Online appendix for the paper

AC-KBO revisited

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Appendix A Omitted Proofs

A.1 Correctness of $>_{\mathsf{ACKBO}}$

First we show that $(=_{\mathsf{AC}}, >_{\mathsf{ACKBO}})$ is an order pair. To facilitate the proof, we decompose $>_{\mathsf{ACKBO}}$ into several orders. We write

- $s >_{01} t$ if $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$ and either $w(s) > w(t)$ or $w(s) = w(t)$ and case 0 or case 1 of Definition 5.1 applies,
- $s >_{23,k} t$ if $|s|, |t| \leqslant k$, $|s|_x \geqslant |t|_x$ for all $x \in \mathcal{V}$, $w(s) = w(t)$, and case 2 or case 3 applies.

The union of $>_{01}$ and $>_{23,k}$ is denoted by $>_k$. The next lemma states straightforward properties.

Lemma A.1
The following statements hold:

1. $>_{\mathsf{ACKBO}} \; = \bigcup \{>_k \mid k \in \mathbb{N}\}$,
2. $(=_{\mathsf{AC}}, >_{01})$ is an order pair, and
3. $(>_{01} \cdot >_k) \cup (>_k \cdot >_{01}) \; \subseteq \; >_{01}$.

Proof

1. The inclusion from right to left is obvious from the definition. For the inclusion from left to right, suppose $s >_{\mathsf{ACKBO}} t$. If either $w(s) > w(t)$, or $w(s) = w(t)$ and case 0 or case 1 of Definition 5.1 applies, then trivially $s >_{01} t$. If case 2 or case 3 applies, then $s >_{23,k} t$ for any k with $k \geqslant \max(|s|, |t|)$.

2. First we show that $>_{01}$ is transitive. Suppose $s >_{01} t >_{01} u$. If $w(s) > w(t)$ or $w(t) > w(u)$, then $w(s) > w(u)$ and $s >_{01} u$. Hence suppose $w(s) = w(t) = w(u)$. Since $s, t \notin \mathcal{V}$, we may write $s = f(s_1, \ldots, s_n)$ and $t = g(t_1, \ldots, t_m)$ with $f > g$. Because of admissibility, g is not a unary symbol with $w(g) = 0$. Thus $u \notin \mathcal{V}$, and we may write $u = h(u_1, \ldots, u_l)$ with $g > h$. By the transitivity of $>$ we obtain $s >_{01} u$. The irreflexivity of $>_{01}$ is obvious from the definition. It remains to show the compatibility condition $=_{\mathsf{AC}} \cdot >_{01} \cdot =_{\mathsf{AC}} \subseteq >_{01}$. This easily follows from the fact that $w(s) = w(t)$ and $\mathsf{root}(s) = \mathsf{root}(t)$ whenever $s =_{\mathsf{AC}} t$.
3. Suppose $s = f(s_1, \ldots, s_n) >_{01} t = g(t_1, \ldots, t_m) >_k u$. If $t >_{01} u$ then $s >_{01} u$ follows from the transitivity of $>_{01}$. Suppose $t >_{23,k} u$. So $w(t) = w(u)$. Thus $w(s) > w(u)$ if $w(s) > w(t)$, and case 1 applies if $w(s) = w(t)$. The inclusion $>_k \cdot >_{01} \subseteq >_k$ is proved in exactly the same way. □

Lemma A.2
Let $>$ be a precedence, $f \in \mathcal{F}$, and $(\succsim, \succ)$ an order pair on terms. Then $(\succsim^f, \succ^f)$ is an order pair.

Proof
We first prove compatibility. Suppose $S \succsim^f T \succ^f U$. From $T \succ^f U$ we infer that $T{\restriction}_f^{\not<} \uplus T{\restriction}_{\mathcal{V}} \succ^{\mathsf{mul}} U{\restriction}_f^{\not<} \uplus U{\restriction}_{\mathcal{V}}$. Hence $S{\restriction}_f^{\not<} \succ^{\mathsf{mul}} U{\restriction}_f^{\not<} \uplus U{\restriction}_{\mathcal{V}} - S{\restriction}_{\mathcal{V}}$ follows from $S \succsim^f T$. Hence also $S\ (\succsim \cdot \succ)^f\ U$. We obtain the desired $S \succ^f U$ from the compatibility of $\succsim$ and $\succ$. Transitivity of $\succsim^f$ and $\succ^f$ is obtained in a very similar way. Reflexivity of $\succsim^f$ and irreflexivity of $\succ^f$ are obvious. □

We employ the following simple criterion to construct order pairs, which enables us to prove correctness in a modular way.

Lemma A.3
Let $(\succsim, \succ_k)$ be order pairs for $k \in \mathbb{N}$ with $\succ_k \subseteq \succ_{k+1}$. If $\succ$ is the union of all $\succ_k$ then $(\succsim, \succ)$ is an order pair.

Proof
The relation $\succsim$ is a preorder by assumption. Suppose $s \succ t \succ u$. By assumption there exist k and l such that $s \succ_k t \succ_l u$. Let $m = \max(k, l)$. We obtain $s \succ_m t \succ_m u$ from the assumptions of the lemma and hence $s \succ_m u$ follows from the fact that $(\succsim, \succ_m)$ is an order pair. Compatibility is an immediate consequence of the assumptions and the irreflexivity of $\succ$ is obtained by an easy induction proof. □

Proof of Lemma 5.4
According to Lemmata A.3 and A.1(1), it is sufficient to prove that $(=_{\mathsf{AC}}, >_k)$ is an order pair for all $k \in \mathbb{N}$. Due to Lemma A.1(2,3) it suffices to prove that $(=_{\mathsf{AC}}, >_{23,k})$ is an order pair, which follows by using induction on k in combination with Lemma A.2 and Theorem 2.2. □

Proof of Theorem 5.12

Let $\mathcal{T}_k$ denote the set of ground terms of size at most k. We use induction on $k \geqslant 1$ to show that $>_{\mathsf{ACKBO}}$ is AC-total on $\mathcal{T}_k$. Let $s, t \in \mathcal{T}_k$. We consider the case where $w(s) = w(t)$ and $\mathsf{root}(s) = \mathsf{root}(t) = f \in \mathcal{F}_{\mathsf{AC}}$. The other cases follow as for standard KBO. Let $S = \triangledown_f(s)$ and $T = \triangledown_f(t)$. Clearly S and T are multisets over $\mathcal{T}_{k-1}$. According to the induction hypothesis, $>_{\mathsf{ACKBO}}$ is AC-total on $\mathcal{T}_{k-1}$ and since multiset extension preserves AC totality, $>^{\mathsf{mul}}_{\mathsf{ACKBO}}$ is AC-total on multisets over $\mathcal{T}_{k-1}$. Hence for any pair of multisets U and V over $\mathcal{T}_{k-1}$, either

$$U >^{\mathsf{mul}}_{\mathsf{ACKBO}} V \quad \text{or} \quad V >^{\mathsf{mul}}_{\mathsf{ACKBO}} U \quad \text{or} \quad U =^{\mathsf{mul}}_{\mathsf{AC}} V$$

Because the precedence $>$ is total and S and T contain neither variables nor terms with f as their root symbol, we have

$$S = S{\restriction}^{\not<}_f \cup S{\restriction}^{<}_f = S{\restriction}^{>}_f \cup S{\restriction}^{<}_f \qquad\qquad T = T{\restriction}^{\not<}_f \cup T{\restriction}^{<}_f = T{\restriction}^{>}_f \cup T{\restriction}^{<}_f$$

If $S{\restriction}^{>}_f >^{\mathsf{mul}}_{\mathsf{ACKBO}} T{\restriction}^{>}_f$ or $T{\restriction}^{>}_f >^{\mathsf{mul}}_{\mathsf{ACKBO}} S{\restriction}^{>}_f$ then case 3(a) of Definition 5.1 is applicable to derive either $s >_{\mathsf{ACKBO}} t$ or $t >_{\mathsf{ACKBO}} s$. Otherwise we must have $S{\restriction}^{>}_f =^{\mathsf{mul}}_{\mathsf{AC}} T{\restriction}^{>}_f$ by AC-totality. If $|S| > |T|$ then we obtain $s >_{\mathsf{ACKBO}} t$ by case 3(b). Similarly, $|S| < |T|$ gives rise to $t >_{\mathsf{ACKBO}} s$.

In the remaining case we have both $S{\restriction}^{>}_f =^{\mathsf{mul}}_{\mathsf{AC}} T{\restriction}^{>}_f$ and $|S| = |T|$. Using case 3(c) of Definition 5.1 we obtain $s >_{\mathsf{ACKBO}} t$ when $S{\restriction}^{<}_f >^{\mathsf{mul}}_{\mathsf{ACKBO}} T{\restriction}^{<}_f$ and $t >_{\mathsf{ACKBO}} s$ when $T{\restriction}^{<}_f >^{\mathsf{mul}}_{\mathsf{ACKBO}} S{\restriction}^{<}_f$. By AC totality there is one case remaining: $S{\restriction}^{<}_f =^{\mathsf{mul}}_{\mathsf{AC}} T{\restriction}^{<}_f$. Combined with $S{\restriction}^{>}_f =^{\mathsf{mul}}_{\mathsf{AC}} T{\restriction}^{>}_f$ we obtain $S =^{\mathsf{mul}}_{\mathsf{AC}} T$. We may write $S = \{s_1, \ldots, s_n\}$ and $T = \{t_1, \ldots, t_n\}$ such that $s_i =_{\mathsf{AC}} t_i$ for all $1 \leqslant i \leqslant n$. Since f is an AC symbol, $s =_{\mathsf{AC}} f(s_1, f(\ldots, s_n)\ldots)$ and $t =_{\mathsf{AC}} f(t_1, f(\ldots, t_n)\ldots)$, from which we conclude $s =_{\mathsf{AC}} t$. □

A.2 Correctness of $>_{\mathsf{KV}'}$

We prove that $>_{\mathsf{KV}'}$ is an AC-compatible simplification order. The proof mimics the one given in Sections 5 and A.1 for $>_{\mathsf{ACKBO}}$, but there are some subtle differences. The easy proof of the following lemma is omitted.

Lemma A.4

The pairs $(=_{\mathsf{AC}}, >_{\mathsf{kv}})$ and $(\geqslant_{\mathsf{kv}'}, >_{\mathsf{kv}})$ are order pairs. □

Lemma A.5

The pair $(=_{\mathsf{AC}}, >_{\mathsf{KV}'})$ is an order pair.

Proof

Similar to the proof of Lemma 5.4, except for case 3 of Definition 4.10, where we need Lemma A.4 and Theorem 2.2. □

The subterm property follows exactly as in the proof of Lemma 5.5; note that the relation $>_{01}$ has the subterm property, and we obviously have $>_{01} \subseteq >_{\mathsf{KV}'}$.

Lemma A.6

The order $>_{\mathsf{KV}'}$ has the subterm property. □

Lemma A.7
The order $>_{\mathsf{KV}'}$ is closed under contexts.

Proof
Suppose $s >_{\mathsf{KV}'} t$. We follow the proof for $>_{\mathsf{ACKBO}}$ in Lemma 5.7 and consider here the case that $w(s) = w(t)$. We will show that one of the cases 3(a,b,c) in Definition 4.10 (4.7) is applicable to $S = \nabla_h(s)$ and $T = \nabla_h(t)$. Let $f = \mathsf{root}(s)$ and $g = \mathsf{root}(t)$. The proof proceeds by case splitting according to the derivation of $s >_{\mathsf{KV}'} t$.

- Suppose $s = f^k(t)$ with $k > 0$ and $t \in \mathcal{V}$. Admissibility enforces $f > h$ and thus $S{\restriction}_h^{\not<} = \{s\} \geqslant_{\mathsf{kv}'}^{\mathsf{mul}} \{t\}$. We have $|S| = |T| = 1$ and $S >_{\mathsf{KV}'}^{\mathsf{mul}} T$. Hence 3(c) applies. (This case breaks down for $>_{\mathsf{KV}}$.)
- Suppose $f = g \notin \mathcal{F}_{\mathsf{AC}}$. We have $S \geqslant_{\mathsf{kv}'}^{\mathsf{mul}} T$, $|S| = |T| = 1$, and $S = \{s\} >_{\mathsf{KV}'}^{\mathsf{mul}} \{t\} = T$. Hence 3(c) applies.
- The remaining cases are similar to the proof of Lemma 5.7, except that we use Lemma 5.6 with $(\geqslant_{\mathsf{kv}'}, >_{\mathsf{kv}})$. □

For closure under substitutions we need to extend Lemma 5.8 with the following case:

3. *If $S \succsim^f T$ and $S' \not\succ^f T'$ then $S' - T' \supseteq S\sigma - T\sigma$ and $T\sigma - S\sigma \supseteq T' - S'$.*

Proof
We continue the proof of Lemma 5.8. From $\nabla_f(U\sigma) = U\sigma$ we infer that $T' = T{\restriction}_{\mathcal{F}}\sigma \uplus U\sigma \uplus \nabla_f(X\sigma)$. On the other hand, $S' = S{\restriction}_{\mathcal{F}}\sigma \uplus \nabla_f(Y\sigma) \uplus \nabla_f(X\sigma)$ with $Y = S{\restriction}_{\mathcal{V}} - X$. Hence

$$\begin{aligned} T' - S' &\subseteq T{\restriction}_{\mathcal{F}}\sigma \uplus U\sigma - S{\restriction}_{\mathcal{F}}\sigma \\ &= T{\restriction}_{\mathcal{F}}\sigma \uplus U\sigma \uplus X\sigma - (S{\restriction}_{\mathcal{F}} \uplus X\sigma) \\ &\subseteq T\sigma - S\sigma \end{aligned}$$

and

$$\begin{aligned} S' - T' &\supseteq S{\restriction}_{\mathcal{F}}\sigma - T{\restriction}_{\mathcal{F}}\sigma - U\sigma \\ &= S{\restriction}_{\mathcal{F}}\sigma \uplus X\sigma - (T{\restriction}_{\mathcal{F}} \uplus U\sigma \uplus X\sigma) \\ &\supseteq S\sigma - T\sigma \end{aligned}$$

establishing the desired inclusions. □

Lemma A.8
The order $>_{\mathsf{KV}'}$ is closed under substitutions.

Proof
By induction on $|s|$ we verify that $s >_{\mathsf{KV}'} t$ implies $s\sigma >_{\mathsf{KV}'} t\sigma$. If $s >_{\mathsf{KV}'} t$ is derived by one of the cases 0, 1, 2, 3(a) or 3(b) in Definition 4.10 (4.7), the proof of Lemma 5.7 goes through. So suppose that $s >_{\mathsf{KV}'} t$ is derived by case 3(c) and

further suppose that $s\sigma >_{\mathsf{KV'}} t\sigma$ can be derived neither by case 3(a) nor 3(b). By definition we have $\nabla_f(s) >^{\mathsf{mul}}_{\mathsf{KV'}} \nabla_f(t)$. This is equivalent[1] to

$$\nabla_f(s) - \nabla_f(t) >^{\mathsf{mul}}_{\mathsf{KV'}} \nabla_f(t) - \nabla_f(s)$$

We obtain $\nabla_f(s)\sigma - \nabla_f(t)\sigma >^{\mathsf{mul}}_{\mathsf{KV'}} \nabla_f(t)\sigma - \nabla_f(s)\sigma$ from the induction hypothesis and thus $\nabla_f(s\sigma) - \nabla_f(t\sigma) >^{\mathsf{mul}}_{\mathsf{KV'}} \nabla_f(t\sigma) - \nabla_f(s\sigma)$ by Lemma 5.8(1). Using the earlier equivalence, we infer $\nabla_f(s\sigma) >^{\mathsf{mul}}_{\mathsf{KV'}} \nabla_f(t\sigma)$ and hence case 3(c) applies to obtain the desired $s\sigma >_{\mathsf{KV'}} t\sigma$. □

The combination of the above results proves Theorem 4.12.

A.3 NP-Hardness of AC-KBO

Next we show NP-hardness of the orientability problem for $>_{\mathsf{ACKBO}}$. To this end we introduce the TRS $\mathcal{R}'_0$ consisting of the rules

$$\mathsf{a}(p_1(\mathsf{c})) \to p_1(\mathsf{a}(\mathsf{c})) \qquad \cdots \qquad \mathsf{a}(p_m(\mathsf{c})) \to p_m(\mathsf{a}(\mathsf{c}))$$

together with a rule $\mathsf{e}^0_i(\mathsf{e}^1_i(\mathsf{c})) \to \mathsf{e}^1_i(\mathsf{e}^0_i(\mathsf{c}))$ for each clause C_i that contains a negative literal. The next property is immediate.

Lemma A.9
If $\mathcal{R}'_0 \subseteq >_{\mathsf{ACKBO}}$ then $\mathsf{e}^0_i > \mathsf{e}^1_i$ for all $1 \leqslant i \leqslant n$ and $\mathsf{a} > p_j$ for all $1 \leqslant j \leqslant m$. □

The TRS $\mathcal{R}_0 \cup \mathcal{R}'_0 \cup \{\ell_i \to r_i \mid 1 \leqslant i \leqslant n\}$ is denoted by $\mathcal{R}'_\phi$.

Lemma A.10
Suppose $\mathsf{a} > + > \mathsf{b}$ and the consequence of Lemma A.9 holds. Then $\mathcal{R}'_\phi \subseteq >_{\mathsf{ACKBO}}$ for some (w, w_0) if and only if for every i there is some p such that $p \in C_i$ with $p \not< +$ or $\neg p \in C_i$ with $+ > p$.

Proof
The "if" direction is analogous to Lemma 6.7. Let us prove the "only if" direction by contradiction. Suppose $+ > p'_j$ for all $1 \leqslant j \leqslant k$, $p''_j \not< +$ for all $1 \leqslant j \leqslant l$, and $\mathcal{R}'_\phi \subseteq >_{\mathsf{ACKBO}}$. As discussed in the proof of Lemma 6.7, for the multisets V and W on page 16 we obtain $V >^{\mathsf{mul}}_{\mathsf{ACKBO}} W$ and all terms in V and W have the same weight. With the help of Lemma A.9 we infer that $\mathsf{a}(\mathsf{e}^0_i(\mathsf{e}^0_i(\mathsf{c}))) \in W$ is greater than every other term in V and W. This contradicts $V >^{\mathsf{mul}}_{\mathsf{ACKBO}} W$. □

Using Lemmata A.9 and A.10, Theorem 6.9 can now be proved in the same way as Theorem 6.8.

[1] This property is well-known for standard multiset extensions (involving a single strict order). It is also not difficult to prove for the multiset extension defined in Definition 2.1.

A.4 AC-RPO

Proof of Lemma 7.5

Because of totality of the precedence, $S{\upharpoonright}_f^{\not<}$ is identified with $S{\upharpoonright}_f^{>}$ in the sequel. First suppose $s >_{\mathsf{ACRPO}} t$ holds by case 4. We may assume that $>_{\mathsf{ACRPO}}$ and $>_{\mathsf{ACRPO'}}$ coincide on smaller terms. The conditions on $\rhd_{\mathsf{emb}}^{f}$ are obviously the same. We distinguish which case applies.

- 4(a) We have $S{\upharpoonright}_f^{>} >_{\mathsf{ACRPO}}^{\mathsf{mul}} T{\upharpoonright}_f^{>} \uplus T{\upharpoonright}_{\mathcal{V}} - S{\upharpoonright}_{\mathcal{V}}$ and thus both $S{\upharpoonright}_f^{>} \uplus S{\upharpoonright}_{\mathcal{V}} \geqslant_{\mathsf{ACRPO}}^{\mathsf{mul}} T{\upharpoonright}_f^{>} \uplus T{\upharpoonright}_{\mathcal{V}}$ and $S{\upharpoonright}_f^{>} >_{\mathsf{ACRPO}}^{\mathsf{mul}} T{\upharpoonright}_f^{>}$. So case 4′(a) is applicable.
- 4(b) We have $|S| > |T|$ and $S =_{\mathsf{AC}}^{f} T$, i.e., $S{\upharpoonright}_f^{>} =_{\mathsf{AC}}^{\mathsf{mul}} T{\upharpoonright}_f^{>} \uplus T{\upharpoonright}_{\mathcal{V}} - S{\upharpoonright}_{\mathcal{V}}$, and in particular $T{\upharpoonright}_{\mathcal{V}} \subseteq S{\upharpoonright}_{\mathcal{V}}$. Thus $S{\upharpoonright}_f^{>} \uplus S{\upharpoonright}_{\mathcal{V}} \geqslant_{\mathsf{ACRPO}}^{\mathsf{mul}} T{\upharpoonright}_f^{>} \uplus T{\upharpoonright}_{\mathcal{V}}$ holds. Since $T{\upharpoonright}_{\mathcal{V}} \subseteq S{\upharpoonright}_{\mathcal{V}}$ and $|S| > |T|$ imply $\#(S) > \#(T)$, case 4′(b) applies.
- 4(c) We obtain $S{\upharpoonright}_f^{>} \uplus S{\upharpoonright}_{\mathcal{V}} \geqslant_{\mathsf{ACRPO}}^{\mathsf{mul}} T{\upharpoonright}_f^{>} \uplus T{\upharpoonright}_{\mathcal{V}}$ as in case 4(b). Together with $|S| = |T|$ this implies $\#(S) \geqslant \#(T)$. As $S = S{\upharpoonright}_f^{>} \uplus S{\upharpoonright}_{\mathcal{V}} \uplus S{\upharpoonright}_f^{<}$ and similar for T, we obtain $S >_{\mathsf{ACRPO}}^{\mathsf{mul}} T$ from the assumption $S{\upharpoonright}_f^{<} >_{\mathsf{ACRPO}}^{\mathsf{mul}} T{\upharpoonright}_f^{<}$. Hence case 4′(c) is applicable.

Now let $s >_{\mathsf{ACRPO'}} t$ by case 4′. Again we assume that $>_{\mathsf{ACRPO}}$ and $>_{\mathsf{ACRPO'}}$ coincide on smaller terms. We have $S{\upharpoonright}_f^{>} \uplus S{\upharpoonright}_{\mathcal{V}} \geqslant_{\mathsf{ACRPO}}^{\mathsf{mul}} T{\upharpoonright}_f^{>} \uplus T{\upharpoonright}_{\mathcal{V}}$ $(*)$.

- 4′(a) We have $S{\upharpoonright}_f^{>} >_{\mathsf{ACRPO}}^{\mathsf{mul}} T{\upharpoonright}_f^{>}$. Suppose $S \not>_{\mathsf{ACRPO}}^{f} T$, i.e., $S{\upharpoonright}_f^{>} >_{\mathsf{ACRPO}}^{\mathsf{mul}} T{\upharpoonright}_f^{>} \uplus T{\upharpoonright}_{\mathcal{V}} - S{\upharpoonright}_{\mathcal{V}}$ does not hold. This is only possible if there is some variable $x \in T{\upharpoonright}_{\mathcal{V}} - S{\upharpoonright}_{\mathcal{V}}$ for which there is no term $s' \in S{\upharpoonright}_f^{>}$ with $s' >_{\mathsf{ACRPO}} x$. This however contradicts $(*)$, so $S >_{\mathsf{ACRPO}}^{f} T$ holds and case 4(a) applies.
- 4′(b) If $S{\upharpoonright}_f^{>} >_{\mathsf{ACRPO}}^{\mathsf{mul}} T{\upharpoonright}_f^{>}$ holds then case 4(a) applies by the reasoning in case 4′(a). Otherwise, due to $(*)$ we must have $S =_{\mathsf{AC}}^{f} T$. Since $\#(S) > \#(T)$ implies $|S| > |T|$, case 4(b) applies.
- 4′(c) If $\#(S) > \#(T)$ is satisfied we argue as in the preceding case. Otherwise $\#(S) \geqslant \#(T)$ and $\#(S) \not> \#(T)$. This implies both $|S| = |T|$ and $S{\upharpoonright}_{\mathcal{V}} \supseteq T{\upharpoonright}_{\mathcal{V}}$. We obtain $S =_{\mathsf{AC}}^{f} T$ as in case 4′(b). From the assumption $S >_{\mathsf{ACRPO}}^{\mathsf{mul}} T$ we infer $S{\upharpoonright}_f^{<} >_{\mathsf{ACRPO}}^{\mathsf{mul}} T{\upharpoonright}_f^{<}$ and thus case 4(c) applies. □