Online appendix for the paper

A Denotational Semantics for Equilibrium Logic

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Appendix A

Proof of Proposition 1

If $v \in (S \cap S') \uparrow$, there exists some $u \in S \cap S'$, $u \leq v$. But then, $u \in S$ and $u \leq v$ implies $v \in S \uparrow$ and the same applies for $u \in S'$, $u \leq v$ implying $v \in S' \uparrow$. The proof for $\downarrow$ is completely analogous. $\square$

Proof of Corollary 1

For left to right, if $v \in S_c \downarrow$ then there exist some $u \in S_c$, $u \geq v$. But then u is classical and, by Proposition 2, $u = v_t$. For right to left, if $v_t \in S$ as v_t is classical, $v_t \in S_c$. Since $v \leq v_t$, we directly get $v \in S_c \downarrow$. $\square$

Proof of Proposition 3

(i) $\Rightarrow$ (ii). Suppose S is total-closed. This means that if we take any $v \in S$, then $v_t \in S$ and, since v_t is classical, $v_t \in S_c$. Moreover, since $v \leq v_t$ we conclude $v \in S_c \downarrow$.
(ii) $\Rightarrow$ (iii). Suppose $S \subseteq S_c \downarrow$ and, for the $\subseteq$ direction, take some $v \in S \uparrow_c$. The latter means that v is classical and there is some $u \in S$ such that $u \leq v$. Furthermore, by Proposition 2, $u_t = v$. Now, as $u \in S \subseteq S_c \downarrow$, by Corollary 1, $u_t (= v) \in S$ and, as u_t is classical, $u_t \in S_c$. For the $\supseteq$ direction, note that $S_c \subseteq S \subseteq S \uparrow$. But at the same time $S_c \subseteq \mathcal{I}_c$ and thus $S_c \subseteq (S \uparrow \cap \mathcal{I}_c) = S \uparrow_c$.
(iii) $\Rightarrow$ (i) Assume $S \uparrow_c = S_c$ and take any $v \in S$. We will prove that $v_t \in S$. As $v \in S$, it is clear that $\{v\} \uparrow \subseteq S \uparrow$ and so, $\{v\} \uparrow_c \subseteq S \uparrow_c$. By Proposition 2, $\{v\} \uparrow_c = \{v_t\}$ and so we get $\{v_t\} \subseteq S \uparrow_c = S_c$ that immediately implies $v_t \in S$, as we wanted to prove. $\square$

Proof of Lemma 1

Suppose $v \in (\overline{S})_c \downarrow$ but $v \in S_c \downarrow$. But now, since both $\overline{S}_c$ and S_c are sets of classical interpretations, by Corollary 1, we respectively get that $v_t \in \overline{S}_c (\subseteq \overline{S})$ and $v_t \in S_c (\subseteq S)$ which is an contradiction. $\square$

Proof of Theorem 1

By structural induction.

- If $\alpha = \bot$ then $v(\alpha) = 0$ and $[\![\bot]\!] = [\![\bot]\!]_c = [\![\bot]\!]_c = \emptyset$ and both equivalences (i) and (ii) become trivially true, as in each case, the two conditions are false.
- If α is some atom $p \in \Sigma$ then, (i) is true by definition of $[\![p]\!]$. For (ii), we have the following chain of equivalences: $v(p) \neq 0 \;\Leftrightarrow\; v_t(p) = 2 \;\Leftrightarrow\; v_t \in [\![p]\!]$.
- Let $\alpha = \varphi \vee \psi$. To prove (i) note that $v(\varphi \vee \psi) = 2$ iff either $v(\varphi) = 2$ or $v(\psi) = 2$. By induction, this is equivalent to $v \in [\![\varphi]\!]$ or $v \in [\![\psi]\!]$ which, in its turn, is equivalent to $v \in [\![\varphi]\!] \cup [\![\psi]\!] = [\![\varphi \vee \psi]\!]$. To prove (ii), $v(\varphi \vee \psi) \neq 0$ iff $v(\varphi) \neq 0$ or $v(\psi) \neq 0$. By induction $v_t \in [\![\varphi]\!]$ or $v_t \in [\![\psi]\!]$, that is $v_t \in [\![\varphi]\!] \cup [\![\psi]\!] = [\![\varphi \vee \psi]\!]$.
- Let $\alpha = \varphi \wedge \psi$. For proving (i), $v(\varphi \wedge \psi) = 2$ iff both $v(\varphi) = 2$ and $v(\psi) = 2$. By induction, this is equivalent to $v \in [\![\varphi]\!]$ and $v \in [\![\psi]\!]$, that is, $v \in [\![\varphi]\!] \cap [\![\psi]\!] = [\![\varphi \wedge \psi]\!]$. To prove (ii), $v(\varphi \wedge \psi) \neq 0$ iff both $v(\varphi) \neq 0$ and $v(\psi) \neq 0$. By induction, $v_t \in [\![\varphi]\!]$ and $v_t \in [\![\psi]\!]$, that is, $v \in [\![\varphi]\!]_c \downarrow \cap [\![\psi]\!] = [\![\varphi \wedge \psi]\!]$.
- Let $\alpha = \varphi \rightarrow \psi$. For proving (i), consider the condition $v \in [\![\varphi \rightarrow \psi]\!]$

$$\begin{aligned} & v \in (\overline{[\![\varphi]\!]} \cup [\![\psi]\!]) \\ \Leftrightarrow\; & v \notin [\![\varphi]\!] \text{ or } v \in [\![\psi]\!] \\ \Leftrightarrow\; & v(\varphi) \neq 2 \text{ or } v(\psi) = 2 \end{aligned}$$

  On the other hand, $v \in (\overline{[\![\varphi]\!]} \cup [\![\psi]\!])_c \downarrow$ iff $v_t \in (\overline{[\![\varphi]\!]} \cup [\![\psi]\!])$ and, using the reasoning above, this means:

$$\begin{aligned} & v_t(\varphi) \neq 2 \text{ or } v_t(\psi) = 2 \\ \Leftrightarrow\; & v(\varphi) = 0 \text{ or } v(\psi) \neq 0 \end{aligned}$$

Thus:

$$
\begin{array}{rl}
 & v \in \llbracket \varphi \to \psi \rrbracket \\
\Leftrightarrow & v \in (\overline{\llbracket \varphi \rrbracket} \cup \llbracket \psi \rrbracket) \cap (\overline{\llbracket \varphi \rrbracket} \cup \llbracket \psi \rrbracket)_c \downarrow \\
\Leftrightarrow & (v(\varphi) \neq 2 \text{ or } v(\psi) = 2) \text{ and } (v(\varphi) = 0 \text{ or } v(\psi) \neq 0) \\
\Leftrightarrow & v(\varphi) = 0 \\
 & \text{or } v(\varphi) \neq 2 \text{ and } v(\psi) \neq 0 \\
 & \text{or } v(\psi) = 2 \text{ and } v(\varphi) = 0 \\
 & \text{or } v(\psi) = 2 \\
\Leftrightarrow & v(\varphi) = 0 \\
 & \text{or } v(\varphi) \neq 2 \text{ and } v(\psi) \neq 0 \\
 & \text{or } v(\psi) = 2 \\
\Leftrightarrow & v(\varphi) \leq v(\psi) \\
\Leftrightarrow & v(\varphi \to \psi) = 2
\end{array}
$$

For proving (ii), note first that $v_t \in \llbracket \varphi \to \psi \rrbracket$ iff $v_t(\varphi \to \psi) = 2$ using the proof for (i) applied to v_t. As v_t is total, the latter is equivalent to $v_t(\varphi) = 0$ or $v_t(\psi) = 2$. This, in its turn, is equivalent to $v(\varphi) = 0$ or $v(\psi) \neq 0$. Finally, looking at the table for implication, this is the same than $v(\varphi \to \psi) \neq 0$.

□

Proof of Proposition 5

$$
\begin{array}{rcll}
\llbracket \alpha \to \beta \rrbracket & = & \big(\overline{\llbracket \alpha \rrbracket} \cup \llbracket \beta \rrbracket\big) \cap \big(\overline{\llbracket \alpha \rrbracket} \cup \llbracket \beta \rrbracket \big)_c \downarrow & \text{by definition} \\
 & = & \big(\overline{\llbracket \alpha \rrbracket} \cup \llbracket \beta \rrbracket\big) \cap \big(\overline{\llbracket \alpha \rrbracket}_c \cup \llbracket \beta \rrbracket_c \big) \downarrow & c/\cup\text{-distributivity} \\
 & = & \big(\overline{\llbracket \alpha \rrbracket} \cup \llbracket \beta \rrbracket\big) \cap \big(\overline{\llbracket \alpha \rrbracket}_c \downarrow \cup \llbracket \beta \rrbracket_c \downarrow \big) & \downarrow/\cup\text{-distributivity} \\
 & = & \big(\overline{\llbracket \alpha \rrbracket} \cap \overline{\llbracket \alpha \rrbracket}_c \downarrow \big) \cup \big(\overline{\llbracket \alpha \rrbracket} \cap \llbracket \beta \rrbracket_c \downarrow \big) & \\
 & & \cup\big(\llbracket \beta \rrbracket \cap \overline{\llbracket \alpha \rrbracket}_c \downarrow \big) \cup \big(\llbracket \beta \rrbracket \cap \llbracket \beta \rrbracket_c \downarrow \big) & \cap/\cup-\text{distributivity}
\end{array}
$$

Now, by Proposition 4, $\overline{\llbracket \alpha \rrbracket}_c \downarrow \subseteq \overline{\llbracket \alpha \rrbracket}$ and, by Proposition 3, $\llbracket \beta \rrbracket \subseteq \llbracket \beta \rrbracket_c \downarrow$, so we get:

$$
\begin{array}{rl}
= & \overline{\llbracket \alpha \rrbracket}_c \downarrow \cup \big(\overline{\llbracket \alpha \rrbracket} \cap \llbracket \beta \rrbracket_c \downarrow \big) \cup \big(\llbracket \beta \rrbracket \cap \overline{\llbracket \alpha \rrbracket}_c \downarrow \big) \cup \llbracket \beta \rrbracket \\
= & \overline{\llbracket \alpha \rrbracket}_c \downarrow \cup \big(\overline{\llbracket \alpha \rrbracket} \cap \llbracket \beta \rrbracket_c \downarrow \big) \cup \llbracket \beta \rrbracket
\end{array}
$$

□

Proof of Theorem 3

1. "$\subseteq$"

   From Proposition 5, we know that $\llbracket \alpha \rrbracket \subseteq \llbracket (\beta \to \alpha) \to \alpha \rrbracket$. It only rests to show that: $\llbracket \alpha \rrbracket \subseteq \llbracket (\alpha \to \beta) \to \beta \rrbracket$. Notice that:

   $$\llbracket \alpha \rrbracket = (\llbracket \alpha \rrbracket \cap \llbracket \beta \rrbracket) \cup (\llbracket \alpha \rrbracket \cap \overline{\llbracket \beta \rrbracket}).$$

   Now

   $$\llbracket \alpha \rrbracket \cap \llbracket \beta \rrbracket \subseteq \llbracket \beta \rrbracket \subseteq \llbracket (\alpha \to \beta) \to \beta \rrbracket$$

and

$$[\![\alpha]\!] \cap \overline{[\![\beta]\!]} \subseteq \overline{[\![\alpha \to \beta]\!]} \subseteq [\![(\alpha \to \beta) \to \beta]\!]$$

since

$$\overline{[\![\alpha \to \beta]\!]} = ([\![\alpha]\!] \cap \overline{[\![\beta]\!]}) \cup \overline{(\overline{[\![\alpha]\!]} \cup [\![\beta]\!])_c \downarrow}.$$

2. "$\supseteq$"
First of all, notice that $\mathcal{I} = [\![\alpha \to \beta]\!] \cup [\![\beta \to \alpha]\!]$ since:

$$\mathcal{I} = [\![\beta]\!] \cup \overline{[\![\beta]\!]} \subseteq [\![\alpha \to \beta]\!] \cup [\![\beta \to \alpha]\!]$$

Then, we have that:

$$\begin{aligned} & [\![(\alpha \to \beta) \to \beta]\!] \cap [\![(\beta \to \alpha) \to \alpha]\!] \\ = \; & ([\![(\alpha \to \beta) \to \beta]\!] \cap [\![\alpha \to \beta]\!] \cap [\![(\beta \to \alpha) \to \alpha]\!]) \\ & \cup([\![(\alpha \to \beta) \to \beta]\!] \cap [\![(\beta \to \alpha) \to \alpha]\!] \cap [\![\beta \to \alpha]\!]) \\ \subseteq \; & [\![\beta]\!] \cup [\![\alpha]\!] \end{aligned}$$

by using again Proposition 5.

□

Proof of Lemma 2
By structural induction. Let us call $P \stackrel{\text{def}}{=} \bigcup_{i=1}^{n} [\![p_i]\!]$. Obviously, $[\![\bot]\!] = \emptyset \subseteq P$ and $[\![p_j]\!] \subseteq P$ for any $j = 1, \ldots, n$. Then, if subformulas α, β satisfy the lemma, i.e. $[\![\alpha]\!] \subseteq P$ and $[\![\beta]\!] \subseteq P$, then their intersection and union too, i.e. $[\![\alpha \wedge \beta]\!] = [\![\alpha]\!] \cap [\![\beta]\!] \subseteq P$ and $[\![\alpha \vee \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!] \subseteq P$. □

Proof of Theorem 4
Suppose it is representable in that language. Take the interpretation $u(p_i) = 0$ for any atom $p_i \in \Sigma$. It is easy to see that $u \in [\![p_1 \to p_2]\!]$. However, $u \notin [\![p_i]\!]$ for all $p_i \in \Sigma$ and this contradicts Lemma 2. □

Proof of Lemma 3
We proceed by structural induction on δ.

1. $\delta = \bot$. It is straightforward since $[\![\bot]\!] = \emptyset$.
2. $\delta = p$. Any interpretation of the form $v = \underline{22\ldots}$ belongs to $[\![p]\!]$. Now, for any of those v, take u equal to v but for $u(q) = 1$. By definition of the order relation, $u < v$, whereas $u \in [\![p]\!]$ because $u(p) = 2$.
3. $\delta = q$. Analogous to the previous case.
4. $\delta = \alpha \vee \beta$. If $v = \underline{22\ldots}$ and $v \in [\![\alpha \vee \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!]$, then suppose that $v \in [\![\alpha]\!]$. By induction we deduce that there exists $u < v$ that coincides with v excepting for p, q and such that $u \in [\![\alpha]\!] \subseteq [\![\delta]\!]$. The same happens if $v \in [\![\beta]\!]$.
5. $\delta = \alpha \to \beta$. Suppose that $v = \underline{22\ldots}$, $v \in [\![\delta]\!]$. We know by Proposition 5, that $v \in \overline{[\![\alpha]\!]} \cup [\![\beta]\!]$. So first suppose that $v \in [\![\beta]\!]$. Since β is a subformula of δ, we know that there exists $u < v \in [\![\beta]\!] \subseteq [\![\delta]\!]$ such that u is equal to v excepting for p, q. In the other case, when $v \in \overline{[\![\alpha]\!]} \subseteq \overline{[\![\alpha]\!]}_c$, we can take $u = \underline{11\ldots}$ equal to v excepting for $u(p) = u(q) = 1$. We have that $u < v$ and so, $u \in \overline{[\![\alpha]\!]}_c \downarrow \subseteq [\![\delta]\!]$ which completes the proof.

□

Proof of Theorem 5

It is easy to see that elements of $[\![p_1 \wedge p_2]\!]$ are exactly those v of the form $v = \underline{22\ldots}$ and that this set is not empty, i.e., we have at least some v in that set. Suppose $p_1 \wedge p_2$ were representable in $\mathcal{L}_\Sigma\{\bot, \vee, \to\}$. Then, Lemma 3 would imply that there exists some $u \in [\![p_1 \wedge p_2]\!]$, such that $u < v$ and u coincides with v excepting for p_1, p_2. But then, either $u(p_1) \neq 2$ or $u(p_2) \neq 2$ and u could not be a model of $p_1 \wedge p_2$. □

Proof of Theorem 6

The fixpoint condition means that the only interpretation in $[\![\alpha]\!] \cap \{v\} \downarrow$ is v. This is the same than saying that the only interpretation that is both model of α and smaller than or equal to v is v itself. □

Proof of Theorem 7

It amounts to observe that a classical model v of α is not in equilibrium iff $v \in ([\![\alpha]\!] \setminus \mathcal{I}_c) \uparrow$. The former means that there is some $u < v$, $u \in [\![\alpha]\!]$. Since v is classical, any $u < v$ must be non-classical. Thus, v is not in equilibrium iff there is some $u < v$, $u \in ([\![\alpha]\!] \setminus \mathcal{I}_c)$. But then, this is equivalent to: $v \in ([\![\alpha]\!] \setminus \mathcal{I}_c) \uparrow$. □

Proof of Lemma 4

Note that $v \in [\![\alpha]\!]_c \subseteq [\![\alpha]\!]$ and that trivially $v \in [\![\gamma_v]\!]$ by definition of γ_v. Moreover, as v is classical, $v \in [\![\alpha \wedge \gamma_v]\!]_c$. To see that v is in equilibrium, note that $u < v$ should assign $u(p) = 1$ to some atom such that $v(p) = 2$. But then, $u \notin [\![\gamma_v]\!]$ and so it cannot be a model of $\alpha \wedge \gamma_v$ either. □

Proposition 7

For any $\alpha, \alpha', \beta, \beta'$:

$$
\begin{array}{rll}
\text{(i)} & [\![\alpha \to \beta]\!] \subseteq [\![\alpha \to \beta']\!] & \text{if} \quad [\![\beta]\!] \subseteq [\![\beta']\!] \\
\text{(ii)} & [\![\alpha \to \beta]\!] \subseteq [\![\alpha' \to \beta]\!] & \text{if} \quad [\![\alpha']\!] \subseteq [\![\alpha]\!]
\end{array}
$$

Proof

(i) is an immediate consequence of Proposition 5. As for (ii), we can also use that proposition and the fact that $\overline{[\![\alpha]\!]} \subseteq \overline{[\![\alpha']\!]}$, and so, $\overline{[\![\alpha]\!]}_c \downarrow \subseteq \overline{[\![\alpha']\!]}_c \downarrow$ too. □