)\ d_1 \cdots d_n$ for $i \in \{1,2\}$. We perform a case analysis on $v = \bigwedge\{x_1, x_2\}$. If $v < F_\alpha$ or $v > T_\alpha$ then $\bigwedge\{x_1,x_2\} = \bigwedge\{y_1,y_2\}$ and thus $\bigwedge\{x_1,x_2\} \sqsubseteq_\alpha \bigwedge\{y_1,y_2\}$. If $v = F_\alpha$ then $F_\alpha \leq \bigwedge\{y_1,y_2\} \leq T_\alpha$ and therefore $\bigwedge\{x_1,x_2\} \sqsubseteq_\alpha \bigwedge\{y_1,y_2\}$. If $v = T_\alpha$ then $\bigwedge\{y_1,y_2\} = T_\alpha$ and thus $\bigwedge\{x_1,x_2\} \sqsubseteq_\alpha \bigwedge\{y_1,y_2\}$. If $F_\alpha < v < T_\alpha$ then $F_\alpha < \bigwedge\{y_1,y_2\} \leq T_\alpha$ and therefore $\bigwedge\{x_1,x_2\} \sqsubseteq_\alpha \bigwedge\{y_1,y_2\}$.

Case $(\sim \mathsf{E}_1)$: Assume $order([\![\mathsf{E}_1]\!]_s(I)) = \alpha$. Then, by induction hypothesis $[\![\mathsf{E}_1]\!]_s(I) \sqsubseteq_\alpha [\![\mathsf{E}_1]\!]_s(J)$ and thus $order([\![\mathsf{E}_1]\!]_s(J)) \geq \alpha$. It follows that $order([\![(\sim \mathsf{E}_1)]\!]_s(I)) > \alpha$ and $order([\![(\sim \mathsf{E}_1)]\!]_s(J)) > \alpha$ and therefore $[\![(\sim \mathsf{E}_1)]\!]_s(I) \sqsubseteq_\alpha [\![(\sim \mathsf{E}_1)]\!]_s(J)$.

Case $(\exists \mathsf{V}.\mathsf{E}_1)$: Assume V is of type ρ. It holds $[\![(\exists \mathsf{V}.\mathsf{E}_1)]\!]_s(I) = \bigvee_{d \in [\![\rho]\!]_{U_\mathsf{P}}} [\![\mathsf{E}_1]\!]_{s[\mathsf{V}/d]}(I)$. It suffices to show $\bigvee_{d \in [\![\rho]\!]_{U_\mathsf{P}}} [\![\mathsf{E}_1]\!]_{s[\mathsf{V}/d]}(I) \sqsubseteq_\alpha \bigvee_{d \in [\![\rho]\!]_{U_\mathsf{P}}} [\![\mathsf{E}_1]\!]_{s[\mathsf{V}/d]}(J)$ which holds by induction hypothesis and Axiom 4. □

Appendix C Proof of Theorem 2

We start by providing some necessary background material from (Ésik and Rondogiannis 2014) on how the $\bigsqcap$ operation on a set of interpretations is actually defined.

Let $x \in V$. For every $X \subseteq (x]_\alpha$ we define $\bigsqcap_\alpha X$ as follows: if $X = \emptyset$ then $\bigsqcap_\alpha X = T_\alpha$, otherwise

$$\bigsqcap_\alpha X = \begin{cases} \bigwedge X & order(\bigwedge X) \leq \alpha \\ T_{\alpha+1} & \text{otherwise} \end{cases}$$

Let P be a program, $I \in \mathcal{I}_\mathsf{P}$ be a Herbrand interpretation of P and $X \subseteq (I]_\alpha$. For all predicate constants p in P of type $\rho_1 \to \cdots \to \rho_n \to o$ and $d_i \in [\![\rho_i]\!]_{U_\mathsf{P}}$ and for all $i = \{1, \dots, n\}$, it holds $\bigsqcap_\alpha X$ as $(\bigsqcap_\alpha X)(\mathsf{p})\ d_1 \cdots\ d_n = \bigsqcap_\alpha \{I(\mathsf{p})\ d_1 \cdots\ d_n : I \in X\}$.

Let X be a nonempty set of Herbrand interpretations. By Lemma 4 we have that $\mathcal{I}_\mathsf{P}$ is a complete lattice with respect to $\leq$ and a basic model. Moreover, by Lemma 1 it follows that $\mathcal{I}_\mathsf{P}$ is also a complete lattice with respect to $\sqsubseteq$. Thus, there exist the least upper bound and greatest lower bound of X for both $\leq$ and $\sqsubseteq$. We denote the greatest lower bound of X as $\bigwedge X$ and $\bigsqcap X$ with respect to relations $\leq$ and $\sqsubseteq$ respectively. Then, $\bigsqcap X$ can be constructed in an symmetric way to the least upper bound construction described in (Ésik and Rondogiannis 2014). More specifically, for each ordinal $\alpha < \kappa$ we define the sets $X_\alpha, Y_\alpha \subseteq X$ and $x_\alpha \in \mathcal{I}_\mathsf{P}$, which are then used in order to obtain $\bigsqcap X$.

Let $Y_0 = X$ and $x_0 = \bigsqcap_0 Y_0$. For every α, with $0 < \alpha < \kappa$ we define $X_\alpha = \{x \in X : \forall \beta \leq \alpha\ x =_\alpha x_\alpha\}$, $Y_\alpha = \bigcap_{\beta<\alpha} X_\beta$; moreover, $x_\alpha = \bigsqcap_\alpha Y_\alpha$ if Y_α is nonempty and $x_\alpha = \bigwedge_{\beta<\alpha} x_\beta$ if Y_α is empty.

Finally, we define $x_\infty = \bigwedge_{\alpha<\kappa} x_\alpha$. In analogy to the proof of (Ésik and Rondogiannis 2014) for the least upper bound it can be shown that $x_\infty = \bigsqcap X$ with respect to the relation $\sqsubseteq$. Moreover, it is easy to prove that by construction it holds $x_\alpha =_\alpha x_\beta$ and $x_\beta \geq x_\alpha$ for all $\beta < \alpha$.

Lemma 7

Let P be a program, $\alpha < \kappa$ and M_α be a Herbrand model of P. Let $\mathcal{M} \subseteq (M_\alpha]_\alpha$ be a nonempty set of Herbrand models of P. Then, $\bigsqcap_\alpha \mathcal{M}$ is also a Herbrand model of P.

Proof
Assume $\bigsqcap_\alpha \mathcal{M}$ is not a model. Then, there exists a clause $\mathsf{p} \leftarrow \mathsf{E}$ in P and $d_i \in [\![\rho_i]\!]_D$ such that $[\![\mathsf{E}]\!](\bigsqcap_\alpha \mathcal{M})\, d_1 \cdots d_n > (\bigsqcap_\alpha \mathcal{M})(\mathsf{p})\, d_1 \cdots d_n$. Since for every $N \in \mathcal{M}$ we have $\bigsqcap_\alpha \mathcal{M} \sqsubseteq_\alpha N$, using Lemma 5 we conclude $[\![\mathsf{E}]\!](\bigsqcap_\alpha \mathcal{M}) \sqsubseteq_\alpha [\![\mathsf{E}]\!](N)$. Let $x = \bigsqcap_\alpha \{N(\mathsf{p})\, d_1 \cdots d_n : N \in \mathcal{M}\}$. By definition, $x = (\bigsqcap_\alpha \mathcal{M})(\mathsf{p})\, d_1 \cdots d_n$.

If $order(x) = \alpha$ then $x = \bigwedge\{N(\mathsf{p})\, d_1 \cdots d_n : N \in \mathcal{M}\}$. If $x = T_\alpha$ then for all $N \in \mathcal{M}$ we have $N(\mathsf{p})\, d_1 \cdots d_n = T_\alpha$. Moreover, $[\![\mathsf{E}]\!](\bigsqcap_\alpha \mathcal{M})\, d_1 \cdots d_n > T_\alpha$ and by α-monotonicity we have $[\![\mathsf{E}]\!](N)\, d_1 \cdots d_n > T_\alpha$ for all $N \in \mathcal{M}$. Then, $N(\mathsf{p})\, d_1 \cdots d_n < [\![\mathsf{E}]\!](N)\, d_1 \cdots d_n$ and therefore N is not a model (contradiction). If $x = F_\alpha$ then there exists $N \in \mathcal{M}$ such that $N(\mathsf{p})\, d_1 \cdots d_n = F_\alpha$ and since N is a model we have $[\![\mathsf{E}]\!](N)\, d_1 \cdots d_n \leq F_\alpha$. But then, it follows $[\![\mathsf{E}]\!](\bigsqcap_\alpha \mathcal{M})\, d_1 \cdots d_n \leq F_\alpha$ and $[\![\mathsf{E}]\!](\bigsqcap_\alpha \mathcal{M})\, d_1 \cdots d_n \leq x$ (contradiction).

If $order(x) < \alpha$ then $x = M_\alpha(\mathsf{p})\, d_1 \cdots d_n$. If $x = T_\beta$ then $[\![\mathsf{E}]\!](\bigsqcap_\alpha \mathcal{M})\, d_1 \cdots d_n > T_\beta$ and $[\![\mathsf{E}]\!](M_\alpha)\, d_1 \cdots d_n > T_\beta$. Then, we have $M_\alpha(\mathsf{p})\, d_1 \cdots d_n < [\![\mathsf{E}]\!](M_\alpha)$ and thus M_α is not a model of P (contradiction). If $x = F_\beta$ then $[\![\mathsf{E}]\!](M_\alpha)\, d_1 \cdots d_n \leq F_\beta$ and by α-monotonicity $[\![\mathsf{E}]\!](\bigsqcap_\alpha \mathcal{M})\, d_1 \cdots d_n \leq F_\beta$. Therefore, $[\![\mathsf{E}]\!](\bigsqcap_\alpha \mathcal{M})\, d_1 \cdots d_n \leq x$ (contradiction).

If $order(x) > \alpha$ then $x = T_{\alpha+1}$ and there exists model $N \in \mathcal{M}$ such that $N(\mathsf{p})\, d_1 \cdots d_n < T_\alpha$. Moreover, we have $[\![\mathsf{E}]\!](\bigsqcap_\alpha \mathcal{M})\, d_1 \cdots d_n \geq T_\alpha$ and by α-monotonicity we conclude $[\![\mathsf{E}]\!](N)\, d_1 \cdots d_n \geq T_\alpha$. But then, $[\![\mathsf{E}]\!](N)\, d_1 \cdots d_n > N(\mathsf{p})\, d_1 \cdots d_n$ and therefore N is not a model of P (contradiction). □

In the following, we will make use of the following lemma that has been shown in (Ésik and Rondogiannis 2014, Lemma 3.18):

Lemma 8
If $\alpha \leq \kappa$ is an ordinal and $(x_\beta)_{\beta<\alpha}$ is a sequence of elements of L such that $x_\beta =_\beta x_\gamma$ and $x_\beta \leq x_\gamma$ ($x_\beta \geq x_\gamma$) whenever $\beta < \gamma < \alpha$, and if $x = \bigvee_{\beta<\alpha} x_\beta$ ($x = \bigwedge_{\beta<\alpha} x_\beta$), then $x_\beta =_\beta x$ holds for all $\beta < \alpha$.

Lemma 9
Let $(M_\alpha)_{\alpha<\kappa}$ be a sequence of Herbrand models of P such that $M_\alpha =_\alpha M_\beta$ and $M_\beta \leq M_\alpha$ for all $\alpha < \beta < \kappa$. Then, $\bigwedge_{\alpha<\kappa} M_\alpha$ is also a Herbrand model of P.

Proof
Let $M_\infty = \bigwedge_{\alpha<\kappa} M_\alpha$ and assume M_∞ is not a model of P. Then, there is a clause $\mathsf{p} \leftarrow \mathsf{E}$ and $d_i \in [\![\rho_i]\!]_D$ such that $[\![\mathsf{E}]\!](M_\infty)\, d_1 \cdots d_n > M_\infty(\mathsf{p})\, d_1 \cdots d_n$. We define $x_\alpha = M_\alpha(\mathsf{p})\, d_1 \cdots d_n$, $x_\infty = M_\infty(\mathsf{p})\, d_1 \cdots d_n$, $y_\alpha = [\![\mathsf{E}]\!](M_\alpha)\, d_1 \cdots d_n$ and $y_\infty = [\![\mathsf{E}]\!](M_\infty)\, d_1 \cdots d_n$ for all $\alpha < \kappa$. It follows from Lemma 8 that $M_\infty =_\alpha M_\alpha$ and thus $x_\infty =_\alpha x_\alpha$ for all $\alpha < \kappa$. Moreover, using α-monotonicity we also have $[\![\mathsf{E}]\!](M_\infty) =_\alpha [\![\mathsf{E}]\!](M_\alpha)$ and thus $y_\infty =_\alpha y_\alpha$ for all $\alpha < \kappa$. We distinguish cases based on the value of x_∞.

Assume $x_\infty = T_\delta$ for some $\delta < \kappa$. It follows by assumption that $y_\infty > T_\delta$. Then, since $x_\infty =_\delta x_\delta$ it follows $x_\delta = T_\delta$. Moreover, since $y_\infty =_\delta y_\delta$ and $order(y_\infty) < \delta$ it follows $y_\delta = y_\infty > T_\delta$. But then, $y_\delta > x_\delta$ (contradiction since M_δ is a model by assumption).

Assume $x_\infty = F_\delta$ for some $\delta < \kappa$. Then, since $x_\infty =_\delta x_\delta$ it follows $x_\delta = F_\delta$. Then, since M_δ is a model it follows $y_\delta \leq x_\delta$ and thus $y_\delta \leq F_\delta$. But then, since $y_\infty =_\delta y_\delta$ it follows $y_\delta = y_\infty \leq F_\delta$. Therefore, $y_\infty \leq x_\infty$ that is a contradiction to our assumption that $y_\infty > x_\infty$.

Assume $x_\infty = 0$. Then, $y_\infty > x_\infty = 0$. Let $y_\infty = T_\beta$ for some $\beta < \kappa$. Then, since

$y_\beta =_\beta y_\infty$ it follows $y_\beta = T_\beta$. Since M_β is a model of P it holds $T_\beta = y_\beta \leq x_\beta$, that is $x_\beta = T_\gamma$ for some $\gamma \leq \beta$. Moreover, since $x_\infty =_\beta x_\beta$ it follows that $x_\infty = T_\gamma$ that is a contradiction to our assumption that $x_\infty = 0$. □

Theorem 2 (*Model Intersection Theorem*)
Let P be a program and $\mathcal{M}$ be a nonempty set of Herbrand models of P. Then, $\bigsqcap \mathcal{M}$ is also a Herbrand model of P.

Proof
We use the construction for $\bigsqcap \mathcal{M}$ described in the beginning of this appendix. More specifically, we define sets $\mathcal{M}_\alpha, Y_\alpha \subseteq \mathcal{M}$ and $M_\alpha \in \mathcal{I}_{\mathsf{P}}$. Let $Y_0 = \mathcal{M}$ and $M_0 = \bigsqcap_0 Y_0$. For every $\alpha > 0$, let $\mathcal{M}_\alpha = \{M \in \mathcal{M} : \forall \beta \leq \alpha\ M =_\alpha M_\alpha\}$ and $Y_\alpha = \bigcap_{\beta<\alpha} \mathcal{M}_\beta$; moreover, $M_\alpha = \bigsqcap_\alpha Y_\alpha$ if Y_α is nonempty and $M_\alpha = \bigwedge_{\beta<\alpha} M_\beta$ if Y_α is empty. Then, $\bigsqcap \mathcal{M} = \bigwedge_{\alpha<\kappa} M_\alpha$. It is easy to see that $M_\alpha =_\alpha M_\beta$ and $\mathcal{M}_\beta \supseteq \mathcal{M}_\alpha$ for all $\beta < \alpha$.

We distinguish two cases. First, consider the case when Y_α is nonempty for all $\alpha < \kappa$. Then, $M_\alpha = \bigsqcap_\alpha Y_\alpha$ and by Lemma 7 it follows that M_α is a model of P. Moreover, by Lemma 9 we get that $M_\infty = \bigwedge_{\alpha<\kappa} M_\alpha$ is also a model of P.

Consider now the case that there exists a least ordinal $\delta < \kappa$ such that Y_δ is empty. It holds (see (Ésik and Rondogiannis 2014)) that $M_\infty = \bigwedge_{\alpha<\delta} M_\delta$. Suppose M_∞ is not a model of P. Then, there is a clause $\mathsf{p} \leftarrow \mathsf{E}$, a Herbrand state s and $d_i \in [\![\rho_i]\!]_D$ such that $[\![\mathsf{E}]\!](M_\infty)\, d_1 \cdots d_n > M_\infty(\mathsf{p})\, d_1 \cdots d_n$. We define $x_\alpha = M_\alpha(\mathsf{p})\, d_1 \cdots d_n$, $x_\infty = M_\infty(\mathsf{p})\, d_1 \cdots d_n$, $y_\alpha = [\![\mathsf{E}]\!](M_\alpha)\, d_1 \cdots d_n$, and $y_\infty = [\![\mathsf{E}]\!](M_\infty)\, d_1 \cdots d_n$ for all $\beta \leq \alpha$. We distinguish cases based on the value of x_∞.

Assume $x_\infty = T_\beta$ for some $\beta < \delta$. It follows by assumption that $y_\infty > x_\infty = T_\beta$. Then, by Lemma 8 it holds that $M_\infty =_\beta M_\beta$ and we get $x_\infty =_\beta x_\beta$ and therefore $x_\beta = T_\beta$. Moreover, by α-monotonicity we get $[\![\mathsf{E}]\!](M_\infty)\, d_1 \cdots d_n =_\beta [\![\mathsf{E}]\!](M_\beta)\, d_1 \cdots d_n$ and it follows that $y_\infty =_\beta y_\beta$. Moreover, since $y_\infty > T_\beta$ it follows $y_\beta = y_\infty > T_\beta$ and $y_\beta > x_\beta$. Since Y_β is not empty by assumption we have that $M_\beta = \bigsqcap_\beta Y_\beta$ and by Lemma 7 we get that M_β is a model of P (contradiction since $y_\beta > x_\beta$).

Assume $x_\infty = F_\beta$ for some $\beta < \delta$. Then, by Lemma 8 it holds $M_\infty =_\beta M_\beta$ and therefore $x_\infty =_\beta x_\beta$. It follows $x_\beta = F_\beta$. Moreover, since Y_β is nonempty by assumption and by Lemma 7 it follows that $M_\beta = \bigsqcap_\beta Y_\beta$ is a model of P and thus $y_\beta \leq x_\beta = F_\beta$. By α-monotonicity we get $[\![\mathsf{E}]\!](M_\infty) =_\beta [\![\mathsf{E}]\!](M_\beta)$ and therefore $y_\infty =_\beta y_\beta \leq F_\beta$. It follows $y_\infty \leq F_\beta = x_\infty$ (contradiction to the initial assumption $y_\infty > x_\infty$).

Assume $x_\infty = T_\delta$. By assumption we have $y_\infty > x_\infty = T_\delta$. Then, let $y_\infty = T_\gamma$ for some $\gamma < \delta$. By Lemma 8 it holds $M_\infty =_\gamma M_\gamma$ and by α-monotonicity it follows $[\![\mathsf{E}]\!](M_\infty) =_\gamma [\![\mathsf{E}]\!](M_\gamma)$ and thus $y_\infty =_\gamma y_\gamma$. It follows that $y_\gamma = T_\gamma$. Moreover, since $\gamma < \delta$ we know by assumption that Y_γ is nonempty and therefore $M_\gamma = \bigsqcap Y_\gamma$ and by Lemma 7 M_γ is a model of P. It follows $T_\gamma = y_\gamma \leq x_\gamma$, that is, $x_\gamma = T_\beta$ for some $\beta \leq \gamma < \delta$. Moreover, since $x_\infty =_\gamma x_\gamma$ it follows $x_\infty = T_\beta$ that is a contradiction (since by assumption $x_\infty = T_\delta$).

Assume $x_\infty = F_\delta$. This case is not possible. Recall that Y_α is not empty for all $\alpha < \delta$ and thus $M_\alpha = \bigsqcap Y_\alpha$. By the definition of $\bigsqcap_\alpha$ we observe that either $\bigsqcap_\alpha Y_\alpha \leq F_\alpha$ or $\bigsqcap_\alpha Y_\alpha \geq T_{\alpha+1}$. Then, since $M_\infty = \bigwedge_{\alpha<\delta} M_\alpha$ it is not possible to have $x_\infty = F_\delta$.

Assume $x_\infty = 0$. This case does not arise. Again, Y_α is not empty for all $\alpha < \delta$ and thus $M_\alpha = \bigsqcap_\alpha Y_\alpha$. Moreover, by definition of $\bigsqcap_\alpha$, $x_\alpha \neq 0$ for all $\alpha < \delta$. Moreover, since $M_\infty = \bigwedge_{\alpha<\delta} M_\alpha$ and since $\delta < \kappa$ it follows that the limit can be at most T_δ. □

Appendix D Proofs of Lemmas 6, 7 and Theorem 3

Lemma 6
Let P be a program. For every predicate constant $\mathsf{p} : \pi$ in P and $I \in \mathcal{I}_\mathsf{P}$, $T_P(I)(\mathsf{p}) \in [\![\pi]\!]_{U_\mathsf{P}}$.

Proof
It follows from the fact that $[\![\pi]\!]_{U_\mathsf{P}}$ is a complete lattice (Lemma 2). □

Lemma 7
Let P be a program. Then, T_P is α-monotonic for all $\alpha < \kappa$.

Proof
Follows directly from Lemma 5 and Proposition 1. □

Lemma 10
Let P be a program. Then, $M \in \mathcal{I}_\mathsf{P}$ is a model of P if and only if $T_\mathsf{P}(M) \leq_{\mathcal{I}_\mathsf{P}} M$.

Proof
An interpretation $I \in \mathcal{I}_\mathsf{P}$ is a model of P iff $[\![\mathsf{E}]\!](I) \leq_\pi I(\mathsf{p})$ for all clauses $\mathsf{p} \leftarrow_\pi \mathsf{E}$ in P iff $\bigvee_{(\mathsf{p}\leftarrow\mathsf{E})\in\mathsf{P}} [\![\mathsf{E}]\!](I) \leq_{\mathcal{I}_\mathsf{P}} I(\mathsf{p})$ iff $T_\mathsf{P}(I) \leq_{\mathcal{I}_\mathsf{P}} I$. □

Proposition 11
Let D be a nonempty set, π be a predicate type and $x, y \in [\![\pi]\!]_D$. If $x \leq_\pi y$ and $x =_\beta y$ for all $\beta < \alpha$ then $x \sqsubseteq_\alpha y$.

Proof
The proof is by structural induction on π.

Induction Basis: If $x =_\beta y$ for all $\beta < \alpha$ then either $x = y$ or $order(x), order(y) \geq \alpha$. If $x = y$ then $x \sqsubseteq_\alpha y$. Suppose $x \neq y$. If $order(x), order(y) > \alpha$ then $x =_\alpha y$. If $x = F_\alpha$ then clearly $x \sqsubseteq_\alpha y$. If $x = T_\alpha$ then $T_\alpha \leq y$ and therefore $y = T_\alpha$. The case analysis for y is similar.

Induction Step: Assume that the statement holds for π. Let $f, g \in [\![\rho \to \pi]\!]_D$ and $\alpha < \kappa$. For all $x \in [\![\rho]\!]_D$ and $\beta < \alpha$, $f(x) \leq g(x)$ and $f(x) =_\beta g(x)$. It follows that $f(x) \sqsubseteq_\alpha g(x)$. Therefore, $f \sqsubseteq_\alpha g$. □

Proposition 12
Let P be a program and I, J be Herbrand interpretations of P. If $I \leq_{\mathcal{I}_\mathsf{P}} J$ and $I =_\beta J$ for all $\beta < \alpha$ then $I \sqsubseteq_\alpha J$.

Proof
Let $I, J \in \mathcal{I}_\mathsf{P}$ and $\alpha < \kappa$. For all predicate constants p and $\beta < \alpha$, $I(\mathsf{p}) \leq J(\mathsf{p})$ and $I(\mathsf{p}) =_\beta J(\mathsf{p})$. It follows by Proposition 11 that $I(\mathsf{p}) \sqsubseteq_\alpha J(\mathsf{p})$ and therefore, $I \sqsubseteq_\alpha J$. □

Lemma 13
Let P be a program. If M is a model of P then $T_{\mathsf{P}}(M) \sqsubseteq M$.

Proof
It follows from Lemma 10 that if M is a Herbrand model of P then $T_{\mathsf{P}}(M) \leq_{\mathcal{I}_{\mathsf{P}}} M$. If $T_{\mathsf{P}}(M) = M$ then the statement is immediate. Suppose $T_{\mathsf{P}}(M) <_{\mathcal{I}_{\mathsf{P}}} M$ and let α denote the least ordinal such that $T_{\mathsf{P}}(M) =_{\alpha} M$ does not hold. Then, $T_{\mathsf{P}}(M) =_{\beta} M$ for all $\beta < \alpha$. Since $T_{\mathsf{P}}(M) <_{\mathcal{I}_{\mathsf{P}}} M$, by Proposition 12 it follows that $T_{\mathsf{P}}(M) \sqsubseteq_{\alpha} M$. Since $T_{\mathsf{P}}(M) =_{\alpha} M$ does not hold, it follows that $T_{\mathsf{P}}(M) \sqsubset_{\alpha} M$. Therefore $T_{\mathsf{P}}(M) \sqsubseteq M$. $\square$

Theorem 3 (*Least Fixed Point Theorem*)
Let P be a program and let $\mathcal{M}$ be the set of all its Herbrand models. Then, T_{P} has a least fixed point M_{P}. Moreover, $M_{\mathsf{P}} = \bigsqcap \mathcal{M}$.

Proof
It follows from Lemma 7 and Theorem 1 that T_{P} has a least pre-fixed point with respect to $\sqsubseteq$ that is also a least fixed point. Let M_{P} be that least fixed point of T_{P}, i.e., $T_{\mathsf{P}}(M_{\mathsf{P}}) = M_{\mathsf{P}}$. It is clear from Lemma 10 that M_{P} is a model of P, i.e., $M_{\mathsf{P}} \in \mathcal{M}$. Then, it follows $\bigsqcap \mathcal{M} \sqsubseteq M_{\mathsf{P}}$. Moreover, from Theorem 2 it is implied that $\bigsqcap \mathcal{M}$ is a model and thus from Lemma 13, $\bigsqcap \mathcal{M}$ is a pre-fixed point of T_{P} with respect to $\sqsubseteq$. Since M_{P} is the least pre-fixed point of P, $M_{\mathsf{P}} \sqsubseteq \bigsqcap \mathcal{M}$ and thus $M_{\mathsf{P}} = \bigsqcap \mathcal{M}$. $\square$