Online appendix for the paper

Putting Logic-Based Distributed Systems on Stable Grounds

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General Remarks

Let $\mathcal{P}$ be a Dedalus program. Recall from Section 5.1.2 that $deduc_{\mathcal{P}} \subseteq \mathcal{P}$ is the subset of all (unmodified) deductive rules. The semantics of $deduc_{\mathcal{P}}$ is given by the stratified semantics. Although the semantics of $deduc_{\mathcal{P}}$ does not depend on the chosen syntactic stratification, for technical convenience in the proofs, we will fix an arbitrary syntactic stratification for $deduc_{\mathcal{P}}$. Whenever we refer to the stratum number of an *idb* relation, we implicitly use this fixed syntactic stratification. Stratum numbers start at 1.

Appendix A Run to Model: Proof Details

In the context of Section 5.2.2, we show that M is a model of $\mathcal{P}$ on input H. Let G abbreviate the ground program $ground_M(C, I)$, where $C = pure(\mathcal{P})$ and $I = decl(H)$. To show that M is a stable model, we have to show $M = N$ where $N = G(decl(H))$. The inclusions $M \subseteq N$ and $N \subseteq M$ are shown respectively in Sections A.1 and A.2. We use the notations of Section 5.2.2.

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A.1 Inclusion $M \subseteq N$

By definition,

$$M = decl(H) \cup \bigcup_{i \in \mathbb{N}} \mathrm{trans}_{\mathcal{R}}^{[i]}.$$

We immediately have $decl(H) \subseteq N$ by the semantics of G. Next, we define for uniformity the set $\mathrm{trans}_{\mathcal{R}}^{[-1]} = \emptyset$. We will show by induction on $i = -1, 0, 1, \ldots$, that $\mathrm{trans}_{\mathcal{R}}^{[i]} \subseteq N$. The base case ($i = -1$) is clear. For the induction hypothesis, let $i \geq 0$, and assume for all $j \in \{-1, 0, \ldots, i-1\}$ that $\mathrm{trans}_{\mathcal{R}}^{[j]} \subseteq N$. We show that $\mathrm{trans}_{\mathcal{R}}^{[i]} \subseteq N$. By definition,

$$\mathrm{trans}_{\mathcal{R}}^{[i]} = \mathrm{caus}_{\mathcal{R}}^{[i]} \cup \mathrm{fin}_{\mathcal{R}}^{[i]} \cup \mathrm{duc}_{\mathcal{R}}^{[i]} \cup \mathrm{snd}_{\mathcal{R}}^{[i]}.$$

We show inclusion of these four sets in N below. Auxiliary claims can be found in Section A.1.5.

A.1.1 Causality

We show that $\mathrm{caus}_{\mathcal{R}}^{[i]} \subseteq N$. Concretely, let $(x, s) \in \mathcal{N} \times \mathbb{N}$ such that $(x, s) \prec_{\mathcal{R}} (x_i, s_i)$. We show $\mathtt{before}(x, s, x_i, s_i) \in N$. We distinguish between the following cases.

Local edge Suppose $(x, s) \prec_{\mathcal{R}} (x_i, s_i)$ is a local edge, i.e., $x = x_i$ and $s_i = s + 1$. Because rule (7) is positive, the following ground rule is always in G:

$$\mathtt{before}(x, s, x, s+1) \leftarrow \mathtt{all}(x),\ \mathtt{tsucc}(s, s+1).$$

The body facts of this ground rule are in $decl(H) \subseteq N$; hence, the rule derives $\mathtt{before}(x, s, x, s+1) = \mathtt{before}(x, s, x_i, s_i) \in N$.

Message edge Suppose $(x, s) \prec_{\mathcal{R}} (x_i, s_i)$ is a message edge, i.e., there is an earlier transition $j < i$ with $j = glob_{\mathcal{R}}(x, s)$, in which x sends a message $\boldsymbol{f}$ to x_i such that $\alpha_{\mathcal{R}}(j, x_i, \boldsymbol{f}) = i$. Denote $\boldsymbol{f} = R(\bar{a})$. Because rules of the form (10) in $pure(\mathcal{P})$ are positive, the following ground rule is always in G:

$$\mathtt{before}(x, s, x_i, s_i) \leftarrow \mathtt{chosen}_R(x, s, x_i, s_i, \bar{a}).$$

We show $\mathtt{chosen}_R(x, s, x_i, s_i, \bar{a}) \in N$, so that $\mathtt{before}(x, s, x_i, s_i) \in N$, as desired. Since $j = glob_{\mathcal{R}}(x, s)$, we have $x_j = x$ and $s_j = s$. Also using $s_i = loc_{\mathcal{R}}(i)$, we have

$$\mathtt{chosen}_R(x, s, x_i, s_i, \bar{a}) \in \mathrm{snd}_{\mathcal{R}}^{[j]} \subseteq \mathrm{trans}_{\mathcal{R}}^{[j]}.$$

Lastly, we have $\mathrm{trans}_{\mathcal{R}}^{[j]} \subseteq N$ by applying the induction hypothesis.

Transitive edge Suppose $(x, s) \prec_{\mathcal{R}} (x_i, s_i)$ is not a local edge nor a message edge. Then we can choose a pair $(z, u) \in \mathcal{N} \times \mathbb{N}$ such that $(x, s) \prec_{\mathcal{R}} (z, u)$ and $(z, u) \prec_{\mathcal{R}} (x_i, s_i)$, but also such that $(z, u) \prec_{\mathcal{R}} (x_i, s_i)$ is a local edge or a message edge. Because rule (8) is positive, the following ground rule is always in G:

$$\mathtt{before}(x, s, x_i, s_i) \leftarrow \mathtt{before}(x, s, z, u),\ \mathtt{before}(z, u, x_i, s_i).$$

We now show that the body of this rule is in N, so that $\texttt{before}(x, s, x_i, s_i) \in N$, as desired. Denote $j = \mathit{glob}_{\mathcal{R}}(z, u)$. First, because $(x, s) \prec_{\mathcal{R}} (z, u)$, we have $\texttt{before}(x, s, z, u) \in \mathrm{caus}^{[j]}_{\mathcal{R}}$. Next, because $(z, u) \prec_{\mathcal{R}} (x_i, s_i)$, we have $j < i$ by Lemma 1. So, by applying the induction hypothesis to j, we have $\texttt{before}(x, s, z, u) \in N$. Secondly, because $(z, u) \prec_{\mathcal{R}} (x_i, s_i)$ is a local edge or a message edge, we have $\texttt{before}(z, u, x_i, s_i) \in N$ as shown in the preceding two cases.

A.1.2 Finite Messages

We show that $\mathrm{fin}^{[i]}_{\mathcal{R}} \subseteq N$. Let $\mathrm{senders}^{[i]}_{\mathcal{R}}$ be as defined in Section 5.2.2. For each of the different kinds of facts in $\mathrm{fin}^{[i]}_{\mathcal{R}}$, we show inclusion in N.

Senders Let $\texttt{hasSender}(x_i, s_i, x, s) \in \mathrm{fin}^{[i]}_{\mathcal{R}}$. We have $(x, s) \in \mathrm{senders}^{[i]}_{\mathcal{R}}$, which means that x during step s sends some message fact $R(\bar{a})$ that arrives in step s_i of x_i. Rules in $\mathit{pure}(\mathcal{P})$ of the form (11) have a negative $\texttt{rcvInf}$-atom in their body. But since we have not added any $\texttt{rcvInf}$-facts to M, including $\texttt{rcvInf}(x_i, s_i)$, the following rule is in G:

$$\texttt{hasSender}(x_i, s_i, x, s) \leftarrow \texttt{chosen}_R(x, s, x_i, s_i, \bar{a})\cdot$$

We are left to show that $\texttt{chosen}_R(x, s, x_i, s_i, \bar{a}) \in N$. Denote $j = \mathit{glob}_{\mathcal{R}}(x, s)$. Using that $x = x_j$ and $s = s_j$, we have $\texttt{chosen}_R(x, s, x_i, s_i, \bar{a}) \in \mathrm{snd}^{[j]}_{\mathcal{R}}$. Because $j < i$ by the operational semantics, we can apply the induction hypothesis to j to know $\mathrm{snd}^{[j]}_{\mathcal{R}} \subseteq N$.

Comparison of timestamps Let $\texttt{isSmaller}(x_i, s_i, x, s) \in \mathrm{fin}^{[i]}_{\mathcal{R}}$. We have $(x, s) \in \mathrm{senders}^{[i]}_{\mathcal{R}}$ and there is a timestamp $s' \in \mathbb{N}$ so that $(x, s') \in \mathrm{senders}^{[i]}_{\mathcal{R}}$ and $s < s'$. Rule (12) is positive and therefore the following ground rule is always in G:

$$\begin{aligned}\texttt{isSmaller}(x_i, s_i, x, s) \quad \leftarrow \quad & \texttt{hasSender}(x_i, s_i, x, s), \texttt{hasSender}(x_i, s_i, x, s'), \\ & s < s'\cdot\end{aligned}$$

We immediately have $(s < s') \in \mathit{decl}(H) \subseteq N$. By construction of $\mathrm{fin}^{[i]}_{\mathcal{R}}$, we also have $\texttt{hasSender}(x_i, s_i, x, s) \in \mathrm{fin}^{[i]}_{\mathcal{R}}$ and $\texttt{hasSender}(x_i, s_i, x, s') \in \mathrm{fin}^{[i]}_{\mathcal{R}}$, and thus both facts are also in N as shown above. Hence the previous ground rule derives $\texttt{isSmaller}(x_i, s_i, x, s) \in N$.

Maximum timestamp Let $\texttt{hasMax}(x_i, s_i, x) \in \mathrm{fin}^{[i]}_{\mathcal{R}}$. Thus x is a sender-node mentioned in $\mathrm{senders}^{[i]}_{\mathcal{R}}$. Let s be the maximum send-timestamp of x in $\mathrm{senders}^{[i]}_{\mathcal{R}}$, which surely exists because $\mathrm{senders}^{[i]}_{\mathcal{R}}$ is finite. We have not added $\texttt{isSmaller}(x_i, s_i, x, s)$ to $\mathrm{fin}^{[i]}_{\mathcal{R}}$, and thus also not to M. Although rule (13) contains a negated $\texttt{isSmaller}$-atom, $\texttt{isSmaller}(x_i, s_i, x, s) \notin M$ implies that the following ground rule is in G:

$$\texttt{hasMax}(x_i, s_i, x) \leftarrow \texttt{hasSender}(x_i, s_i, x, s)\cdot$$

Moreover, $(x, s) \in \mathrm{senders}^{[i]}_{\mathcal{R}}$ implies $\texttt{hasSender}(x_i, s_i, x, s) \in N$, and thus the previous ground rule derives $\texttt{hasMax}(x_i, s_i, x) \in N$, as desired.

A.1.3 Deductive

We show that $\mathrm{duc}_{\mathcal{R}}^{[i]} \subseteq N$. By definition, $\mathrm{duc}_{\mathcal{R}}^{[i]} = D_i^{\Uparrow x_i, s_i}$, where D_i is the output of subprogram $deduc_{\mathcal{P}}$ during transition i. Recall from Section 5.1.3 that $deduc_{\mathcal{P}}$ is given the following input during transition i:

$$st_i(x_i) \cup untag(m_i),$$

where st_i denotes the state at the beginning of transition i, and m_i is the set of (tagged) messages delivered during transition i. If we can show that $(st_i(x_i) \cup untag(m_i))^{\Uparrow x_i, s_i} \subseteq N$, then we can apply Claim 1 to know that $D_i^{\Uparrow x_i, s_i} \subseteq N$, as desired.

State We first show $st_i(x_i)^{\Uparrow x_i, s_i} \subseteq N$. There are two cases:

- Suppose $s_i = 0$, i.e., i is the first transition of $\mathcal{R}$ with active node x_i. Then $st_i(x_i) = H(x_i)$ by the operational semantics, which gives $st_i(x_i)^{\Uparrow x_i, s_i} \subseteq decl(H) \subseteq N$ by definition of $decl(H)$.
- Suppose $s_i > 0$. Then we can consider the last transition j of x_i that came before i. By the operational semantics, we have $st_i(x_i) = st_{j+1}(x_i)$, where st_{j+1} is the state resulting from transition j. More concretely, $st_i(x_i) = H(x_i) \cup induc_{\mathcal{P}}(D_j)$, with D_j the output of $deduc_{\mathcal{P}}$ during transition j. As in the previous case, we already know $H(x_i)^{\Uparrow x_i, s_i} \subseteq decl(H)$. Now, by applying the induction hypothesis to j, we have $\mathrm{duc}_{\mathcal{R}}^{[j]} \subseteq \mathrm{trans}_{\mathcal{R}}^{[j]} \subseteq N$. Next, by applying Claim 3, and by using $s_i = s_j + 1$, we obtain

$$\begin{aligned} st_i(x_i)^{\Uparrow x_i, s_i} &= H(x_i)^{\Uparrow x_i, s_i} \cup induc_{\mathcal{P}}(D_j)^{\Uparrow x_i, s_j+1} \\ &\subseteq N. \end{aligned}$$

Messages Now we show $untag(m_i)^{\Uparrow x_i, s_i} \subseteq N$. Let $\boldsymbol{f} \in untag(m_i)$. We have to show that $\boldsymbol{f}^{\Uparrow x_i, s_i} \in N$. First, because $\boldsymbol{f} \in untag(m_i)$, there is a transition k with $k < i$ such that $(k, \boldsymbol{f}) \in m_i$, i.e., the fact $\boldsymbol{f}$ was sent to x_i during transition k (by node x_k). Denote $\boldsymbol{f} = R(\bar{a})$. So, there must be an asynchronous rule with head-predicate R in $\mathcal{P}$, which has a corresponding rule in $pure(\mathcal{P})$ of the form (6). Rules of the form (6) are positive and thus the following ground rule is always in G:

$$R(x_i, s_i, \bar{a}) \leftarrow \mathtt{chosen}_R(x_k, s_k, x_i, s_i, \bar{a}).$$

We show $\mathtt{chosen}_R(x_k, s_k, x_i, s_i, \bar{a}) \in N$, so that the rule derives $\boldsymbol{f}^{\Uparrow x_i, s_i} \in N$, as desired. Because x_k sends $\boldsymbol{f}$ to x_i during transition k, and i is the transition in which this message is delivered to x_i, we have $\mathtt{chosen}_R(x_k, s_k, x_i, s_i, \bar{a}) \in \mathrm{snd}_{\mathcal{R}}^{[k]} \subseteq \mathrm{trans}_{\mathcal{R}}^{[k]}$. By applying the induction hypothesis to k, we have $\mathrm{snd}_{\mathcal{R}}^{[k]} \subseteq N$.

A.1.4 Sending

We show that $\mathrm{snd}_{\mathcal{R}}^{[i]} \subseteq N$. For each kind of fact in $\mathrm{snd}_{\mathcal{R}}^{[i]}$ we show inclusion in N.

Candidates Let $\mathtt{cand}_R(x_i, s_i, y, t, \bar{a}) \in \mathrm{snd}_{\mathcal{R}}^{[i]}$. We have $R(y, \bar{a}) \in \mathrm{mesg}_{\mathcal{R}}^{[i]}$, $t \in \mathbb{N}$ and $(y, t) \not\prec_{\mathcal{R}} (x_i, s_i)$. Since $D_i^{\Uparrow x_i, s_i} \subseteq N$ (see above), we can use Claim 4 to obtain $\mathtt{cand}_R(x_i, s_i, y, t, \bar{a}) \in N$, as desired.

Chosen Let $\texttt{chosen}_R(x_i, s_i, y, t, \bar{a}) \in \text{snd}^{[i]}_{\mathcal{R}}$. We have $R(y, \bar{a}) \in \text{mesg}^{[i]}_{\mathcal{R}}$ and $t = loc_{\mathcal{R}}(j)$ with $j = \alpha_{\mathcal{R}}(i, y, R(\bar{a}))$. Because $R(y, \bar{a}) \in \text{mesg}^{[i]}_{\mathcal{R}}$, this fact was produced by $async_{\mathcal{P}}$, and thus there is an asynchronous rule in $\mathcal{P}$ with head-predicate R. This asynchronous rule has a corresponding rule in $pure(\mathcal{P})$ of the form (4), that contains a negated $\texttt{other}_R$-atom in the body. But by construction of $\text{snd}^{[i]}_{\mathcal{R}}$, we have not added $\texttt{other}_R(x_i, s_i, y, t, \bar{a})$ to $\text{snd}^{[i]}_{\mathcal{R}}$, and thus also not to M. Therefore the following ground rule of the form (4) is in G:

$$\texttt{chosen}_R(x_i, s_i, y, t, \bar{a}) \leftarrow \texttt{cand}_R(x_i, s_i, y, t, \bar{a})\cdot$$

Because $j > i$ by the operational semantics, we have $(y, t) \not\prec_{\mathcal{R}} (x_i, s_i)$ by Lemma 1. Thus, by construction of $\text{snd}^{[i]}_{\mathcal{R}}$, we have $\texttt{cand}_R(x_i, s_i, y, t, \bar{a}) \in \text{snd}^{[i]}_{\mathcal{R}}$, in which case $\texttt{cand}_R(x_i, s_i, y, t, \bar{a}) \in N$ (shown above). Hence, the previous ground rule derives $\texttt{chosen}_R(x_i, s_i, y, t, \bar{a}) \in N$, as desired.

Other Let $R(y, \bar{a})$ and t be from above. Let $\texttt{other}_R(x_i, s_i, y, u, \bar{a}) \in \text{snd}^{[i]}_{\mathcal{R}}$. We have $u \in \mathbb{N}$, $(y, u) \not\prec_{\mathcal{R}} (x_i, s_i)$ and $u \neq t$. Because rule (5) is positive, the following ground rule is in G:

$$\begin{aligned}\texttt{other}_R(x_i, s_i, y, u, \bar{a}) \quad \leftarrow \quad & \texttt{cand}_R(x_i, s_i, y, u, \bar{a}), \texttt{chosen}_R(x_i, s_i, y, t, \bar{a}),\\ & u \neq t\cdot\end{aligned}$$

We immediately have $(u \neq t) \in decl(H) \subseteq N$. Now we show that the other body facts are in N, so the rule derives $\texttt{other}_R(x_i, s_i, y, u, \bar{a}) \in N$, as desired. Because $(y, u) \not\prec_{\mathcal{R}} (x_i, s_i)$, by construction of $\text{snd}^{[i]}_{\mathcal{R}}$, we have $\texttt{cand}_R(x_i, s_i, y, u, \bar{a}) \in \text{snd}^{[i]}_{\mathcal{R}}$ and thus $\texttt{cand}_R(x_i, s_i, y, u, \bar{a}) \in N$ (shown above). Moreover, it was shown above that $\texttt{chosen}_R(x_i, s_i, y, t, \bar{a}) \in N$.

A.1.5 Subclaims

Claim 1
Let i be a transition of $\mathcal{R}$. If $(st_i(x_i) \cup untag(m_i))^{\Uparrow x_i, s_i} \subseteq N$, then $D_i^{\Uparrow x_i, s_i} \subseteq N$.

Proof
Abbreviate $I_i = st_i(x_i) \cup untag(m_i)$. Recall that $D_i = deduc_{\mathcal{P}}(I_i)$, which is computed with the stratified semantics.

For $k \in \mathbb{N}$, we write $D_i^{\to k}$ to denote the set obtained by adding to I_i all facts derived in stratum 1 up to stratum k during the computation of D_i. For the largest stratum number n of $deduc_{\mathcal{P}}$, we have $D_i^{\to n} = D_i$. Also, because stratum numbers start at 1, we have $D_i^{\to 0} = I_i$. We show by induction on $k = 0, 1, 2, \ldots, n$, that $(D_i^{\to k})^{\Uparrow x_i, s_i} \subseteq N$.

Base case For the base case, $k = 0$, the property holds by the given assumption $I_i^{\Uparrow x_i, s_i} \subseteq N$.

Induction hypothesis For the induction hypothesis, assume for some stratum number k with $k \geq 1$ that $(D_i^{\to k-1})^{\Uparrow x_i, s_i} \subseteq N$.

Inductive step For the inductive step, we show that $(D_i^{\to k})^{\Uparrow x_i,s_i} \subseteq N$. Recall that the input of stratum k in $deduc_{\mathcal{P}}$ is the set $D_i^{\to k-1}$, and the semantics is given by the fixpoint semantics of semi-positive Datalog$^{\neg}$ (see Section 3.2.2). So, we can consider $D_i^{\to k}$ to be a fixpoint, i.e., as the set $\bigcup_{l\in\mathbb{N}} A_l$ with $A_0 = D_i^{\to k-1}$ and $A_l = T(A_{l-1})$ for each $l \geq 1$, where T is the immediate consequence operator of stratum k. We show by inner induction on $l = 0$, 1, etc, that

$$(A_l)^{\Uparrow x_i,s_i} \subseteq N.$$

For the base case ($l = 0$), we have $A_0 = D_i^{\to k-1}$, for which we can apply the outer induction hypothesis to know that $(D_i^{\to k-1})^{\Uparrow x_i,s_i} = (A_0)^{\Uparrow x_i,s_i} \subseteq N$, as desired. For the inner induction hypothesis, we assume for some $l \geq 1$ that $(A_{l-1})^{\Uparrow x_i,s_i} \subseteq N$. For the inner inductive step, we show that $(A_l)^{\Uparrow x_i,s_i} \subseteq N$. Let $\boldsymbol{f} \in A_l \setminus A_{l-1}$. Let $\varphi \in deduc_{\mathcal{P}}$ and V be a rule from stratum k and valuation respectively that have derived $\boldsymbol{f}$. Let φ' be the rule in $pure(\mathcal{P})$ obtained by applying the transformation (1) to φ. Let V' be V extended to assign x_i and s_i to the new variables in φ' that represent the location and timestamp respectively. Note in particular that $V'(pos_{\varphi'}) = V(pos_{\varphi})^{\Uparrow x_i,s_i}$ and $V'(neg_{\varphi'}) = V(neg_{\varphi})^{\Uparrow x_i,s_i}$. Let ψ be the *positive* ground rule obtained by applying V' to φ' and by subsequently removing all negative (ground) body atoms. We show that $\psi \in G$ and that its body is in N, so that ψ derives $head_{\psi} = \boldsymbol{f}^{\Uparrow x_i,s_i} \in N$, as desired.

- In order for ψ to be in G, it is required that $V'(neg_{\varphi'}) \cap M = \emptyset$. Because V is satisfying for φ, and negation in φ is only applied to lower strata, we have $V(neg_{\varphi}) \cap D_i^{\to k-1} = \emptyset$. Moreover, since a relation is computed in only one stratum of $deduc_{\mathcal{P}}$, we overall have $V(neg_{\varphi}) \cap D_i = \emptyset$. Then by Claim 2 we have $V(neg_{\varphi})^{\Uparrow x_i,s_i} \cap M = \emptyset$. Hence,

$$V'(neg_{\varphi'}) \cap M = \emptyset.$$

- Now we show that $pos_{\psi} \subseteq N$. Because V is satisfying for φ, we have $V(pos_{\varphi}) \subseteq A_{l-1}$, and by applying the inner induction hypothesis we have $V(pos_{\varphi})^{\Uparrow x_i,s_i} \subseteq N$. Therefore, $pos_{\psi} = V'(pos_{\varphi'}) \subseteq N$.

□

Claim 2
Let i be a transition of $\mathcal{R}$. Let I be a set of facts over $sch(\mathcal{P})$. If $I \cap D_i = \emptyset$ then $I^{\Uparrow x_i,s_i} \cap M = \emptyset$.

Proof
If a fact $\boldsymbol{f} \in M$ is over schema $sch(\mathcal{P})^{\mathrm{LT}}$ and has location specifier x_i and timestamp s_i then $\boldsymbol{f} \in \mathrm{duc}_{\mathcal{R}}^{[i]}$ because *(i)* for any transition j there are no facts over $sch(\mathcal{P})^{\mathrm{LT}}$ in $\mathrm{caus}_{\mathcal{R}}^{[j]}$, $\mathrm{fin}_{\mathcal{R}}^{[j]}$ or $\mathrm{snd}_{\mathcal{R}}^{[j]}$; *(ii)* we only add facts with location specifier x_i to $\mathrm{duc}_{\mathcal{R}}^{[j]}$ if j is a transition of node x_i; and, *(iii)* for every transition j of node x_i, if $i \neq j$ then $loc_{\mathcal{R}}(j) \neq s_i$.

Hence, it suffices to show $I^{\Uparrow x_i,s_i} \cap \mathrm{duc}_{\mathcal{R}}^{[i]} = \emptyset$. But this is immediate from $I \cap D_i = \emptyset$ because $\mathrm{duc}_{\mathcal{R}}^{[i]}$ equals $D_i^{\Uparrow x_i,s_i}$ by definition. □

Claim 3
Let j be a transition of $\mathcal{R}$. Let D_j be the output of $deduc_{\mathcal{P}}$ during transition j. Suppose $\mathrm{duc}_{\mathcal{R}}^{[j]} \subseteq N$. We have $induc_{\mathcal{P}}(D_j)^{\Uparrow x_j, s_j+1} \subseteq N$.

Proof
Let $\boldsymbol{f} \in induc_{\mathcal{P}}(D_j)$. Let $\varphi \in induc_{\mathcal{P}}$ and V respectively be a rule and valuation that have derived $\boldsymbol{f}$. Let φ' be the rule in $pure(\mathcal{P})$ that is obtained after applying transformation (2) to φ. Thus, besides the additional location variable, the rule φ' has two timestamp variables, one in the body and one in the head. Moreover, the body contains an additional positive `tsucc`-atom. Let V' be V extended to assign x_j to the location variable, and to assign timestamps s_j and $s_j + 1$ to the body and head timestamp variables respectively. Let ψ be the *positive* ground rule obtained from φ' by applying valuation V' and by subsequently removing all negative (ground) body atoms. We show that $\psi \in G$ and that its body is in N, so that ψ derives $head_{\psi} = \boldsymbol{f}^{\Uparrow x_j, s_j+1} \in N$, as desired.

- For ψ to be in G, we require $V'(neg_{\varphi'}) \cap M = \emptyset$. Since $V'(neg_{\varphi'}) = V(neg_{\varphi})^{\Uparrow x_j, s_j}$, it suffices to show $V(neg_{\varphi})^{\Uparrow x_j, s_j} \cap M = \emptyset$. Because V is satisfying for φ, we have $V(neg_{\varphi}) \cap D_j = \emptyset$. Then, by Claim 2 we have $V(neg_{\varphi})^{\Uparrow x_j, s_j} \cap M = \emptyset$.
- Now we show $V'(pos_{\varphi'}) \subseteq N$. The set $V'(pos_{\varphi'})$ consists of the facts $V(pos_{\varphi})^{\Uparrow x_j, s_j}$ and the fact $\mathtt{tsucc}(s_j, s_j + 1)$. The latter fact is in $decl(H)$ and thus in N. For the other facts, because V is satisfying for φ, we have $V(pos_{\varphi}) \subseteq D_j$ and thus $V(pos_{\varphi})^{\Uparrow x_j, s_j} \subseteq D_j^{\Uparrow x_j, s_j} = \mathrm{duc}_{\mathcal{R}}^{[j]}$. And by using the given assumption $\mathrm{duc}_{\mathcal{R}}^{[j]} \subseteq N$, we obtain the inclusion in N.

□

Claim 4
Let i be a transition of $\mathcal{R}$. Suppose $D_i^{\Uparrow x_i, s_i} \subseteq N$. For each $R(y, \bar{a}) \in \mathrm{mesg}_{\mathcal{R}}^{[i]}$ and timestamp $t \in \mathbb{N}$ with $(y, t) \not\prec_{\mathcal{R}} (x_i, s_i)$ we have

$$\mathtt{cand}_R(x_i, s_i, y, t, \bar{a}) \in N \cdot$$

Proof
By definition of $\mathrm{mesg}_{\mathcal{R}}^{[i]}$, we have $R(y, \bar{a}) \in async_{\mathcal{P}}(D_i)$. Let $\varphi \in async_{\mathcal{P}}$ and V be a rule and valuation that have produced $R(y, \bar{a})$. Let $\varphi' \in \mathcal{P}$ be the original asynchronous rule on which φ is based. Let $\varphi'' \in pure(\mathcal{P})$ be the rule obtained from φ' by applying transformation (9). Let V'' be valuation V extended to assign x_i and s_i to respectively the sender location and sender timestamp of φ'', and to assign y and t respectively to the addressee location and addressee arrival timestamp. Let ψ denote the *positive* ground rule that is obtained from φ'' by applying valuation V'' and by subsequently removing all negative (ground) body atoms. We show that $\psi \in G$ and that its body is in N, so that ψ derives $head_{\psi} = \mathtt{cand}_R(x_i, s_i, y, t, \bar{a}) \in N$, as desired.

- For ψ to be in G, we require $V''(neg_{\varphi''}) \cap M = \emptyset$. By construction of φ'', the set $V''(neg_{\varphi''})$ consists of the facts $V(neg_{\varphi})^{\Uparrow x_i, s_i}$ and the fact $\texttt{before}(y, t, x_i, s_i)$. First, because V is satisfying for φ, we have $V(neg_{\varphi}) \cap D_i = \emptyset$, and thus $V(neg_{\varphi})^{\Uparrow x_i, s_i} \cap M = \emptyset$ by Claim 2. Moreover, we are given that $(y, t) \not\prec_{\mathcal{R}} (x_i, s_i)$, and thus we have not added $\texttt{before}(y, t, x_i, s_i)$ to $\text{caus}^{[i]}_{\mathcal{R}}$, and by extension also not to M (since $\text{caus}^{[i]}_{\mathcal{R}}$ is the only part of M where we add before-facts with last two components x_i and s_i). Thus overall $V''(neg_{\varphi''}) \cap M = \emptyset$, as desired.
- Now we show $V''(pos_{\varphi''}) \subseteq N$. By construction of φ'', the set $V''(pos_{\varphi''})$ consists of the facts $V(pos_{\varphi})^{\Uparrow x_i, s_i}$, $\texttt{all}(y)$ and $\texttt{time}(t)$. First, we immediately have $\texttt{time}(t) \in decl(H) \subseteq N$. Also, by definition of $\text{mesg}^{[i]}_{\mathcal{R}}$, y is a valid addressee and thus $\texttt{all}(y) \in decl(H) \subseteq N$. Finally, because V is satisfying for φ, we have $V(pos_{\varphi}) \subseteq D_i$. Thus $V(pos_{\varphi})^{\Uparrow x_i, s_i} \subseteq D_i^{\Uparrow x_i, s_i}$, and we are given that $D_i^{\Uparrow x_i, s_i} \subseteq N$. Thus overall $V''(pos_{\varphi''}) \subseteq N$.

□

A.2 Inclusion $N \subseteq M$

In this section we show that $N \subseteq M$. By definition, $N = G(decl(H))$. Following the semantics of positive Datalog$^{\neg}$ programs in Section 3.2.1, we can view N as a fixpoint, i.e., $N = \bigcup_{l \in \mathbb{N}} N_l$, where $N_0 = decl(H)$, and for each $l \geq 1$ the set N_l is obtained by applying the immediate consequence operator of G to N_{l-1}. This implies $N_{l-1} \subseteq N_l$ for each $l \geq 1$. We show by induction on $l = 0, 1, \ldots$, that $N_l \subseteq M$. For the base case ($l = 0$), we immediately have $N_0 = decl(H) \subseteq M$. For the induction hypothesis, we assume for some $l \geq 1$ that $N_{l-1} \subseteq M$. For the inductive step, we show that $N_l \subseteq N$. Specifically, we divide the facts of $N_l \setminus N_{l-1}$ into groups based on their predicate, and for each group we show inclusion in M. As for terminology, we call a ground rule $\psi \in G$ *active* on N_{l-1} if $pos_{\psi} \subseteq N_{l-1}$. The numbered claims we will refer to can be found in Section A.2.5.

A.2.1 Causality

Let $\texttt{before}(x, s, y, t) \in N_l \setminus N_{l-1}$. It is sufficient to show that $(x, s) \prec_{\mathcal{R}} (y, t)$ because then $\texttt{before}(x, s, y, t) \in \text{caus}^{[i]}_{\mathcal{R}} \subseteq M$ where $i = glob_{\mathcal{R}}(y, t)$. We have the following cases:

Local edge The before-fact was derived by a ground rule in G of the form (7) (local edge). This implies $x = y$ and $t = s + 1$. Then $(x, s) \prec_{\mathcal{R}} (y, t)$ by definition of $\prec_{\mathcal{R}}$.

Message edge The before-fact was derived by a ground rule in G of the form (10) (message edge):

$$\texttt{before}(x, s, y, t) \leftarrow \texttt{chosen}_R(x, s, y, t, \bar{a}) \cdot$$

Since this rule is active on N_{l-1}, we have $\texttt{chosen}_R(x, s, y, t, \bar{a}) \in N_{l-1}$. By applying the induction hypothesis, we have $\texttt{chosen}_R(x, s, y, t, \bar{a}) \in M$. Denoting

$j = glob_{\mathcal{R}}(x, s)$, the set $\text{snd}_{\mathcal{R}}^{[j]}$ is the only part of M where we could have added this fact. This implies that x during its step s sends a message to y, and this message arrives at local step t of y. Then $(x, s) \prec_{\mathcal{R}} (y, t)$ by definition of $\prec_{\mathcal{R}}$.

Transitive edge The `before`-fact was derived by a ground rule in G of the form (8) (transitive edge):

$$\texttt{before}(x, s, y, t) \leftarrow \texttt{before}(x, s, z, u), \texttt{before}(z, u, y, t)\cdot$$

Since this rule is active on N_{l-1}, its body facts are in N_{l-1}. By applying the induction hypothesis, we have $\texttt{before}(x, s, z, u) \in M$ and $\texttt{before}(z, u, y, t) \in M$. The only places we could have added these facts to M are in the sets $\text{caus}_{\mathcal{R}}^{[j]}$ and $\text{caus}_{\mathcal{R}}^{[k]}$ respectively, where $j = glob_{\mathcal{R}}(z, u)$ and $k = glob_{\mathcal{R}}(y, t)$. By construction of the sets $\text{caus}_{\mathcal{R}}^{[j]}$ and $\text{caus}_{\mathcal{R}}^{[k]}$ we respectively have that $(x, s) \prec_{\mathcal{R}} (z, u)$ and $(z, u) \prec_{\mathcal{R}} (y, t)$, and thus by transitivity $(x, s) \prec_{\mathcal{R}} (y, t)$, as desired.

A.2.2 Finite Messages

Senders Let $\texttt{hasSender}(x, s, y, t) \in N_l \setminus N_{l-1}$. This fact can only have been derived by a ground rule in G of the form (11):

$$\texttt{hasSender}(x, s, y, t) \leftarrow \texttt{chosen}_R(y, t, x, s, \bar{a})\cdot$$

Since this rule is active on N_{l-1}, we have $\texttt{chosen}_R(y, t, x, s, \bar{a}) \in N_{l-1}$. By applying the induction hypothesis, we have $\texttt{chosen}_R(y, t, x, s, \bar{a}) \in M$. We can only have added this fact in the set $\text{snd}_{\mathcal{R}}^{[i]}$ with $i = glob_{\mathcal{R}}(y, t)$. This means that y during its step t sends a message $R(\bar{a})$ to x, and this message arrives during step s of x. Hence, denoting $j = glob_{\mathcal{R}}(x, s)$, we have $(y, t) \in \text{senders}_{\mathcal{R}}^{[j]}$ (with $\text{senders}_{\mathcal{R}}^{[j]}$ as defined in Section 5.2.2). Thus we have added the fact $\texttt{hasSender}(x, s, y, t) \in \text{fin}_{\mathcal{R}}^{[j]} \subseteq M$, as desired.

Comparison of timestamps Let $\texttt{isSmaller}(x, s, y, t) \in N_l \setminus N_{l-1}$. This fact can only have been derived by a ground rule in G of the form (12):

$$\texttt{isSmaller}(x, s, y, t) \quad \leftarrow \quad \texttt{hasSender}(x, s, y, t), \texttt{hasSender}(x, s, y, t'), \\ t < t'\cdot$$

Since this rule is active on N_{l-1}, its body facts are in N_{l-1}. By applying the induction hypothesis, we have $\texttt{hasSender}(x, s, y, t) \in M$ and $\texttt{hasSender}(x, s, y, t') \in M$. The only part of M where we could have added these facts is the set $\text{fin}_{\mathcal{R}}^{[i]}$ with $i = glob_{\mathcal{R}}(x, s)$. By construction of the set $\text{fin}_{\mathcal{R}}^{[i]}$, this implies that $(y, t) \in \text{senders}_{\mathcal{R}}^{[i]}$ and $(y, t') \in \text{senders}_{\mathcal{R}}^{[i]}$. Because $(t < t') \in N_{l-1}$, we more specifically know that $(t < t') \in decl(H)$, which implies $t < t'$. Thus we have added $\texttt{isSmaller}(x, s, y, t) \in \text{fin}_{\mathcal{R}}^{[i]}$, as desired.

Maximum timestamp Let $\texttt{hasMax}(x, s, y) \in N_l \setminus N_{l-1}$. This fact can only have been derived by a ground rule in G of the form (13):

$$\texttt{hasMax}(x, s, y) \leftarrow \texttt{hasSender}(x, s, y, t)\cdot$$

Since this rule is active on N_{l-1}, we have $\texttt{hasSender}(x,s,y,t) \in N_{l-1}$. By applying the induction hypothesis, we have $\texttt{hasSender}(x,s,y,t) \in M$. The only part of M where we could have added this fact, is the set $\mathrm{fin}_{\mathcal{R}}^{[i]}$ with $i = glob_{\mathcal{R}}(x,s)$. Thus $(y,t) \in \mathrm{senders}_{\mathcal{R}}^{[i]}$, and y is a sender-node mentioned in $\mathrm{senders}_{\mathcal{R}}^{[i]}$. Hence, we have added $\texttt{hasMax}(x,s,y) \in \mathrm{fin}_{\mathcal{R}}^{[i]} \subseteq M$, as desired.

Receive infinite Let $\texttt{rcvInf}(x,s) \in N_l \setminus N_{l-1}$. This fact can only have been derived by a ground rule in G of the form (14):

$$\texttt{rcvInf}(x,s) \leftarrow \texttt{hasSender}(x,s,y,t)\cdot$$

Since this rule is active on N_{l-1}, we have $\texttt{hasSender}(x,s,y,t) \in N_{l-1}$. By applying the induction hypothesis, we have $\texttt{hasSender}(x,s,y,t) \in M$. The only part of M where we could have added this fact, is the set $\mathrm{fin}_{\mathcal{R}}^{[i]}$ with $i = glob_{\mathcal{R}}(x,s)$. Thus $(y,t) \in \mathrm{senders}_{\mathcal{R}}^{[i]}$. Moreover, because the rule (14) contains a negative $\texttt{hasMax}$-atom in the body, and the above ground rule is in G, it must be that $\texttt{hasMax}(x,s,y) \notin M$, and thus $\texttt{hasMax}(x,s,y) \notin \mathrm{fin}_{\mathcal{R}}^{[i]}$. But since y is a sender-node mentioned in $\mathrm{senders}_{\mathcal{R}}^{[i]}$, the absence of $\texttt{hasMax}(x,s,y)$ from $\mathrm{fin}_{\mathcal{R}}^{[i]}$ is impossible. Therefore this case can not occur.

A.2.3 Regular Facts

Let $R(x,s,\bar{a}) \in (N_l \setminus N_{l-1})|_{sch(\mathcal{P})^{\mathrm{LT}}}$. The fact $R(x,s,\bar{a})$ has been derived by a ground rule $\psi \in G$ that is active on N_{l-1}. Because $\psi \in G$, there is a rule $\varphi \in pure(\mathcal{P})$ and valuation V such that ψ is obtained from φ by applying V and by subsequently removing the negative (ground) body atoms, and such that $V(neg_{\varphi}) \cap M = \emptyset$. We have the following cases:

Deductive Rule φ is of the form (1). Let $\varphi' \in deduc_{\mathcal{P}}$ be the original deductive rule corresponding to φ. By construction of φ out of φ', we can apply valuation V to φ' as well. Denote $i = glob_{\mathcal{R}}(x,s)$. We will show now that V is satisfying for φ' during transition i, which causes $V(head_{\varphi'}) = R(\bar{a}) \in D_i$ to be derived, and we obtain as desired:

$$R(x,s,\bar{a}) \in D_i^{\Uparrow x,s} = D_i^{\Uparrow x_i,s_i} = \mathrm{duc}_{\mathcal{R}}^{[i]} \subseteq M\cdot$$

By definition of syntactic stratification, relations mentioned in $pos_{\varphi'}$ are never computed in a stratum higher than R, and relations mentioned in $neg_{\varphi'}$ are computed in a strictly lower stratum than R. Thus, it is sufficient to show that $V(pos_{\varphi'}) \subseteq D_i$ and $V(neg_{\varphi'}) \cap D_i = \emptyset$.

First we show $V(pos_{\varphi'}) \subseteq D_i$. Because φ is of the form (1), all facts in $V(pos_{\varphi})$ are over $sch(\mathcal{P})^{\mathrm{LT}}$ and have location specifier x and timestamp s. Moreover, since ψ is active on N_{l-1}, we have $pos_{\psi} = V(pos_{\varphi}) \subseteq N_{l-1}$. By applying the induction hypothesis, we have $V(pos_{\varphi}) \subseteq M$, and thus $V(pos_{\varphi})^{\Downarrow} \subseteq D_i$ by Claim 5. We thus obtain $V(pos_{\varphi'}) \subseteq D_i$ since $V(pos_{\varphi})^{\Downarrow} = V(pos_{\varphi'})$.

Next we show $V(neg_{\varphi'}) \cap D_i = \emptyset$. Because φ is of the form (1), all facts in $V(neg_{\varphi})$ are over $sch(\mathcal{P})^{\mathrm{LT}}$ and have location specifier x and timestamp s. Moreover, by choice of φ and V, we have $V(neg_\varphi) \cap M = \emptyset$, and thus $V(neg_\varphi)^{\Downarrow} \cap D_i = \emptyset$ by Claim 6. We thus obtain $V(neg_{\varphi'}) \cap D_i = \emptyset$ since $V(neg_\varphi)^{\Downarrow} = V(neg_{\varphi'})$.

Inductive Rule φ is of the form (2). Let $\varphi' \in induc_{\mathcal{P}}$ be the rule corresponding to φ. First, ψ contains in its body a fact of the form $\texttt{tsucc}(r, s)$. Since ψ is active on N_{l-1}, we have $\texttt{tsucc}(r, s) \in N_{l-1}$ and more specifically, $\texttt{tsucc}(r, s) \in decl(H)$. This implies that $s = r + 1$. Denote $i = glob_{\mathcal{R}}(x, r)$ and $j = glob_{\mathcal{R}}(x, s)$. Since $s = r + 1$, there are no transitions of node x between i and j. By the relationship between φ and φ', we can apply V to φ', and we will now show that V is satisfying for φ' during transition i. This results in $V(head_{\varphi'}) = R(\bar{a}) \in induc_{\mathcal{P}}(D_i) \subseteq st_{i+1}(x)$, and since $st_{i+1}(x) = st_j(x) \subseteq D_j$, we obtain $R(x, s, \bar{a}) \in D_j^{\Uparrow x,s} = \text{duc}_{\mathcal{R}}^{[j]} \subseteq M$, as desired.

First we show $V(pos_{\varphi'}) \subseteq D_i$. Denote $I = V(pos_\varphi)|_{sch(\mathcal{P})^{\text{LT}}}$, which allows us to exclude the extra $\texttt{tsucc}$-fact in the body. All facts in I have location specifier x and timestamp r. Because ψ is active on N_{l-1}, we have $I \subseteq pos_\psi \subseteq N_{l-1}$, and by applying the induction hypothesis, we have $I \subseteq M$. Thus $I^{\Downarrow} \subseteq D_i$ by Claim 5. Hence, $V(pos_{\varphi'}) = I^{\Downarrow} \subseteq D_i$.

Secondly, showing that $V(neg_{\varphi'}) \cap D_i = \emptyset$ is like in the previous case, where φ is deductive.

Delivery Rule φ is of the form (6). Then ψ concretely looks as follows, where $(y, t) \in \mathcal{N} \times \mathbb{N}$:

$$R(x, s, \bar{a}) \leftarrow \texttt{chosen}_R(y, t, x, s, \bar{a})\cdot$$

Since ψ is active on N_{l-1}, we have $\texttt{chosen}_R(y, t, x, s, \bar{a}) \in N_{l-1}$, and by applying the induction hypothesis, we have $\texttt{chosen}_R(y, t, x, s, \bar{a}) \in M$. The only part of M where we could have added this fact, is $\text{snd}_{\mathcal{R}}^{[i]}$ with $i = glob_{\mathcal{R}}(y, t)$. This implies that x will receive $R(\bar{a})$ during its local step s, thus during transition $j = glob_{\mathcal{R}}(x, s)$. Then, by the operational semantics, we have $R(\bar{a}) \in untag(m_j) \subseteq D_j$. Hence, $R(x, s, \bar{a}) \in D_j^{\Uparrow x,s} = \text{duc}_{\mathcal{R}}^{[j]} \subseteq M$.

A.2.4 Sending

For a transition i of $\mathcal{R}$, let D_i denote the output of subprogram $deduc_{\mathcal{P}}$ during transition i.

Candidates Let $\texttt{cand}_R(x, s, y, t, \bar{a}) \in N_l \setminus N_{l-1}$. The fact $\texttt{cand}_R(x, s, y, t, \bar{a})$ is derived by a ground rule $\psi \in G$ of the form (9) that is active on N_{l-1}. Because $\psi \in G$, there is a rule $\varphi \in pure(\mathcal{P})$ and a valuation V such that ψ is obtained from φ by applying valuation V and by subsequently removing the negative (ground) body atoms, and so that $V(neg_\varphi) \cap M = \emptyset$. Denote $i = glob_{\mathcal{R}}(x, s)$. It is sufficient to show that $R(y, \bar{a}) \in \text{mesg}_{\mathcal{R}}^{[i]}$ and $(y, t) \not\prec_{\mathcal{R}} (x, s)$, because then $\texttt{cand}_R(x, s, y, t, \bar{a}) \in \text{snd}_{\mathcal{R}}^{[i]} \subseteq M$, as desired.

First, we show $(y, t) \not\prec_{\mathcal{R}} (x, s)$. Because there is a negative $\texttt{before}$-atom in φ, the existence of ψ in G implies that $\mathtt{before}(y,t,x,s)\notin M$. Hence, $\mathtt{before}(y,t,x,s)\notin \mathrm{caus}^{[i]}_{\mathcal{R}}$. Then by construction of $\mathrm{caus}^{[i]}_{\mathcal{R}}$ we obtain $(y,t)\not\prec_{\mathcal{R}}(x,s)$.

Secondly, we show $R(y,\bar{a})\in \mathrm{mesg}^{[i]}_{\mathcal{R}}$. Let $\varphi'\in\mathcal{P}$ be the original asynchronous rule on which φ is based. Let $\varphi''\in \mathit{async}_{\mathcal{P}}$ be the rule corresponding to φ'. It follows from the constructions of φ out of φ' and φ'' out of φ' that valuation V can be applied to φ''. Note, $V(\mathit{head}_{\varphi''})=R(y,\bar{a})$. We show that V is satisfying for φ'' during transition i on D_i, which gives $R(y,\bar{a})\in \mathit{async}_{\mathcal{P}}(D_i)$. Moreover, the body of ψ contains the fact $\mathtt{all}(y)\in \mathit{decl}(H)$, and thus $y\in\mathcal{N}$, making y a valid addressee. Hence, $R(y,\bar{a})\in \mathrm{mesg}^{[i]}_{\mathcal{R}}$, as desired.

We have to show $V(\mathit{pos}_{\varphi''})\subseteq D_i$ and $V(\mathit{neg}_{\varphi''})\cap D_i=\emptyset$. Abbreviate $I_1 = V(\mathit{pos}_{\varphi})|_{\mathit{sch}(\mathcal{P})^{\mathrm{LT}}}$ and $I_2 = V(\mathit{neg}_{\varphi})|_{\mathit{sch}(\mathcal{P})^{\mathrm{LT}}}$. Note, $I_1^{\Downarrow}=V(\mathit{pos}_{\varphi''})$ and $I_2^{\Downarrow}=V(\mathit{neg}_{\varphi''})$. All facts in $I_1\cup I_2$ have location specifier x and timestamp s.

- Because ψ is active on N_{l-1}, we have $I_1\subseteq \mathit{pos}_{\psi}\subseteq N_{l-1}$, and thus $I_1\subseteq M$ by the induction hypothesis. Then $V(\mathit{pos}_{\varphi''})=I_1^{\Downarrow}\subseteq D_i$ by Claim 5.
- By choice of φ and V, we have $I_2\cap M=\emptyset$. Then $I_2^{\Downarrow}\cap D_i=\emptyset$ by Claim 6, giving $V(\mathit{neg}_{\varphi''})\cap D_i=\emptyset$.

Chosen Let $\mathtt{chosen}_R(x,s,y,t,\bar{a})\in N_l\setminus N_{l-1}$. This fact is derived by a ground rule ψ in G of the form (4):

$$\mathtt{chosen}_R(x,s,y,t,\bar{a})\leftarrow \mathtt{cand}_R(x,s,y,t,\bar{a})\cdot$$

Denote $i=\mathit{glob}_{\mathcal{R}}(x,s)$. We show that $R(y,\bar{a})\in \mathrm{mesg}^{[i]}_{\mathcal{R}}$ and that t is the actual arrival timestamp of this message at y. Then $\mathtt{chosen}_R(x,s,y,t,\bar{a})\in \mathrm{snd}^{[i]}_{\mathcal{R}}\subseteq M$, as desired.

First, since ψ is active on N_{l-1}, we have $\mathtt{cand}_R(x,s,y,t,\bar{a})\in N_{l-1}$, and thus $\mathtt{cand}_R(x,s,y,t,\bar{a})\in M$ by the induction hypothesis. The set $\mathrm{snd}^{[i]}_{\mathcal{R}}$ is the only part of M where we could have added this fact, which implies $R(y,\bar{a})\in \mathrm{mesg}^{[i]}_{\mathcal{R}}$ and $(y,t)\not\prec_{\mathcal{R}}(x,s)$.

We are left to show that t is the actual arrival timestamp of the message. Because $\psi\in G$, there is a rule $\varphi\in \mathit{pure}(\mathcal{P})$ and valuation V such that ψ is obtained from φ by applying V and by subsequently removing the negative (ground) body atoms, and so that $V(\mathit{neg}_{\varphi})\cap M=\emptyset$. Now, because rule φ contains a negative $\mathtt{other}_R$-atom in its body, we have $\mathtt{other}_R(x,s,y,t,\bar{a})\notin M$ and thus $\mathtt{other}_R(x,s,y,t,\bar{a})\notin \mathrm{snd}^{[i]}_{\mathcal{R}}$. Since $R(y,\bar{a})\in \mathrm{mesg}^{[i]}_{\mathcal{R}}$ and $(y,t)\not\prec_{\mathcal{R}}(x,s)$ (see above), the absence of this $\mathtt{other}_R$-fact from $\mathrm{snd}^{[i]}_{\mathcal{R}}$ can only be explained by the following: $t=\mathit{loc}_{\mathcal{R}}(j)$ with $j=\alpha_{\mathcal{R}}(i,y,R(\bar{a}))$, as desired.

Other Let $\mathtt{other}_R(x,s,y,t,\bar{a})\in N_l\setminus N_{l-1}$. This fact is derived by a ground rule ψ of the form (5):

$$\begin{aligned}\mathtt{other}_R(x,s,y,t,\bar{a}) \quad\leftarrow\quad & \mathtt{cand}_R(x,s,y,t,\bar{a}),\ \mathtt{chosen}_R(x,s,y,t',\bar{a}),\\ & t\neq t'\cdot\end{aligned}$$

We have $\mathtt{cand}_R(x,s,y,t,\bar{a})\in N_{l-1}$ and $\mathtt{chosen}_R(x,s,y,t',\bar{a})\in N_{l-1}$ since ψ is active on N_{l-1}, and these facts are thus also in M by the induction hypothesis.

Denote $i = glob_{\mathcal{R}}(x, s)$. The only part of M where we could have added these $\mathtt{cand}_R$- and $\mathtt{chosen}_R$-facts to M, is the set $\mathrm{snd}_{\mathcal{R}}^{[i]}$. First, $\mathtt{cand}_R(x, s, y, t, \bar{a}) \in \mathrm{snd}_{\mathcal{R}}^{[i]}$ implies that $R(y, \bar{a}) \in \mathrm{mesg}_{\mathcal{R}}^{[i]}$ and $(y, t) \not\prec_{\mathcal{R}} (x, s)$. Second, $\mathtt{chosen}_R(x, s, y, t', \bar{a}) \in \mathrm{snd}_{\mathcal{R}}^{[i]}$ implies that t' is the real arrival timestamp of the message $R(\bar{a})$ at y. Finally, since ψ is active, we have $(t \neq t') \in decl(H)$, and thus $t \neq t'$. Therefore we have added $\mathtt{other}_R(x, s, y, t, \bar{a})$ to $\mathrm{snd}_{\mathcal{R}}^{[i]} \subseteq M$, as desired.

A.2.5 Subclaims

Claim 5

Let I be a set of facts over $sch(\mathcal{P})^{\mathrm{LT}}$, all having the same location specifier $x \in \mathcal{N}$ and timestamp $s \in \mathbb{N}$. Denote $i = glob_{\mathcal{R}}(x, s)$. If $I \subseteq M$ then $I^{\Downarrow} \subseteq D_i$, where D_i denotes the output of subprogram $deduc_{\mathcal{P}}$ during transition i of $\mathcal{R}$.

Proof

The only part of M where we add facts over $sch(\mathcal{P})^{\mathrm{LT}}$ with location specifier x and timestamp s is $\mathrm{duc}_{\mathcal{R}}^{[i]}$. Hence $I \subseteq \mathrm{duc}_{\mathcal{R}}^{[i]} = D_i^{\Uparrow x,s}$ and thus $I^{\Downarrow} \subseteq D_i$. □

Claim 6

Let I be a set of facts over $sch(\mathcal{P})^{\mathrm{LT}}$, all having the same location specifier $x \in \mathcal{N}$ and timestamp $s \in \mathbb{N}$. Denote $i = glob_{\mathcal{R}}(x, s)$. If $I \cap M = \emptyset$ then $I^{\Downarrow} \cap D_i = \emptyset$, where D_i denotes the output of subprogram $deduc_{\mathcal{P}}$ during transition i of $\mathcal{R}$.

Proof

First, $I \cap M = \emptyset$ implies $I \cap \mathrm{duc}_{\mathcal{R}}^{[i]} = \emptyset$ because $\mathrm{duc}_{\mathcal{R}}^{[i]} \subseteq M$. And since $\mathrm{duc}_{\mathcal{R}}^{[i]} = D_i^{\Uparrow x,s}$, we have $I \cap D_i^{\Uparrow x,s} = \emptyset$. Finally, since the facts in $I \cup D_i^{\Uparrow x,s}$ all have the same location specifier x and timestamp s, we obtain $I^{\Downarrow} \cap D_i = \emptyset$. □

Appendix B Model to Run: Proof Details

Consider the definitions and notations from Section 5.3. In this section we show that $\mathcal{R}$ is a run of $\mathcal{P}$ on input H, and that $trace(\mathcal{R}) = M|_{sch(\mathcal{P})^{\mathrm{LT}}}$. We do this in several parts, where each part is placed in its own subsection:

- in Section B.2 we show $\rho_0 = start(\mathcal{P}, H)$;
- in Section B.3 we show that every transition of $\mathcal{R}$ is valid; and,
- in Section B.4 we show $trace(\mathcal{R}) = M|_{sch(\mathcal{P})^{\mathrm{LT}}}$.

Before we start, the next subsection gives definitions and notations. The numbered claims we will refer to can be found in Section B.5.

B.1 Definitions and Notations

Using notations of Section 3.2.3, let G be the ground program $ground_M(C, I)$ where $C = pure(\mathcal{P})$ and $I = decl(H)$. By definition of M as a stable model, we have $M = G(I)$.

Let $\varphi \in pure(\mathcal{P})$ be a rule having its head atom over $sch(\mathcal{P})^{\mathrm{LT}}$. From the construction of $pure(\mathcal{P})$, we know that φ belongs to exactly one of the following three cases:

- φ is of the form (1), i.e., *deductive*, recognizable as a rule in which only atoms over $sch(\mathcal{P})^{\mathrm{LT}}$ are used, and in which the location and timestamp variable in the head are the same as in the body;
- φ is of the form (2), i.e., *inductive*, recognizable as a rule with a head atom over $sch(\mathcal{P})^{\mathrm{LT}}$ and a `tsucc`-atom in the body;
- φ is of the form (6), i.e., a *delivery*, recognizable as a rule with a head atom over $sch(\mathcal{P})^{\mathrm{LT}}$ and a $\texttt{chosen}_R$-fact in the body (with R the head-predicate).

The same classification of deductive, inductive and delivery rules can also be applied to the (positive) ground rules in G that have a ground head atom over $sch(\mathcal{P})^{\mathrm{LT}}$.

Recall from the general remarks at the beginning of the appendix that we are working with a fixed (but arbitrary) syntactic stratification for the deductive rules. Stratum numbers start at 1. If $\varphi \in pure(\mathcal{P})$ is deductive, we can uniquely identify its stratum number as the stratum number of the original deductive rule in $\mathcal{P}$ on which φ is based. Similarly, for deductive ground rules, we can also uniquely identify the stratum number as the stratum number of a corresponding non-ground rule in $pure(\mathcal{P})$.[1]

We call a ground rule $\psi \in G$ *active* if $pos_\psi \subseteq M$, which implies that $head_\psi \in M$ because M is stable. Now we define the following subsets of M:

- $M^{\mathrm{duc},k}$: the head facts of all active deductive rules in G with stratum number less than or equal to k;
- M^{ind}: the head facts of all active inductive rules in G;
- M^{deliv}: the head facts of all active delivery rules in G.

This allows us to classify the facts in $M|_{sch(\mathcal{P})^{\mathrm{LT}}}$ as being derived in a deductive manner, an inductive manner or being message deliveries. We also define:

$$M^{\blacktriangle} = M|_{edb(\mathcal{P})^{\mathrm{LT}}} \cup M^{\mathrm{ind}} \cup M^{\mathrm{deliv}}.$$

For $(x, s) \in \mathcal{N} \times \mathbb{N}$, we write $I|^{x,s}$ to abbreviate $(I|_{sch(\mathcal{P})^{\mathrm{LT}}})|^{x,s}$. So intuitively, when we select the facts with location specifier x and timestamp s, we are only interested in facts that provide these two components, which are the facts over $sch(\mathcal{P})^{\mathrm{LT}}$.

Intuitively, for $i \in \mathbb{N}$, the set $(M^{\blacktriangle})|^{x_i,s_i}$ is the input for the deductive rules during local step s_i of node x_i, consisting of *(i)* the *edb*-facts; *(ii)* the facts derived

[1] We say *a* rather than *the* corresponding rule because there could be more than one. Indeed, multiple original deductive rules in $pure(\mathcal{P})$ could be mapped to the same positive ground rule after applying a valuation and removing their negative ground body atoms. But in any case, these non-ground rules will have the same head predicate. Hence, they have the same stratum.

by inductive rules during a previous step (if any) of x_i; and, *(iii)* the delivered messages. The deductive rules then complete this information by deriving some new facts, that are visible within step s_i of x_i.

For a transition number i of $\mathcal{R}$, *(i)* we denote the source-configuration of transition i as $\rho_i = (st_i, bf_i)$; *(ii)* we denote the set of (tagged) messages delivered in transition i as m_i; and, *(iii)* we denote $D_i = deduc_{\mathcal{P}}(st_i(x_i) \cup untag(m_i))$. For a number $k \in \mathbb{N}$, we write $D_i^{\to k}$ to denote the set of facts obtained by adding to $st_i(x_i) \cup untag(m_i)$ all facts derived in stratum 1 up to stratum k during the computation of D_i. To mirror this notation, we write $M^{\to k}$ to denote the set $M^{\blacktriangle} \cup M^{\mathrm{duc},k}$. For uniformity in the proofs, we will consider the case $k = 0$, which is an invalid stratum number, and this gives $D_i^{\to 0} = st_i(x_i) \cup untag(m_i)$ and $M^{\to 0} = M^{\blacktriangle}$.

B.2 Valid Start

We show that $\rho_0 = start(\mathcal{P}, H)$. Denote $\rho_0 = (st_0, bf_0)$. Let $x \in \mathcal{N}$. First we show $st_0(x) = H(x)$. By definition,

$$st_0(x) = \big((M|_{edb(\mathcal{P})^{\mathrm{LT}}})|^{x,s} \cup M^{\mathrm{ind}}|^{x,s}\big)^{\Downarrow}$$

with $s = loc_M(0, x)$. Note, $s = 0$ because no elements of $\mathcal{N} \times \mathbb{N}$ with first component x have an ordinal strictly less than 0 in the total order $<_M$. Now, there can be no ground inductive rules in G that derive facts with head timestamp 0 because it follows from the construction of $decl(H)$ that the second component of a `tsucc`-fact is always strictly larger than 0. Therefore $M^{\mathrm{ind}}|^{x,s} = \emptyset$, and thus $st_0(x) = \big((M|_{edb(\mathcal{P})^{\mathrm{LT}}})|^{x,s}\big)^{\Downarrow}$. Then by Claim 7 we have $st_0(x) = (H(x)^{\Uparrow x,s})^{\Downarrow} = H(x)$, as desired.

Now we show $bf_0(x) = \emptyset$. By definition, $bf_0(x)$ is

$$\{(glob_M(y,t),\ R(\bar{a})) \mid \ \exists u : \texttt{chosen}_R(y,t,x,u,\bar{a}) \in M,\ glob_M(y,t) < 0 \leq glob_M(x,u)\}\cdot$$

By definition of function $glob_M(\cdot)$, all facts of the form $\texttt{chosen}_R(y,t,x,u,\bar{a}) \in M$ satisfy $glob_M(y,t) \geq 0$. Hence, $bf_0(x) = \emptyset$.

We conclude that $\rho_0 = start(\mathcal{P}, H)$.

B.3 Valid Transition

Let $i \in \mathbb{N}$. We show that $(\rho_i, x_i, m_i, i, \rho_{i+1})$ is a valid transition. Denote $\rho_i = (st_i, bf_i)$ and $\rho_{i+1} = (st_{i+1}, bf_{i+1})$.

We start by showing $m_i \subseteq bf_i(x_i)$. Let $(j, \boldsymbol{f}) \in m_i$. By definition of m_i, there is a fact of the form $\texttt{chosen}_R(y,t,z,u,\bar{a}) \in M$ with $glob_M(z,u) = i$ such that $j = glob_M(y,t)$ and $\boldsymbol{f} = R(\bar{a})$. Note, $glob_M(z,u) = i$ implies $z = x_i$ and $u = s_i$. Now, because rules in $pure(\mathcal{P})$ of the form (10) are always positive, the following ground rule is in G, which is of the form (10):

$$\texttt{before}(y,t,x_i,s_i) \leftarrow \texttt{chosen}_R(y,t,x_i,s_i,\bar{a})\cdot$$

Since its body is in M, this rule derives $\texttt{before}(y,t,x_i,s_i) \in M$. Hence $(y,t) \prec_M$

(x_i, s_i) by definition of $\prec_M$. Moreover, $<_M$ respects $\prec_M$, and thus $(y,t) <_M (x_i, s_i)$, which implies $glob_M(y,t) < glob_M(x_i, s_i)$. And since $glob_M(x_i, s_i) = i$, we overall have

$$glob_M(y,t) < i \leq glob_M(x_i, s_i).$$

Therefore $(j, \boldsymbol{f}) \in bf_i(x_i)$.

Now, because $m_i \subseteq bf_i(x_i)$, and because transitions are deterministic once the active node and delivered messages are fixed, we can consider the unique result configuration $\rho = (st, bf)$ such that $(\rho_i, x_i, m_i, i, \rho)$ is a valid transition. We are left to show $\rho_{i+1} = \rho$. We divide the work in two parts: for each $x \in \mathcal{N}$, we show that *(i)* $st_{i+1}(x) = st(x)$, and *(ii)* $bf_{i+1}(x) = bf(x)$.

B.3.1 State

Let $x \in \mathcal{N}$. We show $st_{i+1}(x) = st(x)$. Denote $s = loc_M(i+1, x)$. By definition,

$$st_{i+1}(x) = \left((M|_{edb(\mathcal{P})^{\mathrm{LT}}})|^{x,s} \cup M^{\mathrm{ind}}|^{x,s}\right)^{\Downarrow}.$$

Case $x \neq x_i$. By definition, $st(x) = st_i(x)$. Hence, it suffices to show $st_{i+1}(x) = st_i(x)$. Since $x \neq x_i$, the number of pairs from $\mathcal{N} \times \mathbb{N}$ containing node x that come strictly before ordinal $i+1$ is the same as the number of pairs containing node x that come strictly before ordinal i. Formally: $s = loc_M(i+1, x) = loc_M(i, x)$. Thus the right-hand side in the previous equation equals $st_i(x)$, and the result is obtained.

Case $x = x_i$. By definition, $st(x) = H(x) \cup induc_{\mathcal{P}}(D_i)$. Referring to the definition of $st_{i+1}(x)$ from above, by Claim 7 we have

$$(M|_{edb(\mathcal{P})^{\mathrm{LT}}})|^{x,s} = H(x)^{\Uparrow x,s}.$$

If we can also show $M^{\mathrm{ind}}|^{x,s} = induc_{\mathcal{P}}(D_i)^{\Uparrow x,s}$, then we overall have, as desired:

$$\begin{aligned} st_{i+1}(x) &= \left((M|_{edb(\mathcal{P})^{\mathrm{LT}}})|^{x,s} \cup M^{\mathrm{ind}}|^{x,s}\right)^{\Downarrow} \\ &= H(x) \cup induc_{\mathcal{P}}(D_i) \\ &= st(x). \end{aligned}$$

Since $x = x_i$, we have $s = loc_M(i+1, x_i) = loc_M(i, x_i) + 1$, and using that $loc_M(i, x_i) = s_i$ (Claim 8), we have $s = s_i + 1$. Now, Claim 9 and Claim 12 together show $M^{\mathrm{ind}}|^{x_i, s_i+1} = induc_{\mathcal{P}}(D_i)^{\Uparrow x_i, s_i+1}$.

B.3.2 Buffer

Let $x \in \mathcal{N}$. We show $bf_{i+1}(x) = bf(x)$. Denote

$$\delta^{i \to x} = \{(i, R(\bar{a})) \mid R(x, \bar{a}) \in async_{\mathcal{P}}(D_i)\}.$$

Like in the operational semantics, $\delta^{i \to x}$ denotes the (tagged) messages that are sent to x during transition i.

Case $x \neq x_i$. By definition, $bf(x) = bf_i(x) \cup \delta^{i \to x}$. We start by showing $bf(x) \subseteq bf_{i+1}(x)$. Let $(j, \boldsymbol{f}) \in bf(x)$. Denote $\boldsymbol{f} = R(\bar{a})$.

- Suppose $(j, \boldsymbol{f}) \in bf_i(x)$. By definition of $bf_i(x)$, there are values $y \in \mathcal{N}$, $t \in \mathbb{N}$ and $u \in \mathbb{N}$ such that $\texttt{chosen}_R(y, t, x, u, \bar{a}) \in M$ and $j = glob_M(y, t) < i \leq glob_M(x, u)$. Now, since $x \neq x_i$, we more specifically have $i < glob_M(x, u)$ and thus $i + 1 \leq glob_M(x, u)$. Therefore $(j, \boldsymbol{f}) \in bf_{i+1}(x)$, as desired.
- Suppose $(j, \boldsymbol{f}) \in \delta^{i \to x}$. By definition of $\delta^{i \to x}$, this implies $j = i$ and $R(x, \bar{a}) \in async_{\mathcal{P}}(D_i)$. Then $(j, \boldsymbol{f}) = (i, R(\bar{a})) \in bf_{i+1}(x)$ by Claim 13, as desired.

Secondly, we show $bf_{i+1}(x) \subseteq bf(x)$. Let $(j, \boldsymbol{f}) \in bf_{i+1}(x)$. Denote $\boldsymbol{f} = R(\bar{a})$. By definition of $bf_{i+1}(x)$, there are values $y \in \mathcal{N}$, $t \in \mathcal{N}$ and $u \in \mathcal{N}$ such that $\texttt{chosen}_R(y, t, x, u, \bar{a}) \in M$ and $j = glob_M(y, t) < i + 1 \leq glob_M(x, u)$. So $j \leq i$. We have the following cases:

- Suppose $j < i$. Thus $glob_M(y, t) < i$. This immediately gives $(j, \boldsymbol{f}) \in bf_i(x) \subseteq bf(x)$, as desired.
- Suppose $j = i$. Then $R(x, \bar{a}) \in async_{\mathcal{P}}(D_i)$ by Claim 14. This implies that $(j, \boldsymbol{f}) = (i, R(\bar{a})) \in \delta^{i \to x} \subseteq bf(x)$, as desired.

Case $x = x_i$. By definition, $bf(x) = (bf_i(x) \setminus m_i) \cup \delta^{i \to x}$. Some parts of the reasoning are similar to the case $x \neq x_i$. We refer to shared subclaims where possible.

We start by showing $bf(x) \subseteq bf_{i+1}(x)$. Let $(j, \boldsymbol{f}) \in bf(x)$. Denote $\boldsymbol{f} = R(\bar{a})$. We have the following cases:

- Suppose $(j, \boldsymbol{f}) \in bf_i(x) \setminus m_i$. Thus $(j, \boldsymbol{f}) \in bf_i(x)$ and $(j, \boldsymbol{f}) \notin m_i$. Here, $(j, \boldsymbol{f}) \in bf_i(x)$ implies there are values $y \in \mathcal{N}$, $t \in \mathbb{N}$ and $u \in \mathbb{N}$ such that $\texttt{chosen}_R(y, t, x, u, \bar{a}) \in M$ and $j = glob_M(y, t) < i \leq glob_M(x, u)$. Also, $(j, \boldsymbol{f}) \notin m_i$ implies $glob_M(x, u) \neq i$. Hence, $i + 1 \leq glob_M(x, u)$ and we obtain $(j, \boldsymbol{f}) \in bf_{i+1}(x)$, as desired.
- Suppose $(j, \boldsymbol{f}) \in \delta^{i \to x}$. By definition of $\delta^{i \to x}$, we have $j = i$ and $R(x, \bar{a}) \in async_{\mathcal{P}}(D_i)$. By Claim 13 we then have $(i, R(\bar{a})) \in bf_{i+1}(x)$, as desired.

Secondly, we show $bf_{i+1}(x) \subseteq bf(x)$. Let $(j, \boldsymbol{f}) \in bf_{i+1}(x)$. Denote $\boldsymbol{f} = R(\bar{a})$. By definition of $bf_{i+1}(x)$, there are values $y \in \mathcal{N}$, $t \in \mathbb{N}$ and $u \in \mathbb{N}$ such that $\texttt{chosen}_R(y, t, x, u, \bar{a}) \in M$ and $j = glob_M(y, t) < i + 1 \leq glob_M(x, u)$. Now we look at the cases for j:

- Suppose $j < i$. This gives us $glob_M(y, t) < i \leq glob_M(x, u)$, which implies $(j, \boldsymbol{f}) \in bf_i(x)$. Moreover, $i + 1 \leq glob_M(x, u)$ gives $glob_M(x, u) \neq i$. Hence, $(j, \boldsymbol{f}) \notin m_i$. Taken together, we now have $(j, \boldsymbol{f}) \in bf_i(x) \setminus m_i \subseteq bf(x)$.
- Suppose $j = i$. Then $(i, R(\bar{a})) \in bf_{i+1}(x)$, and by Claim 14 we obtain that $R(x, \bar{a}) \in async_{\mathcal{P}}(D_i)$. Therefore $(j, \boldsymbol{f}) = (i, R(\bar{a})) \in \delta^{i \to x} \subseteq bf(x)$, as desired.

B.4 Trace

In this section we show $\mathit{trace}(\mathcal{R}) = M|_{\mathit{sch}(\mathcal{P})^{\mathrm{LT}}}$. Recall from Section 5.1.5 that

$$\mathit{trace}(\mathcal{R}) = \bigcup_{i \in \mathbb{N}} (D_i)^{\Uparrow x_i,\, \mathit{loc}_{\mathcal{R}}(i)}.$$

For each $i \in \mathbb{N}$, $\mathit{loc}_{\mathcal{R}}(i)$ is the number of transitions in $\mathcal{R}$ before i in which x_i is also the active node. From the construction of $\mathcal{R}$ we know $\mathit{loc}_{\mathcal{R}}(i) = \mathit{loc}_M(i, x_i)$; indeed, $\mathit{loc}_M(i, x_i)$ counts the number of pairs in $\mathcal{N} \times \mathbb{N}$ with node x_i that have an ordinal strictly smaller than i, which is precisely the number of transitions in $\mathcal{R}$ with active node x_i that come before i. Moreover, by Claim 8 we have $\mathit{loc}_M(i, x_i) = s_i$. Hence,

$$\mathit{trace}(\mathcal{R}) = \bigcup_{i \in \mathbb{N}} (D_i)^{\Uparrow x_i, s_i}.$$

Thus, by Claim 15:

$$\mathit{trace}(\mathcal{R}) = \bigcup_{i \in \mathbb{N}} M|^{x_i, s_i}.$$

For the next step, let us denote $A = \{(x_i, s_i) \mid i \in \mathbb{N}\}$. We show $A = \mathcal{N} \times \mathbb{N}$. First, we have $A \subseteq \mathcal{N} \times \mathbb{N}$ because $x_i \in \mathcal{N}$ and $s_i \in \mathbb{N}$ for each $i \in \mathbb{N}$. Now, let $(x, s) \in \mathcal{N} \times \mathbb{N}$. Denote $i = \mathit{glob}_M(x, s)$. By definition, $x_i = x$ and $s_i = s$. Hence $(x, s) = (x_i, s_i) \in A$. Now we may write:

$$\begin{aligned} \mathit{trace}(\mathcal{R}) &= \bigcup_{(x,s) \in A} M|^{x,s} \\ &= \bigcup_{(x,s) \in \mathcal{N} \times \mathbb{N}} M|^{x,s}. \end{aligned}$$

Finally, because M is well-formed (see Section 5.3), for each $R(v, w, \bar{a}) \in M|_{\mathit{sch}(\mathcal{P})^{\mathrm{LT}}}$ we have $v \in \mathcal{N}$ and $w \in \mathbb{N}$. We obtain, as desired:

$$\mathit{trace}(\mathcal{R}) = M|_{\mathit{sch}(\mathcal{P})^{\mathrm{LT}}}.$$

B.5 Subclaims

Claim 7
Let $x \in \mathcal{N}$ and $s \in \mathbb{N}$. We have $(M|_{\mathit{edb}(\mathcal{P})^{\mathrm{LT}}})|^{x,s} = H(x)^{\Uparrow x,s}$.

Proof
First, by construction of $\mathit{decl}(H)$ we have $(\mathit{decl}(H)|_{\mathit{edb}(\mathcal{P})^{\mathrm{LT}}})|^{x,s} = H(x)^{\Uparrow x,s}$. Because $\mathit{decl}(H) \subseteq M$, and because facts over $\mathit{edb}(\mathcal{P})^{\mathrm{LT}}$ can not be derived by rules in $\mathit{pure}(\mathcal{P})$, we have $M|_{\mathit{edb}(\mathcal{P})^{\mathrm{LT}}} = \mathit{decl}(H)|_{\mathit{edb}(\mathcal{P})^{\mathrm{LT}}}$. Hence,

$$(M|_{\mathit{edb}(\mathcal{P})^{\mathrm{LT}}})|^{x,s} = (\mathit{decl}(H)|_{\mathit{edb}(\mathcal{P})^{\mathrm{LT}}})|^{x,s} = H(x)^{\Uparrow x,s}.$$

□

Claim 8
Let $i \in \mathbb{N}$. We have $s_i = loc_M(i, x_i)$.

Proof
Recall that $(x_i, s_i) \in \mathcal{N} \times \mathbb{N}$ is the unique pair at ordinal i in $<_M$, i.e., $glob_M(x_i, s_i) = i$. Suppose we would know for all $s \in \mathbb{N}$ and $t \in \mathbb{N}$ that $s < t$ implies $glob_M(x_i, s) < glob_M(x_i, t)$. Then $loc_M(i, x_i)$, which is

$$|\{s \in \mathbb{N} \mid glob_M(x, s) < i\}|,$$

is precisely

$$|\{s \in \mathbb{N} \mid s < s_i\}|\cdot$$

The latter is just s_i.

We are left to show for any $s \in \mathbb{N}$ and $t \in \mathbb{N}$ that $s < t$ implies $glob_M(x_i, s) < glob_M(x_i, t)$. It is actually sufficient to show for any $s \in \mathbb{N}$ that $(x_i, s) \prec_M (x_i, s+1)$. Indeed, this would imply for any $t \in \mathbb{N}$ with $s < t$ that

$$(x_i, s) \prec_M (x_i, s+1) \prec_M (x_i, s+2) \prec_M \ldots \prec_M (x_i, t)\cdot$$

And since $\prec_M$ is a partial order, it is transitive, and thus $(x_i, s) \prec_M (x_i, t)$. Next, since $<_M$ respects $\prec_M$, we obtain $(x_i, s) <_M (x_i, t)$ and thus $glob_M(x_i, s) < glob_M(x_i, t)$, as desired. To show $(x_i, s) \prec_M (x_i, s+1)$, we observe that the rule (7) in $pure(\mathcal{P})$ is positive. Hence, for any $s \in \mathbb{N}$, the following ground rule is always in G, and it derives $\texttt{before}(x_i, s, x_i, s+1) \in M$ because $\texttt{all}(x_i) \in decl(H)$ and $\texttt{tsucc}(s, s+1) \in decl(H)$:

$$\texttt{before}(x_i, s, x_i, s+1) \leftarrow \texttt{all}(x_i),\ \texttt{tsucc}(s, s+1)\cdot$$

Thus $(x_i, s) \prec_M (x_i, s+1)$ by definition of $\prec_M$. $\square$

Claim 9
Let $i \in \mathbb{N}$. We have $M^{\text{ind}}|^{x_i, s_i+1} \subseteq induc_{\mathcal{P}}(D_i)^{\Uparrow x_i, s_i+1}$.

Proof
Let $\boldsymbol{f} \in M^{\text{ind}}|^{x_i, s_i+1}$. We show $\boldsymbol{f} \in induc_{\mathcal{P}}(D_i)^{\Uparrow x_i, s_i+1}$.

By definition of M^{ind}, there is an active *inductive* ground rule $\psi \in G$ with $head_\psi = \boldsymbol{f}$. Because $\psi \in G$, there is a rule $\varphi \in pure(\mathcal{P})$ and a valuation V so that ψ can be obtained from φ by applying V and by subsequently removing all negative (ground) body literals, and so that $V(neg_\varphi) \cap M = \emptyset$. The rule φ must be of the form (2), which implies that V must assign x_i and s_i to the body location and timestamp variable respectively, and that it must assign x_i and $s_i + 1$ to the head location and timestamp variable respectively.

Let $\varphi' \in \mathcal{P}$ be the original inductive rule on which φ is based. Let $\varphi'' \in induc_{\mathcal{P}}$ be the rule corresponding to φ'. It follows from the construction of φ out of φ' and φ'' out of φ' that valuation V can also be applied to rule φ''. Indeed, rule φ just has more variables for the location and timestamps. We show that V is satisfying for φ'' with respect to D_i, so that φ'' and V together derive $V(head_{\varphi''}) = \boldsymbol{f}^{\Downarrow} \in induc_{\mathcal{P}}(D_i)$, which gives $\boldsymbol{f} \in induc_{\mathcal{P}}(D_i)^{\Uparrow x_i, s_i+1}$, as desired.

We must concretely show $V(pos_{\varphi''}) \subseteq D_i$ and $V(neg_{\varphi''}) \cap D_i = \emptyset$. We start by showing $V(pos_{\varphi''}) \subseteq D_i$. From the relationship between ψ, φ and φ'', we know that

$$pos_{\psi}|_{sch(\mathcal{P})^{\mathrm{LT}}} = V(pos_{\varphi})|_{sch(\mathcal{P})^{\mathrm{LT}}} = V(pos_{\varphi''})^{\Uparrow x_i, s_i}.$$

Since ψ is active with respect to M, we have $pos_{\psi} \subseteq M$, and thus $V(pos_{\varphi''})^{\Uparrow x_i, s_i} \subseteq M$. Then by Claim 10 we have $V(pos_{\varphi''}) \subseteq D_i$, as desired.

Now we show that $V(neg_{\varphi''}) \cap D_i = \emptyset$. By the relationship of φ and φ'', we have $V(neg_{\varphi''})^{\Uparrow x_i, s_i} = V(neg_{\varphi})$. By choice of φ and V, we have $V(neg_{\varphi}) \cap M = \emptyset$. Hence, $V(neg_{\varphi''})^{\Uparrow x_i, s_i} \cap M = \emptyset$. Finally, by Claim 11, we have $V(neg_{\varphi''}) \cap D_i = \emptyset$, as desired. □

Claim 10
Let $i \in \mathbb{N}$. Let I be a set of facts over $sch(\mathcal{P})^{\mathrm{LT}}$ that all have location specifier x_i and timestamp s_i. If $I \subseteq M$ then $I^{\Downarrow} \subseteq D_i$, with D_i as defined in Section B.1.

Proof
We are given $I \subseteq M$. By the assumptions on I, we more specifically have $I \subseteq M|^{x_i, s_i}$. Then by Claim 15 we have $I \subseteq (D_i)^{\Uparrow x_i, s_i}$. Hence $I^{\Downarrow} \subseteq D_i$, as desired. □

Claim 11
Let $i \in \mathbb{N}$. Let I be a set of facts over $sch(\mathcal{P})^{\mathrm{LT}}$ that all have location specifier x_i and timestamp s_i. If $I \cap M = \emptyset$ then $I^{\Downarrow} \cap D_i = \emptyset$, with D_i as defined in Section B.1.

Proof
We are given that $I \cap M = \emptyset$. This implies $I \cap M|^{x_i, s_i} = \emptyset$. By Claim 15 we have $I \cap (D_i)^{\Uparrow x_i, s_i} = \emptyset$. Hence, by the assumptions on I, we have $I^{\Downarrow} \cap D_i = \emptyset$, as desired. □

Claim 12
Let $i \in \mathbb{N}$. We have $induc_{\mathcal{P}}(D_i)^{\Uparrow x_i, s_i+1} \subseteq M^{\mathrm{ind}}|^{x_i, s_i+1}$.

Proof
Let $\boldsymbol{f} \in induc_{\mathcal{P}}(D_i)$. We show that $\boldsymbol{f}^{\Uparrow x_i, s_i+1} \in M^{\mathrm{ind}}|^{x_i, s_i+1}$.

Recall the semantics for $induc_{\mathcal{P}}$ from Section 5.1.2. Let $\varphi \in induc_{\mathcal{P}}$ and V be the rule and valuation that together derived $\boldsymbol{f} \in induc_{\mathcal{P}}(D_i)$. Let $\varphi' \in \mathcal{P}$ be the original inductive rule on which φ is based. Let $\varphi'' \in pure(\mathcal{P})$ be the inductive rule that in turn is based on φ', which is of the form (2). Let V'' be the valuation for φ'' that is obtained by extending V to assign x_i and s_i to respectively the location and timestamp variables in the body, and to assign $s_i + 1$ to the head timestamp variable. Let ψ be the positive ground rule obtained from φ'' by applying the valuation V'', and by subsequently removing the negative (ground) body literals. Note that $head_\psi = V(head_\varphi)^{\Uparrow x_i, s_i+1} = \boldsymbol{f}^{\Uparrow x_i, s_i+1}$. We will show that $\psi \in G$ and that $pos_\psi \subseteq M$, so that this ground rule derives $\boldsymbol{f}^{\Uparrow x_i, s_i+1} \in M$. And since ψ is inductive, we more specifically have $\boldsymbol{f}^{\Uparrow x_i, s_i+1} \in M^{\text{ind}}|^{x_i, s_i+1}$, as desired.

- For $\psi \in G$, we require $V''(neg_{\varphi''}) \cap M = \emptyset$. From the construction of rule φ'', we have $V''(neg_{\varphi''}) = V(neg_\varphi)^{\Uparrow x_i, s_i}$. We show $V(neg_\varphi)^{\Uparrow x_i, s_i} \cap M = \emptyset$. Because V is satisfying for φ with respect to D_i, we have $V(neg_\varphi) \cap D_i = \emptyset$. This gives $V(neg_\varphi)^{\Uparrow x_i, s_i} \cap (D_i)^{\Uparrow x_i, s_i} = \emptyset$. Then $V(neg_\varphi)^{\Uparrow x_i, s_i} \cap M|^{x_i, s_i} = \emptyset$ by Claim 15. Next, we obtain $V(neg_\varphi)^{\Uparrow x_i, s_i} \cap M = \emptyset$ since $V(neg_\varphi)^{\Uparrow x_i, s_i}$ contains only facts over $sch(\mathcal{P})^{\text{LT}}$ with location specifier x_i and timestamp s_i.
- Now we show $pos_\psi \subseteq M$. From the construction of rule φ'', we have

  $$pos_\psi = V''(pos_{\varphi''}) = V(pos_\varphi)^{\Uparrow x_i, s_i} \cup \{\texttt{tsucc}(s_i, s_i + 1)\}.$$

  We immediately have $\texttt{tsucc}(s_i, s_i + 1) \in decl(H) \subseteq M$. Moreover, since V is satisfying for φ with respect to D_i, we have $V(pos_\varphi) \subseteq D_i$. Hence $V(pos_\varphi)^{\Uparrow x_i, s_i} \subseteq (D_i)^{\Uparrow x_i, s_i}$. By Claim 15 we then have $V(pos_\varphi)^{\Uparrow x_i, s_i} \subseteq M|^{x_i, s_i} \subseteq M$, as desired.

□

Claim 13
Let $i \in \mathbb{N}$. Let $x \in \mathcal{N}$. For each $R(x, \bar{a}) \in async_{\mathcal{P}}(D_i)$, we have $(i, R(\bar{a})) \in bf_{i+1}(x)$.

Proof
The main approach of this proof is as follows. We will show there is a timestamp $u \in \mathbb{N}$ such that $\texttt{chosen}_R(x_i, s_i, x, u, \bar{a}) \in M$. Next, because rules of the form (10) are positive, in G there is always the following ground rule:

$$\texttt{before}(x_i, s_i, x, u) \leftarrow \texttt{chosen}_R(x_i, s_i, x, u, \bar{a}).$$

Thus if $\texttt{chosen}_R(x_i, s_i, x, u, \bar{a}) \in M$ then $\texttt{before}(x_i, s_i, x, u) \in M$, which implies $(x_i, s_i) \prec_M (x, u)$ by definition of $\prec_M$. Since $<_M$ respects $\prec_M$, we obtain $(x_i, s_i) <_M (x, u)$ and thus $glob_M(x_i, s_i) < glob_M(x, u)$. Also, since $glob_M(x_i, s_i) = i$, we overall get

$$glob_M(x_i, s_i) < i + 1 \leq glob_M(x, u),$$

which together with $\texttt{chosen}_R(x_i, s_i, x, u, \bar{a}) \in M$ gives $(glob_M(x_i, s_i), R(\bar{a})) = (i, R(\bar{a})) \in bf_{i+1}(x)$, as desired.

Now we are left to show that such a timestamp u exists. Recall the semantics for $async_{\mathcal{P}}$ from Section 5.1.2. Let $\varphi \in async_{\mathcal{P}}$ and V be a rule and valuation that together have derived $R(x, \bar{a}) \in async_{\mathcal{P}}(D_i)$. Let $\varphi' \in \mathcal{P}$ be the original asynchronous rule on which φ is based. Let $\varphi'' \in pure(\mathcal{P})$ be the rule obtained by applying transformation (9) to φ'. To continue, because $\prec_M$ is well-founded, there are only a finite number of timestamps $v \in \mathbb{N}$ of node x such that $(x, v) \prec_M (x_i, s_i)$. So, there exists a timestamp $u \in \mathbb{N}$ such that $(x, u) \not\prec_M (x_i, s_i)$. Now, let V'' be the valuation for φ'' that is the extension of valuation V to assign x_i and s_i to the body location variable and timestamp variable respectively (both belonging to the sender), and to assign u to the addressee arrival timestamp. Note that from the construction of φ'' we also know that V (and thus V'') assigns the value x to the addressee location variable and the tuple $\bar{a}$ to the message contents. Let ψ denote the ground rule obtained by applying V'' to φ'', and by subsequently removing the negative (ground) body literals. We will first show that $\psi \in G$, and then we show that $pos_\psi \subseteq M$, meaning that ψ derives $head_\psi = \texttt{cand}_R(x_i, s_i, x, u, \bar{a}) \in M$. Then Claim 17 can be applied to know that there is a timestamp u', with possibly $u' = u$, such that $\texttt{chosen}_R(x_i, s_i, x, u', \bar{a}) \in M$, as desired.

In order for ψ to be in G, we require $V''(neg_{\varphi''}) \cap M = \emptyset$. It follows from the construction of φ'' out of φ' and φ out of φ' that

$$V''(neg_{\varphi''}) = V(neg_\varphi)^{\Uparrow x_i, s_i} \cup \{\texttt{before}(x, u, x_i, s_i)\}.$$

We have $\texttt{before}(x, u, x_i, s_i) \notin M$ because $(x, u) \not\prec_M (x_i, s_i)$ by choice of u. Next, we show that $V(neg_\varphi)^{\Uparrow x_i, s_i} \cap M = \emptyset$. Because V is satisfying for φ with respect to D_i, we have $V(neg_\varphi) \cap D_i = \emptyset$, and thus

$$V(neg_\varphi)^{\Uparrow x_i, s_i} \cap (D_i)^{\Uparrow x_i, s_i} = \emptyset.$$

Then, by Claim 15,

$$V(neg_\varphi)^{\Uparrow x_i, s_i} \cap M|^{x_i, s_i} = \emptyset.$$

Since $V(neg_\varphi)^{\Uparrow x_i, s_i}$ contains only facts over $sch(\mathcal{P})^{\mathrm{LT}}$ with location specifier x_i and timestamp s_i, we have

$$V(neg_\varphi)^{\Uparrow x_i, s_i} \cap M = \emptyset.$$

We now show $pos_\psi \subseteq M$. Note, $pos_\psi = V''(pos_{\varphi''})$. From the construction of φ'' we have

$$V''(pos_{\varphi''}) = V(pos_\varphi)^{\Uparrow x_i, s_i} \cup \{\texttt{all}(x), \texttt{time}(u)\}.$$

Because $x \in \mathcal{N}$ and $u \in \mathbb{N}$, we immediately have $\{\texttt{all}(x), \texttt{time}(u)\} \subseteq decl(H) \subseteq M$. We are left to show $V(pos_\varphi)^{\Uparrow x_i, s_i} \subseteq M$. Because V is satisfying for φ with respect to D_i, we have $V(pos_\varphi) \subseteq D_i$. Hence $V(pos_\varphi)^{\Uparrow x_i, s_i} \subseteq (D_i)^{\Uparrow x_i, s_i}$. By again using Claim 15 we then obtain $V(pos_\varphi)^{\Uparrow x_i, s_i} \subseteq M|^{x_i, s_i} \subseteq M$, as desired. ☐

Claim 14
Let $i \in \mathbb{N}$ and $x \in \mathcal{N}$. For each $(i, R(\bar{a})) \in bf_{i+1}(x)$, we have $R(x, \bar{a}) \in async_{\mathcal{P}}(D_i)$.

Proof
By definition of $bf_{i+1}(x)$, the pair $(i, R(\bar{a})) \in bf_{i+1}(x)$ implies that there are values $y \in \mathcal{N}$, $t \in \mathbb{N}$ and $u \in \mathbb{N}$ such that $\texttt{chosen}_R(y, t, x, u, \bar{a}) \in M$, $glob_M(y, t) = i$ and $glob_M(y, t) < i + 1 \leq glob_M(x, u)$. And $glob_M(y, t) = i$ gives us that $y = x_i$ and $t = s_i$. Thus $\texttt{chosen}_R(x_i, s_i, x, u, \bar{a}) \in M$.

All ground rules in G that can derive $\texttt{chosen}_R(x_i, s_i, x, u, \bar{a}) \in M$ are of the form (4), and hence $\texttt{cand}_R(x_i, s_i, x, u, \bar{a}) \in M$. Let $\psi \in G$ be an active ground rule with head $\texttt{cand}_R(x_i, s_i, x, u, \bar{a})$. Because $\psi \in G$, there is a rule $\varphi \in \mathit{pure}(\mathcal{P})$ and a valuation V so that ψ is obtained from φ by applying V and by subsequently removing all negative (ground) body literals, and so that $V(\mathit{neg}_\varphi) \cap M = \emptyset$. The rule φ is of the form (9), which implies that V must assign x_i and s_i respectively to the body location and timestamp variable that correspond to the sender, and that it must assign x and u respectively to the location and timestamp variable that correspond to the addressee. Let $\varphi' \in \mathcal{P}$ be the original asynchronous rule on which φ is based. Let φ'' be the corresponding rule in $\mathit{async}_\mathcal{P}$. From the construction of φ out of φ' and φ'' out of φ', it follows that V can also be applied to φ''. Note, $V(\mathit{head}_{\varphi''}) = R(x, \bar{a})$. We now show that V is satisfying for φ'' with respect to D_i, which causes $R(x, \bar{a}) \in \mathit{async}_\mathcal{P}(D_i)$, as desired. Specifically, we have to show $V(\mathit{pos}_{\varphi''}) \subseteq D_i$ and $V(\mathit{neg}_{\varphi''}) \cap D_i = \emptyset$.

First we show $V(\mathit{pos}_{\varphi''}) \subseteq D_i$. By construction of φ and φ'', we have

$$\mathit{pos}_\psi|_{\mathit{sch}(\mathcal{P})^{\mathrm{LT}}} = V(\mathit{pos}_\varphi)|_{\mathit{sch}(\mathcal{P})^{\mathrm{LT}}} = V(\mathit{pos}_{\varphi''})^{\Uparrow x_i, s_i}.$$

Since ψ is active, we have $\mathit{pos}_\psi|_{\mathit{sch}(\mathcal{P})^{\mathrm{LT}}} \subseteq M$, and therefore $V(\mathit{pos}_{\varphi''})^{\Uparrow x_i, s_i} \subseteq M$. Then, because the facts in $V(\mathit{pos}_{\varphi''})^{\Uparrow x_i, s_i}$ are over $\mathit{sch}(\mathcal{P})^{\mathrm{LT}}$ and have location specifier x_i and timestamp s_i, we can apply Claim 10 to know that $V(\mathit{pos}_{\varphi''}) \subseteq D_i$, as desired.

Now we show $V(\mathit{neg}_{\varphi''}) \cap D_i = \emptyset$. By construction of φ and φ'', we have

$$V(\mathit{neg}_\varphi)|_{\mathit{sch}(\mathcal{P})^{\mathrm{LT}}} = V(\mathit{neg}_{\varphi''})^{\Uparrow x_i, s_i}.$$

By choice of φ and V, we have $V(\mathit{neg}_\varphi) \cap M = \emptyset$. Hence, $V(\mathit{neg}_{\varphi''})^{\Uparrow x_i, s_i} \cap M = \emptyset$. Then, because the facts in $V(\mathit{neg}_{\varphi''})^{\Uparrow x_i, s_i}$ are over $\mathit{sch}(\mathcal{P})^{\mathrm{LT}}$ and have location specifier x_i and timestamp s_i, we can apply Claim 11 to know that $V(\mathit{neg}_{\varphi''}) \cap D_i = \emptyset$, as desired. □

Claim 15

Let $i \in \mathbb{N}$. We have $M|^{x_i, s_i} = (D_i)^{\Uparrow x_i, s_i}$. Intuitively, this means that the operational deductive fixpoint D_i during transition i, corresponding to step s_i of node x_i, is represented by M in an exact way.

Proof

Recall the notations from Section B.1. Let n denote the largest stratum number of the deductive rules of $\mathcal{P}$. We show by induction on $k = 0, 1, \ldots, n$ that

$$(M^{\to k})|^{x_i, s_i} = (D_i^{\to k})^{\Uparrow x_i, s_i}.$$

This will give us $(M^{\to n})|^{x_i, s_i} = (D_i^{\to n})^{\Uparrow x_i, s_i} = (D_i)^{\Uparrow x_i, s_i}$. Moreover, Claim 18 says that $(M^{\to n})|^{x_i, s_i} = M|^{x_i, s_i}$, and thus we obtain $M|^{x_i, s_i} = (D_i)^{\Uparrow x_i, s_i}$, as desired.

Base case (k = 0) By definition,

$$M^{\to 0} = M^{\blacktriangle} \cup M^{\mathrm{duc},0}.$$

But since there are no deductive ground rules in G with stratum 0, we have $M^{\mathrm{duc},0} = \emptyset$. Hence,

$$\begin{aligned}(M^{\to 0})|^{x_i,s_i} &= (M^{\blacktriangle})|^{x_i,s_i} \\ &= (M|_{edb(\mathcal{P})^{\mathrm{LT}}})|^{x_i,s_i} \cup M^{\mathrm{ind}}|^{x_i,s_i} \cup M^{\mathrm{deliv}}|^{x_i,s_i}. \qquad \text{(B1)}\end{aligned}$$

Using Claim 16 and Claim 19, we can rewrite expression (B1) to the desired equality:

$$\begin{aligned}(M^{\to 0})|^{x_i,s_i} &= st_i(x_i)^{\Uparrow x_i,s_i} \cup untag(m_i)^{\Uparrow x_i,s_i} \\ &= (st_i(x_i) \cup untag(m_i))^{\Uparrow x_i,s_i} \\ &= (D_i^{\to 0})^{\Uparrow x_i,s_i}.\end{aligned}$$

Induction hypothesis For the induction hypothesis, we assume for a stratum number $k \geq 1$ that

$$(M^{\to k-1})|^{x_i,s_i} = (D_i^{\to k-1})^{\Uparrow x_i,s_i}.$$

Inductive step We show that

$$(M^{\to k})|^{x_i,s_i} = (D_i^{\to k})^{\Uparrow x_i,s_i}.$$

We show both inclusions separately, in Claims 20 and 21. □

Claim 16
Let $i \in \mathbb{N}$. We have $st_i(x_i)^{\Uparrow x_i,s_i} = (M|_{edb(\mathcal{P})^{\mathrm{LT}}})|^{x_i,s_i} \cup M^{\mathrm{ind}}|^{x_i,s_i}$.

Proof
By definition,

$$st_i(x_i) = \left((M|_{edb(\mathcal{P})^{\mathrm{LT}}})|^{x_i,s} \cup M^{\mathrm{ind}}|^{x_i,s}\right)^{\Downarrow},$$

where $s = loc_M(i, x_i)$. Using Claim 8, we have $s = s_i$. Therefore,

$$st_i(x_i)^{\Uparrow x_i,s_i} = (M|_{edb(\mathcal{P})^{\mathrm{LT}}})|^{x_i,s_i} \cup M^{\mathrm{ind}}|^{x_i,s_i}.$$

□

Claim 17
For each fact $\texttt{cand}_R(x, s, y, u, \bar{a}) \in M$, there is a timestamp $u' \in \mathbb{N}$ such that $\texttt{chosen}_R(x, s, y, u', \bar{a}) \in M$, with possibly $u' = u$.

Proof
Towards a proof by contradiction, suppose there is no such timestamp u'. Now, because $\texttt{cand}_R(x, s, y, u, \bar{a}) \in M$, the following ground rule, which is of the form (4), can not be in G, because otherwise $\texttt{chosen}_R(x, s, y, u, \bar{a}) \in M$, which is assumed not to be possible:

$$\texttt{chosen}_R(x, s, y, u, \bar{a}) \leftarrow \texttt{cand}_R(x, s, y, u, \bar{a})\cdot$$

Because rules of the form (4) contain a negative $\texttt{other}_{\ldots}$-atom in their body, the absence of the above ground rule from G implies $\texttt{other}_R(x, s, y, u, \bar{a}) \in M$. This $\texttt{other}_R$-fact must be derived by a ground rule of the form (5):

$$\texttt{other}_R(x, s, y, u, \bar{a}) \leftarrow \texttt{cand}_R(x, s, y, u, \bar{a}),\ \texttt{chosen}_R(x, s, y, u', \bar{a}),\ u \neq u'\cdot$$

But this implies that $\texttt{chosen}_R(x, s, y, u', \bar{a}) \in M$, which is a contradiction. □

Claim 18
Let $i \in \mathbb{N}$. Let n denote the largest stratum number of the deductive rules of $\mathcal{P}$. We have $(M^{\to n})|^{x_i, s_i} = M|^{x_i, s_i}$.

Proof
First, since $M^{\to n} \subseteq M$, we immediately have $(M^{\to n})|^{x_i, s_i} \subseteq M|^{x_i, s_i}$.

Now, let $\boldsymbol{f} \in M|^{x_i, s_i}$. We show $\boldsymbol{f} \in (M^{\to n})|^{x_i, s_i}$. Since $\boldsymbol{f}$ has location specifier x_i and timestamp s_i, we are left to show $\boldsymbol{f} \in M^{\to n}$. We have the following cases:

- Suppose $\boldsymbol{f} \in M|_{edb(\mathcal{P})^{\mathrm{LT}}}$. Then $\boldsymbol{f} \in M^{\blacktriangle} \subseteq M^{\to n}$.
- Suppose $\boldsymbol{f} \in M|_{idb(\mathcal{P})^{\mathrm{LT}}}$. Then there is an active ground rule $\psi \in G$ with $head_\psi = \boldsymbol{f}$. As seen in Section B.1, rule ψ can be of three types: deductive, inductive and delivery. The last two cases would respectively imply $\boldsymbol{f} \in M^{\mathrm{ind}}$ and $\boldsymbol{f} \in M^{\mathrm{deliv}}$, giving $\boldsymbol{f} \in M^{\blacktriangle} \subseteq M^{\to n}$. In the deductive case, rule ψ has a stratum number no larger than n, and hence $\boldsymbol{f} \in M^{\mathrm{duc}, n} \subseteq M^{\to n}$.

□

Claim 19
Let $i \in \mathbb{N}$. We have $M^{\mathrm{deliv}}|^{x_i, s_i} = untag(m_i)^{\Uparrow x_i, s_i}$.

Proof
Let $\boldsymbol{f} \in M^{\mathrm{deliv}}|^{x_i, s_i}$. We show $\boldsymbol{f} \in untag(m_i)^{\Uparrow x_i, s_i}$. Denote $\boldsymbol{f} = R(x_i, s_i, \bar{a})$. By definition of M^{deliv}, there is an active delivery rule $\psi \in G$ that derives $\boldsymbol{f}$:

$$R(x_i, s_i, \bar{a}) \leftarrow \texttt{chosen}_R(y, t, x_i, s_i, \bar{a})\cdot$$

Because this rule is active, we have $\texttt{chosen}_R(y, t, x_i, s_i, \bar{a}) \in M$. Now, by definition of x_i and s_i, we have $glob_M(x_i, s_i) = i$. Hence, $(glob_M(y, t), R(\bar{a})) \in m_i$ and thus $R(\bar{a}) \in untag(m_i)$. Finally, we obtain $\boldsymbol{f} = R(x_i, s_i, \bar{a}) \in untag(m_i)^{\Uparrow x_i, s_i}$, as desired.

Let $\boldsymbol{f} \in untag(m_i)^{\Uparrow x_i, s_i}$. We show $\boldsymbol{f} \in M^{\mathrm{deliv}}|^{x_i, s_i}$. Denote $\boldsymbol{f} = R(x_i, s_i, \bar{a})$. We have $R(\bar{a}) \in \mathit{untag}(m_i)$. Thus, there is some tag $j \in \mathbb{N}$ such that $(j, R(\bar{a})) \in m_i$. By definition of m_i, there are values $y \in \mathcal{N}$, $t \in \mathbb{N}$, $z \in \mathcal{N}$ and $u \in \mathbb{N}$ such that

$$\texttt{chosen}_R(y, t, z, u, \bar{a}) \in M,$$

where $\mathit{glob}_M(y, t) = j$ and $\mathit{glob}_M(z, u) = i$. Here, $\mathit{glob}_M(z, u) = i$ implies $z = x_i$ and $u = s_i$. Hence, $\texttt{chosen}_R(y, t, x_i, s_i, \bar{a}) \in M$. Now, the following ground rule ψ is in G because (delivery) rules of the form (6) are always positive:

$$R(x_i, s_i, \bar{a}) \leftarrow \texttt{chosen}_R(y, t, x_i, s_i, \bar{a}).$$

This rule derives $\boldsymbol{f} = R(x_i, s_i, \bar{a}) \in M$ because its body-fact is in M. Hence, $\boldsymbol{f} \in M^{\mathrm{deliv}}|^{x_i,s_i}$, as desired. $\square$

Claim 20

Let $i \in \mathbb{N}$. Let k be a stratum number (thus $k \geq 1$). Suppose that

$$(M^{\to k-1})|^{x_i,s_i} = (D_i^{\to k-1})^{\Uparrow x_i,s_i}.$$

We have

$$(M^{\to k})|^{x_i,s_i} \subseteq (D_i^{\to k})^{\Uparrow x_i,s_i}.$$

Proof

We consider the fixpoint computation of M, i.e., $M = \bigcup_{l\in\mathbb{N}} M_l$ with $M_0 = \mathit{decl}(H)$ and $M_l = T(M_{l-1})$ for each $l \geq 1$, where T is the immediate consequence operator of G. By the semantics of operator T, we have $M_{l-1} \subseteq M_l$.

We show by induction on $l = 0, 1, 2, \ldots$, that

$$(M_l \cap M^{\to k})|^{x_i,s_i} \subseteq (D_i^{\to k})^{\Uparrow x_i,s_i}.$$

This will imply that

$$\left(\left(\bigcup_{l\in\mathbb{N}} M_l\right) \cap M^{\to k}\right)|^{x_i,s_i} \subseteq (D_i^{\to k})^{\Uparrow x_i,s_i}.$$

Hence, we obtain, as desired

$$(M \cap M^{\to k})|^{x_i,s_i} = (M^{\to k})|^{x_i,s_i} \subseteq (D_i^{\to k})^{\Uparrow x_i,s_i}.$$

Before we start with the induction, recall from Section B.1 that

$$\begin{aligned} M^{\to k} &= M^{\blacktriangle} \cup M^{\mathrm{duc},k} \\ &= M|_{\mathit{edb}(\mathcal{P})^{\mathrm{LT}}} \cup M^{\mathrm{ind}} \cup M^{\mathrm{deliv}} \cup M^{\mathrm{duc},k}. \end{aligned}$$

Base case ($l = 0$) We have $M_0 = decl(H)$. Thus M_0 contains no facts derived by deductive, inductive or delivery ground rules. Therefore,

$$M_0 \cap M^{\to k} \quad = \quad M|_{edb(\mathcal{P})^{\mathrm{LT}}}.$$

Hence,

$$\begin{aligned}(M_0 \cap M^{\to k})|^{x_i,s_i} &\subseteq (M^{\blacktriangle})|^{x_i,s_i} \\ &\subseteq (M^{\to k-1})|^{x_i,s_i}.\end{aligned}$$

And by using the given equality $(M^{\to k-1})|^{x_i,s_i} = (D_i^{\to k-1})^{\Uparrow x_i,s_i}$, we obtain, as desired:

$$\begin{aligned}(M_0 \cap M^{\to k})|^{x_i,s_i} &\subseteq (D_i^{\to k-1})^{\Uparrow x_i,s_i} \\ &\subseteq (D_i^{\to k})^{\Uparrow x_i,s_i}.\end{aligned}$$

Induction hypothesis Let $l \geq 1$. We assume

$$(M_{l-1} \cap M^{\to k})|^{x_i,s_i} \subseteq (D_i^{\to k})^{\Uparrow x_i,s_i}.$$

Inductive step We show

$$(M_l \cap M^{\to k})|^{x_i,s_i} \subseteq (D_i^{\to k})^{\Uparrow x_i,s_i}.$$

Let $\boldsymbol{f} \in (M_l \cap M^{\to k})|^{x_i,s_i}$. If $\boldsymbol{f} \in M_{l-1}$ then $\boldsymbol{f} \in (M_{l-1} \cap M^{\to k})|^{x_i,s_i}$ and the induction hypothesis can be immediately applied. Now suppose that $\boldsymbol{f} \in M_l \setminus M_{l-1}$. Then there is a ground rule $\psi \in G$ with $head_\psi = \boldsymbol{f}$ that is active on M_{l-1}. We have $pos_\psi \subseteq M_{l-1}$. As we have seen in Section B.1, rule ψ can be of three types: deductive, inductive or a delivery. If ψ is an inductive rule or a delivery rule then

$$\begin{aligned}\boldsymbol{f} &\in M^{\mathrm{ind}}|^{x_i,s_i} \cup M^{\mathrm{deliv}}|^{x_i,s_i} \\ &\subseteq (M^{\blacktriangle})|^{x_i,s_i} \subseteq (M^{\to k-1})|^{x_i,s_i} \\ &= (D_i^{\to k-1})^{\Uparrow x_i,s_i} \subseteq (D_i^{\to k})^{\Uparrow x_i,s_i}.\end{aligned}$$

Now suppose ψ is deductive. If ψ has stratum less than or equal to $k-1$, then $\boldsymbol{f} \in (M^{\to k-1})|^{x_i,s_i}$. In that case, the given equality $(M^{\to k-1})|^{x_i,s_i} = (D_i^{\to k-1})^{\Uparrow x_i,s_i}$ gives $\boldsymbol{f} \in (D_i^{\to k-1})^{\Uparrow x_i,s_i} \subseteq (D_i^{\to k})^{\Uparrow x_i,s_i}$, as desired. Now suppose that ψ has stratum k. Because $\psi \in G$, there is a rule $\varphi \in pure(\mathcal{P})$ and valuation V so that ψ is obtained from φ by applying valuation V and subsequently removing the negative (ground) body literals, and so that $V(neg_\varphi) \cap M = \emptyset$. Let $\varphi' \in \mathcal{P}$ be the original deductive rule on which φ is based. Thus $\varphi' \in deduc_{\mathcal{P}}$ (see Section 5.1.2). By construction of φ out of φ', valuation V can also be applied to rule φ'. We now show that V is satisfying for φ' during the computation of D_i, in stratum k. Since $V(head_\varphi) = head_\psi = \boldsymbol{f}$, this results in the derivation of $V(head_{\varphi'}) = \boldsymbol{f}^{\Downarrow} \in D_i^{\to k}$ and thus $\boldsymbol{f} \in (D_i^{\to k})^{\Uparrow x_i,s_i}$, as desired. It is sufficient to show $V(pos_{\varphi'}) \subseteq D_i^{\to k}$ and $V(neg_{\varphi'}) \cap D_i^{\to k-1} = \emptyset$ because by the syntactic stratification, if φ' uses relations positively then those relations are in stratum k or lower, and if φ' uses relations negatively then those relations are in a stratum strictly lower than k.

- We show $V(pos_{\varphi'}) \subseteq D_i^{\to k}$. First, by the relationship between φ and φ', and because valuation V assigns x_i and s_i to respectively the body location variable and body timestamp variable of φ, we have $pos_\psi = V(pos_\varphi) = V(pos_{\varphi'})^{\Uparrow x_i,s_i}$. By choice of ψ, we already know $pos_\psi \subseteq M_{l-1}$. If we could show $pos_\psi \subseteq M^{\to k}$ then $pos_\psi \subseteq (M_{l-1} \cap M^{\to k})|^{x_i,s_i}$, to which the induction hypothesis can be applied to obtain $pos_\psi = V(pos_{\varphi'})^{\Uparrow x_i,s_i} \subseteq (D_i^{\to k})^{\Uparrow x_i,s_i}$, resulting in $V(pos_{\varphi'}) \subseteq D_i^{\to k}$, as desired.
  Now we show $pos_\psi \subseteq M^{\to k}$. Let $\boldsymbol{g} \in pos_\psi$. If $\boldsymbol{g} \in M^\blacktriangle$ then we immediately have $\boldsymbol{g} \in M^{\to k}$. Now suppose that $\boldsymbol{g} \notin M^\blacktriangle$. Since $pos_\psi \subseteq M|^{x_i,s_i}$, we have $\boldsymbol{g} \in M|^{x_i,s_i} \setminus M^\blacktriangle$. Then Claim 18 implies there is an active deductive ground rule $\psi' \in G$ with $head_{\psi'} = \boldsymbol{g}$. But we are working with a syntactic stratification, and thus the stratum of ψ' can not be higher than the stratum of ψ, which is k. Hence $\boldsymbol{g} \in M^{\mathrm{duc},k} \subseteq M^{\to k}$.
- We show $V(neg_{\varphi'}) \cap D_i^{\to k-1} = \emptyset$. By choice of φ and V, we have $V(neg_\varphi) \cap M = \emptyset$. So,

  $$V(neg_\varphi) \cap (M^{\to k-1})|^{x_i,s_i} = \emptyset.$$

  By applying the given equality $(M^{\to k-1})|^{x_i,s_i} = (D_i^{\to k-1})^{\Uparrow x_i,s_i}$, we then have $V(neg_\varphi) \cap (D_i^{\to k-1})^{\Uparrow x_i,s_i} = \emptyset$. By the relationship between φ and φ', we have $V(neg_\varphi) = V(neg_{\varphi'})^{\Uparrow x_i,s_i}$. Thus $V(neg_{\varphi'}) \cap D_i^{\to k-1} = \emptyset$, as desired.

□

Claim 21
Let $i \in \mathbb{N}$. Let k be a stratum number (thus $k \geq 1$). Suppose that

$$(M^{\to k-1})|^{x_i,s_i} = (D_i^{\to k-1})^{\Uparrow x_i,s_i}.$$

We have

$$(D_i^{\to k})^{\Uparrow x_i,s_i} \subseteq (M^{\to k})|^{x_i,s_i}.$$

Proof
Recall that the semantics of stratum k in $deduc_\mathcal{P}$ is that of semi-positive Datalog$^\neg$, with input $D_i^{\to k-1}$. So, we can consider $D_i^{\to k}$ to be a fixpoint, i.e., as the set $\bigcup_{l\in\mathbb{N}} A_l$ with $A_0 = D_i^{\to k-1}$ and $A_l = T(A_{l-1})$ for each $l \geq 1$, where T is the immediate consequence operator of stratum k in $deduc_\mathcal{P}$. We show by induction on $l = 0$, 1, 2, etc, that

$$(A_l)^{\Uparrow x_i,s_i} \subseteq (M^{\to k})|^{x_i,s_i}.$$

This then gives us the desired result.

Base case ($l = 0$) We have $A_0 = D_i^{\to k-1}$. By applying the given equality, we obtain

$$(A_0)^{\Uparrow x_i,s_i} = (D_i^{\to k-1})^{\Uparrow x_i,s_i} = (M^{\to k-1})|^{x_i,s_i} \subseteq (M^{\to k})|^{x_i,s_i}.$$

Induction hypothesis Let $l \geq 1$. We assume

$$(A_{l-1})^{\Uparrow x_i,s_i} \subseteq (M^{\to k})|^{x_i,s_i}.$$

Inductive step Let $\boldsymbol{f} \in A_l$. We show $\boldsymbol{f}^{\Uparrow x_i,s_i} \in (M^{\to k})|^{x_i,s_i}$. If $\boldsymbol{f} \in A_{l-1}$ then the induction hypothesis can be applied to obtain the desired result. Now suppose $\boldsymbol{f} \in A_l \setminus A_{l-1}$. Let $\varphi \in deduc_{\mathcal{P}}$ and V be respectively a rule with stratum k and a valuation that together have derived $\boldsymbol{f} \in A_l$. Let $\varphi' \in pure(\mathcal{P})$ be the rule obtained from φ by applying transformation (1). Let V' be the extension of V to assign x_i and s_i respectively to the body location and timestamp variable of φ', which are also both used in the head of φ'. Let ψ be the ground rule obtained from φ' by applying valuation V' and by subsequently removing all negative body literals. We show $\psi \in G$ and $pos_\psi \subseteq M$, which then implies

$$head_\psi = V'(head_{\varphi'}) = V(head_\varphi)^{\Uparrow x_i,s_i} = \boldsymbol{f}^{\Uparrow x_i,s_i} \in M.$$

Moreover, because φ (and thus φ') has stratum k, rule ψ is an active deductive ground rule with stratum k, and thus $\boldsymbol{f}^{\Uparrow x_i,s_i} \in (M^{\mathrm{duc},k})|^{x_i,s_i} \subseteq (M^{\to k})|^{x_i,s_i}$, as desired.

- To show $\psi \in G$, we require $V'(neg_{\varphi'}) \cap M = \emptyset$. Because V is satifying for φ, and because negation is only applied to lower strata, we have

  $$V(neg_\varphi) \cap D_i^{\to k-1} = \emptyset.$$

  Thus

  $$V(neg_\varphi)^{\Uparrow x_i,s_i} \cap (D_i^{\to k-1})^{\Uparrow x_i,s_i} = \emptyset.$$

  By the relationship between φ and φ', we have $V(neg_\varphi)^{\Uparrow x_i,s_i} = V'(neg_{\varphi'})$, which gives us

  $$V'(neg_{\varphi'}) \cap (D_i^{\to k-1})^{\Uparrow x_i,s_i} = \emptyset.$$

  And by using the given equality $(M^{\to k-1})|^{x_i,s_i} = (D_i^{\to k-1})^{\Uparrow x_i,s_i}$, we have

  $$V'(neg_{\varphi'}) \cap (M^{\to k-1})|^{x_i,s_i} = \emptyset.$$

  Now, for the last step, we work towards a contradiction: suppose that there is a fact $\boldsymbol{g} \in V'(neg_{\varphi'}) \cap M$. From the construction of φ', we know that $\boldsymbol{g}$ is over $sch(\mathcal{P})^{\mathrm{LT}}$ and has location specifier x_i and timestamp s_i.

  — If $\boldsymbol{g}$ is over $edb(\mathcal{P})^{\mathrm{LT}}$ then $\boldsymbol{g} \in (M|_{edb(\mathcal{P})^{\mathrm{LT}}})|^{x_i,s_i}$. Thus $\boldsymbol{g} \in (M^{\blacktriangle})|^{x_i,s_i} \subseteq (M^{\to k-1})|^{x_i,s_i}$, which is a contradiction.

  — If $\boldsymbol{g}$ is over $idb(\mathcal{P})^{\mathrm{LT}}$ then there is an active ground rule $\psi' \in G$ with $head_{\psi'} = \boldsymbol{g}$. As seen in Section B.1, rule ψ' is either deductive, inductive or a delivery. The last two cases would imply that $\boldsymbol{g} \in (M^{\mathrm{ind}} \cup M^{\mathrm{deliv}})|^{x_i,s_i} \subseteq (M^{\blacktriangle})|^{x_i,s_i}$, which gives a contradiction like in the previous case. Now suppose that ψ' is deductive. Because the predicate of $\boldsymbol{g}$ is used negatively in φ' and thus negatively in φ, the syntactic stratification assigns a smaller stratum number to ψ' than the stratum number of ψ, which is k. Hence, $\boldsymbol{g} \in (M^{\to k-1})|^{x_i,s_i}$, which is again a contradiction.

We conclude that $V'(neg_{\varphi'}) \cap M = \emptyset$.

- We show $pos_\psi \subseteq M$. Because V is satisfying for φ, we have

$$V(pos_\varphi) \subseteq A_{l-1}.$$

By the relationship between φ and φ' (and ψ), we have $V(pos_\varphi)^{\Uparrow x_i, s_i} = V'(pos_{\varphi'}) = pos_\psi$. Thus

$$pos_\psi \subseteq (A_{l-1})^{\Uparrow x_i, s_i}.$$

By now applying the induction hypothesis, we obtain, as desired:

$$pos_\psi \subseteq (M^{\to k})|^{x_i, s_i} \subseteq M.$$

□
