Online appendix for the paper

ASP for Minimal Entailment in a Rational Extension of SROEL

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Laura Giordano and Daniele Theseider Dupré
DISIT - Università del Piemonte Orientale, Alessandria, Italy
(*e-mail:* laura.giordano@uniupo.it, dtd@di.unipmn.it)

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Appendix A Proofs of Theorems 1 and 2

Theorem 1 (Small Rank)
Let K=(TBox,RBox,ABox) be a normalized SROEL$(\sqcap,\times)^{\mathbf{R}}\mathbf{T}$ *knowledge base. Given any model* $\mathscr{M} = (\Delta, <, \cdot^I)$ *of K, there exists a model* $\mathscr{M}' = (\Delta, <', \cdot^{I'})$ *of K (over the extended language) such that, for all* $x \in \Delta'$*: (i)* $k_{\mathscr{M}'}(x) \leq max_K$*; (ii) for all* $C \in N_C$*,* $x \in C^{I'}$ *iff* $x \in C^I$*; and (iii) for all* $C \in C_K$*,* $x \in (\mathbf{T}(C))^{I'}$ *iff* $x \in (\mathbf{T}(C))^I$.

Proof
We define the model $\mathscr{M}'$ over the domain Δ by letting $\cdot^{I'} = \cdot^I$, while changing the rank of the elements in Δ. What is preserved from $\mathscr{M}$ is the relative order of the ranks of the typical C elements, for $C \in C_K$. Remember that, from the definition of the rank of a concept in a model, $k_{\mathscr{M}}(C)$ is equal to the rank of all the typical C's in $\mathscr{M}$ (which must have all the same rank). Let us partition the set C_K according to the ranks of the concepts in $\mathscr{M}$:
$H_0 = \{ C \in C_K \mid \text{there is no } D \in C_K \text{ with } k_{\mathscr{M}}(D) < k_{\mathscr{M}}(C)\}$
$H_i = \{ C \in C_K - (H_0 \cup \ldots \cup H_{i-1}) \mid \text{there is no } D \in C_K - (H_0 \cup \ldots \cup H_{i-1}) \text{ with } k_{\mathscr{M}}(D) < k_{\mathscr{M}}(C)\}$
As the set C_K is finite and its cardinality is max_K, there is some minimum $n < max_K$, such that $H_{n+1} = \emptyset$.

We define the relation $<'$ by setting the rank of all the domain elements in $\mathscr{M}'$ between 0 and $n+1$. In particular, we want to let the rank of all the typical C elements to be i, if $C \in H_i$. For all $x \in \Delta$:
- if $k_{\mathscr{M}}(x) \leq k_{\mathscr{M}}(C)$ for some $C \in H_0$, then let $k_{\mathscr{M}'}(x) = 0$;
- if $k_{\mathscr{M}}(B) < k_{\mathscr{M}}(x) \leq k_{\mathscr{M}}(C)$ for some $B \in H_{i-1}$ and $C \in H_i$ $(0 < i \leq n)$, then let $k_{\mathscr{M}'}(x) = i$;
- if $k_{\mathscr{M}}(B) < k_{\mathscr{M}}(x)$ for some $B \in H_n$, then let $k_{\mathscr{M}'}(x) = n+1$.

In particular, we let the rank of all the typical C elements to be i, if $C \in H_i$. In fact, if $x \in (\mathbf{T}(C))^I$ then $k_{\mathscr{M}}(x) = k_{\mathscr{M}}(C)$. In case $C \in H_i$, then $k_{\mathscr{M}'}(x) = i$.

Changing the ranks as above cannot make a domain element, which is a typical C (for some $C \in C_K$), become a nontypical C element. In fact, if $x \in (\mathbf{T}(C))^I$, then for all y such that $k_{\mathscr{M}}(y) < k_{\mathscr{M}}(x)$, $y \notin C$. Suppose a typical C element x gets the rank i in $\mathscr{M}'$ (as $C \in H_i$). Some y can get in $\mathscr{M}'$ the same rank as x if $k_{\mathscr{M}}(B) < k_{\mathscr{M}}(y) \leq k_{\mathscr{M}}(C)$, for some $B \in H_{i-1}$. However, even if the rank of y becomes i, x remains a typical C element. Also, it is not the case that a nontypical C element z (for $C \in C_K$) can become a typical C element. In fact, one such z must have a rank

$k_{\mathcal{M}}(z)$ greater than the rank of any typical C element x, i.e., $k_{\mathcal{M}}(x) < k_{\mathcal{M}}(z)$. If x gets rank i in $\mathcal{M}'$, since $C \in H_i$, then (by definition of $\mathcal{M}'$) z gets a rank higher then i. Of course, this is not true for the concepts $C \notin C_K$. However, we can include as well in the set C_K all the concepts C such that $\mathbf{T}(C)$ might occur in a query. $\square$

Theorem 2 Let K be a knowledge base satisfying the following conditions:
(i) a canonical model of K exists;
(ii) the ranking $K_{\mathcal{M}}$ of each canonical model $\mathcal{M}$ of K is the same as the one determined by the Rational Closure construction;
and let Q be an inclusion $\mathbf{T}(C) \sqsubseteq D$ (where C and D are non-extended concepts). Then, $K \models_{\mathbf{T}min} \mathbf{T}(C) \sqsubseteq D$ iff $K \models_{min} \mathbf{T}(C) \sqsubseteq D$.

Proof
(If) By contraposition. Suppose that $K \not\models_{\mathbf{T}min} Q$, i.e. there is a $\mathbf{T}$-minimal model $\mathcal{M}$ of K which falsifies Q. Let us consider any minimal canonical model $\mathcal{M}'$ of K (there is one by (i)). $\mathcal{M}'$ must give the same ranks as $\mathcal{M}$ to the concepts $C \in \mathcal{T}_{K,Q}$. First it is not the case that $\mathcal{M}' \prec_{\mathbf{T}} \mathcal{M}$, otherwise $\mathcal{M}$ would not be a $\mathbf{T}$-minimal model. Also, it is not the case that there is a concept $C \in \mathcal{T}_{K,Q}$ such that $k_{\mathcal{M}}(C) < k_{\mathcal{M}'}(C) = rank(C)$, as the rank of a concept in any model of K cannot be lower than $rank(C)$, the rank of C in the Rational Closure[1] (this property holds for $SROEL(\sqcap,\times)^{\mathbf{R}}\mathbf{T}$ as it holds for $ALC + \mathbf{T_R}$ (Giordano et al. 2015) and for $SHIQ^{\mathbf{R}}\mathbf{T}$ (Giordano et al. 2014)). If there is a concept $C \in \mathcal{T}_{K,Q}$ such that $k_{\mathcal{M}'}(C) < k_{\mathcal{M}}(C) = rank(C)$, then as we have excluded that $\mathcal{M}' \prec_{\mathbf{T}} \mathcal{M}$, there must be a concept $C' \in \mathcal{T}_{K,Q}$ such that $k_{\mathcal{M}}(C') < k_{\mathcal{M}'}(C') = rank(C')$, (i.e., the two models $\mathcal{M}$ and $\mathcal{M}'$ must be incomparable wrt. $\prec_{\mathbf{T}}$). But we have already seen that it not possible that the rank of C' in a model is lower than the rank of C' in the rational closure. Thus, the minimal canonical model $\mathcal{M}'$ assigns to the concepts in $\mathcal{T}_{K,Q}$ the same rank as $\mathcal{M}$.

We have to show that $\mathcal{M}'$ falsifies the query Q. Let Q be $\mathbf{T}(C) \sqsubseteq D$. As $\mathcal{M}$ falsifies $\mathbf{T}(C) \sqsubseteq D$, there is an element $x \in \Delta$ such that $x \in (T(C))^I$ (x is a typical C element in $\mathcal{M}$) and $x \notin D^I$. Hence, $x \in (C \sqcap \neg D)^I$. Let $k_{\mathcal{M}}(x) = i$ (and hence $k_{\mathcal{M}}(C) = i$). As $\mathcal{M}'$ is a canonical model, $\mathcal{M}'$ must contain a domain element $y \in (C \sqcap \neg D)^{I'}$. Clearly, $k_{\mathcal{M}'}(C \wedge \neg D) \geq k_{\mathcal{M}'}(C)$. If $k_{\mathcal{M}'}(C \wedge \neg D) = i$, then $y \in \mathbf{T}(C)^{I'}$ (as C has the same rank i in $\mathcal{M}$ and in $\mathcal{M}'$), and $\mathcal{M}'$ falsifies Q. We show that assuming that $k_{\mathcal{M}'}(C \wedge \neg D) = j > i$, leads to a contradiction. By hypothesis (ii) $\mathcal{M}'$ assigns to concepts the same rank as the rational closure, hence $rank(C \wedge \neg D) = j > i$ in the rational closure. This contradicts the fact that $k_{\mathcal{M}}(C \wedge \neg D) = i$, as the rank of a concept in a model of K cannot be lower than the rank of that concept in the Rational Closure.

(Only If) By contraposition. Let $\mathcal{M}$ is a minimal canonical model of K falsifying Q. We want to show that there is a $\mathbf{T}$-minimal model $\mathcal{M}'$ falsifying Q. We can show that $\mathcal{M}$ is itself a $\mathbf{T}$-minimal model of K (falsifying Q). Clearly, $\mathcal{M}$ is a $\mathbf{T}$-complete model of K. If $\mathcal{M}$ were non-minimal wrt. $\prec_{\mathbf{T}}$, there would be a model $\mathcal{M}' \prec_{\mathbf{T}} \mathcal{M}$. In this case, there would be a $C \in \mathcal{T}_{K,Q}$ such that $k_{\mathcal{M}'}(C) < k_{\mathcal{M}}(C)$. This is not possible, due to the property that the rank of a concept C in a model of K cannot be lower than $rank(C)$, the rank of the concept C in the Rational Closure. As, from hypothesis (ii), $k_{\mathcal{M}}(C) = rank(C)$, it is not the case that $k_{\mathcal{M}'}(C) < k_{\mathcal{M}}(C)$. $\square$

[1] Observe that, the rank of a concept C can be determined in the rational closure construction for a KB in $SROEL(\sqcap,\times)^{\mathbf{R}}\mathbf{T}$, by iteratively verifying exceptionality of the concept C with respect to a set of inclusions E_i according to the iterative construction in (Giordano et al. 2015): *C is exceptional wrt. E_i* iff $E_i \models_{sroelrt} \mathbf{T}(\top) \sqcap C \sqsubseteq \bot$. For a concept $C \wedge \neg D$, where C and D are non extended concepts, *$C \wedge \neg D$ is exceptional wrt. E_i* iff $E_i \models_{sroelrt} \mathbf{T}(\top) \sqcap C \sqsubseteq D$.

Appendix B Proof of Theorem 3: Lower Bound for T-minimal entailment

In this section we show that the problem of deciding instance checking under the **T**-minimal model semantics is a Π^P_2-hard problem for $\mathcal{SROEL}(\sqcap,\times)^{\mathbf{R}}\mathbf{T}$ knowledge bases. To show this, we provide a reduction of the minimal entailment problem of *positive disjunctive logic programs*, which has been proved to be a Π^P_2-hard problem by Eiter and Gottlob in (Eiter and Gottlob 1995). A similar reduction has been used to prove Π^P_2-hardness of entailment for Circumscribed Left Local $\mathcal{EL}^\perp$ knowledge bases in (Bonatti et al. 2011).

Let $PV = \{p_1,\dots,p_n\}$ be a set of propositional variables. A clause is formula $l_1 \vee \dots \vee l_h$, where each literal l_j is either a propositional variable p_i or its negation $\neg p_i$. A positive disjunctive logic program (PDLP) is a set of clauses $S = \{\gamma_1,\dots,\gamma_m\}$, where each γ_j contains at least one positive literal. A truth valuation for S is a set $I \subseteq PV$, containing the propositional variables which are true. A truth valuation is a model of S if it satisfies all clauses in S. For a literal l, we write $S \models_{min} l$ if and only if every minimal model (with respect to subset inclusion) of S satisfies l. The minimal-entailment problem can be then defined as follows: given a PDLP S and a literal l, determine whether $S \models_{min} l$. In the following we sketch the reduction of the minimal-entailment problem for a PDLP S to the instance checking problem under **T**-minimal entailment, from a knowledge base K constructed from S.

We define a KB $K = (TBox, RBox, ABox)$ in $\mathcal{SROEL}(\sqcap,\times)^{\mathbf{R}}\mathbf{T}$ as follows. We introduce a concept name $P_h \in N_C$ for each variable $p_h \in PV$ ($h = 1,\dots,n$). Also, we introduce in N_C an auxiliary concept H, a concept name D_S associated with the set of clauses S, and a concept name D_j associated with each clause γ_j in S ($j = 1,\dots,m$). We let $a \in N_I$ be an individual name, and we define K as follows:

$RBox = \emptyset$,

$ABox = \{P_h(a), h = 1,\dots,n\} \cup \{\mathbf{T}(H)(a), D_S(a)\}$,

and $TBox$ contains the following inclusions (where C^j_i and $\overline{C^j_i}$ are concepts associated with each literal l^j_i occurring in $\gamma_j = l^j_1 \vee \dots \vee l^j_k$, as defined below):

(1) $\mathbf{T}(\top) \sqcap H \sqsubseteq \bot$

(2) $\{a\} \sqcap C^j_i \sqsubseteq D_j$ for all $\gamma_j = l^j_1 \vee \dots \vee l^j_k$ in S

(3) $\{a\} \sqcap D_j \sqcap \overline{C^j_1} \sqcap \dots \sqcap \overline{C^j_k} \sqsubseteq \bot$ for all $\gamma_j = l^j_1 \vee \dots \vee l^j_k$ in S

(4) $\{a\} \sqcap D_1 \sqcap \dots \sqcap D_m \sqsubseteq D_S$

(5) $\{a\} \sqcap D_S \sqsubseteq D_1 \sqcap \dots \sqcap D_m$

for each $h = 1,\dots,n$, $j = 1,\dots,m$, and where C^j_i is defined as follows:

$$C^j_i = \begin{cases} \mathbf{T}(P_h) & \text{if } l^j_i = p_h \\ \exists U.(\mathbf{T}(\top) \sqcap P_h) & \text{if } l^j_i = \neg p_h \end{cases}$$

$$\overline{C^j_i} = \begin{cases} \exists U.(\mathbf{T}(\top) \sqcap P_h) & \text{if } l^j_i = p_h \\ \mathbf{T}(P_h) & \text{if } l^j_i = \neg p_h \end{cases}$$

where U is the universal role. Let us consider any model $\mathcal{M} = \langle \Delta, <, \cdot^I \rangle$ of K. Observe that, all the $\mathbf{T}(\top)$ elements are all $\neg H$ elements. Hence, a^I (being a typical H) must have rank greater then 0, and it will have rank 1 in all **T**-minimal models. The **T**-minimal models of K satisfying $D_S(a)$ are intended to correspond to the (propositional) minimal interpretations J satisfying S. Roughly speaking, the concepts P_h such that $a^I \in (\mathbf{T}(P_h))^I$ in $\mathcal{M}$ correspond to the variables p_h in the minimal interpretation J satisfying S. In any **T**-minimal model of K, either P_h has rank 0 (and a is not a typical P_h), or P_h has rank 1 (and a is a typical P_h). Clearly, by $\mathbf{T}$-minimality, a model of K in which the ranking of a set of P_h's is 0, is preferred to the models in which the ranking of some of those P_h's is higher (i.e. 1). This captures the subset inclusion minimality in the interpretations of the positive disjunctive logic program S. Inclusions (2)-(5) bind the truth values of the $P_h(a)$ to the truth values of the clauses in S and of their conjunction. The assertion $D_S(a)$ in *ABox* is required to select only those interpretations satisfying the set S of disjunctions. Observe also that any $\mathbf{T}$-minimal model must contain al least a P_h element, for each $h = 1, \dots, n$, as P_h is a consistent concept.

In any minimal canonical model $\mathscr{M}$ of K satisfying $D_S(a)$:

$$\text{either } a^I \in (\mathbf{T}(P_h))^I \text{ or } a^I \in (\exists U.(\mathbf{T}(\top) \sqcap \mathbf{T}(P_h)))^I$$

Hence, for a^I the two concepts in the definition of C_i^j are disjoint and complementary, and $\overline{C_i^j}$ is actually the concept representing the complement of C_i^j. Given a set S of clauses and a literal L, the following holds:

Proposition 4
Given a set S of clauses and a literal L,

$$S \models_{min} L \quad \text{if and only if} \quad K \models_{\mathbf{T}min} C_L(a)$$

where C_L is the concept associated with L, i.e., $C_L = \mathbf{T}(p_h)$ if $L = p_h$, and $C_L = \exists U.(\mathbf{T}(\top) \sqcap P_h)$ if $L = \neg p_h$.

From the reduction above and the fact that minimal entailment for PDLP is Π_2^p-hard (Eiter and Gottlob 1995), it follows that minimal entailment under $\mathbf{T}$-minimal model semantics is Π_2^p-hard, i.e. Theorem 3 holds.

Appendix C Calculus for instance checking in *SROEL*$(\sqcap, \times)$

We report the calculus for *SROEL*$(\sqcap, \times)$ instance checking from (Krötzsch 2010a) used in section 5 and, with a small variant, in section 4. The representation of a knowledge base (*input translation*) is as follows, where, to keep a DL-like notation, we do not follow the ASP convention where variable names start with uppercase; in particular, A, B C, and R, S, T, are intended as ASP constants corresponding to the same class/role names in K:

$$\begin{array}{rl}
a \in N_I & \mapsto nom(a) \\
C \in N_C & \mapsto cls(C) \\
R \in N_R & \mapsto rol(R) \\
C(a) & \mapsto subClass(a, C) \\
R(a,b) & \mapsto supEx(a, R, b, b) \\
\top \sqsubseteq C & \mapsto top(C) \\
A \sqsubseteq \bot & \mapsto bot(A) \\
\{a\} \sqsubseteq C & \mapsto subClass(a, C) \\
A \sqsubseteq \{c\} & \mapsto subClass(A, c) \\
A \sqsubseteq C & \mapsto subClass(A, C) \\
A \sqcap B \sqsubseteq C & \mapsto subConj(A, B, C) \\
\exists R.Self \sqsubseteq C & \mapsto subSelf(R, C) \\
A \sqsubseteq \exists R.Self & \mapsto supSelf(A, R)
\end{array}$$

$$
\begin{array}{rl}
\exists R.A \sqsubseteq C & \mapsto subEx(R,A,C) \\
A \sqsubseteq \exists R.B & \mapsto supEx(A,R,B,aux_i) \\
R \sqsubseteq T & \mapsto subRole(R,T) \\
R \circ S \sqsubseteq T & \mapsto subRChain(R,S,T) \\
R \sqsubseteq C \times D & \mapsto supProd(R,C,D) \\
A \times B \sqsubseteq R & \mapsto subProd(A,B,R) \\
R \sqcap S \sqsubseteq T & \mapsto subRConj(R,S,T)
\end{array}
$$

In the translation of $A \sqsubseteq \exists R.B$, aux_i is a new constant, different for each axiom of this form.

The *inference rules* (included in Π_{IR} in section 4) are the following[2]:

(1) $inst(x,x) \leftarrow nom(x)$
(2) $self(x,v) \leftarrow nom(x), triple(x,v,x)$
(3) $inst(x,z) \leftarrow top(z), inst(x,z')$
(4) $\bot \leftarrow bot(z), inst(u,z)$
(5) $inst(x,z) \leftarrow subClass(y,z), inst(x,y)$
(6) $inst(x,z) \leftarrow subConj(y1,y2,z), inst(x,y1), inst(x,y2)$
(7) $inst(x,z) \leftarrow subEx(v,y,z), triple(x,v,x'), inst(x',y)$
(8) $inst(x,z) \leftarrow subEx(v,y,z), self(x,v), inst(x,y)$
(9) $triple(x,v,x') \leftarrow supEx(y,v,z,x'), inst(x,y)$
(10) $inst(x',z) \leftarrow supEx(y,v,z,x'), inst(x,y)$
(11) $inst(x,z) \leftarrow subSelf(v,z), self(x,v)$
(12) $self(x,v) \leftarrow supSelf(y,v), inst(x,y)$
(13) $triple(x,w,x') \leftarrow subRole(v,w), triple(x,v,x')$
(14) $self(x,w) \leftarrow subRole(v,w), self(x,v)$
(15) $triple(x,w,x'') \leftarrow subRChain(u,v,w), triple(x,u,x'), triple(x',v,x'')$
(16) $triple(x,w,x') \leftarrow subRChain(u,v,w), self(x,u), triple(x,v,x')$
(17) $triple(x,w,x') \leftarrow subRChain(u,v,w), triple(x,u,x'), self(x',v)$
(18) $triple(x,w,x) \leftarrow subRChain(u,v,w), self(x,u), self(x,v)$
(19) $triple(x,w,x') \leftarrow subRConj(v1,v2,w), triple(x,v1,x'), triple(x,v2,x')$
(20) $self(x,w) \leftarrow subRConj(v1,v2,w), self(x,v1), self(x,v2)$
(21) $triple(x,w,x') \leftarrow subProd(y1,y2,w), inst(x,y1), inst(x',y2)$
(22) $self(x,w) \leftarrow subProd(y1,y2,w), inst(x,y1), inst(x,y2)$
(23) $inst(x,z1) \leftarrow supProd(v,z1,z2), triple(x,v,x')$
(24) $inst(x,z1) \leftarrow supProd(v,z1,z2), self(x,v)$
(25) $inst(x',z2) \leftarrow supProd(v,z1,z2), triple(x,v,x')$
(26) $inst(x,z2) \leftarrow supProd(v,z1,z2), self(x,v)$
(27) $inst(y,z) \leftarrow inst(x,y), nom(y), inst(x,z)$
(28) $inst(x,z) \leftarrow inst(x,y), nom(y), inst(y,z)$
(29) $triple(z,u,y) \leftarrow inst(x,y), nom(y), triple(z,u,x)$

The version of the calculus in (Krötzsch 2010a), used in Section 5, contains the rule:
(4*b*) $inst(x,y) \leftarrow bot(z), inst(u,z), inst(x,z'), cls(y)$
instead of rule (4) above.

[2] Here, u,v,x,y,z,w, possibly with suffixes, are ASP variables.

Appendix D Proofs for Section 4

D.1 Proof of Proposition 2

Proposition 2. Given a normalized knowledge base K and a query Q, if there is an answer set S of the ASP program $\Pi(K) \cup \{-\pi_Q\}$, then there is a model $\mathscr{M} = (\Delta, <, \cdot^I)$ of K such that Q is not satisfied in $\mathscr{M}$.

The proof is similar to the one for Lemma 3 in (Krötzsch 2010b), which proves the completeness of the materialization calculus for $\mathit{SROEL}(\sqcap, \times)$ by contraposition, building a model of the KB from the minimal Herbrand model of the Datalog encoding. Here, given the answer set S of the program $\Pi(K) \cup \{-\pi_Q\}$ we build the model $\mathscr{M}$ falsifying Q exploiting the information in S.

In particular, we construct the domain of $\mathscr{M}$ from the set *Const* including all the name constants $c \in N_I$ as well as all the auxiliary constants occurring in the ASP program $\Pi(KB, Q)$, defining an equivalence relation over constants and using equivalence classes to define domain elements. For readability, we write $aux^{A \sqsubseteq \exists R.C}$ and aux_C, respectively, for the constants associated with inclusions $A \sqsubseteq \exists R.C$ and with the typicality concepts $\mathbf{T}(C)$. Observe that the answer set S contains all the details about the definition of the ranking of the domain elements that can be used to build the model $\mathscr{M}$.

First, let us define a relation $\approx$ between the constants in *Const*:

Definition 7
Let $\approx$ be the reflexive, symmetric and transitive closure of the relation $\{(c,d) \mid \mathit{inst}(c,d) \in S$, for $c \in \mathit{Const}$ and $d \in N_I\}$.

It can be proved that:

Lemma 1
Given a constant c such that $c \approx a$ for $a \in N_I$, if $\mathit{inst}(c,A)$ ($\mathit{triple}(c,R,d), \mathit{triple}(d,R,c), \mathit{self}(c,R), \mathit{rank}(c,k)$) is in S, then $\mathit{inst}(a,A)$ ($\mathit{triple}(a,R,d), \mathit{triple}(d,R,a), \mathit{self}(a,R), \mathit{rank}(a,k)$) is in S.

The proof is similar to the proof of Lemma 2 in (Krötzsch 2010b). For the predicate *rank*, the proof exploits rule (46). The vice-versa of Lemma 1 only holds for some of the predicates, namely:

Lemma 2
Given a constant c such that $c \approx a$ for $a \in N_I$, if $\mathit{inst}(a,A)$ ($triple(a,R,d)$, $\mathit{rank}(a,k)$) is in S, then $\mathit{inst}(c,A)$ ($triple(c,R,d)$, $\mathit{rank}(c,k)$) is in S.

Now, let $[c] = \{d \mid d \approx c\}$ denote the equivalence class of c; we define the domain Δ of the interpretation $\mathscr{M}$ as follows: $\Delta = \{[c] \mid c \in N_I\} \cup \{w_1^{A \sqsubseteq \exists R.C}, w_2^{A \sqsubseteq \exists R.C} \mid \mathit{inst}(aux^{A \sqsubseteq \exists R.C}, e) \in S$ for some e and there is no $d \in N_I$ such that $aux^{A \sqsubseteq \exists R.C} \approx d\} \cup \{z_C^1, z_C^2 \mid \mathit{inst}(aux_C, e) \in S$ for some e and there is no $d \in N_I$ such that $aux_C \approx d\}$. Two copies of auxiliary constants are introduced, as in (Krötzsch 2010b), to handle *Self* statements.

For each element $e \in \Delta$, we define a projection $\iota(e)$ to *Const* as follows:
- $\iota([c]) = c$;
- $\iota(w_i^{A \sqsubseteq \exists R.C}) = aux^{A \sqsubseteq \exists R.C}$, i=1,2;
- $\iota(z_C^i) = aux_C, i = 1,2$;

We define the interpretation of individual constants, concepts and roles over Δ as follows:
- for all $c \in N_I$, $c^I = [c]$;
- for all $d \in \Delta$, $d \in A^I$ iff $\mathit{inst}(\iota(d),A) \in S$;

- for all $d,e \in \Delta$, $(d,e) \in R^I$ iff $(triple(\iota(d),R,\iota(e)) \in S$ and $d \neq e)$
or $(self(\iota(d),R) \in S$ and $d = e)$.

We define the rank of the domain elements in Δ in agreement with the extension of the *rank* predicate in S:

- for all $d \in \Delta$, $k_{\mathscr{M}}(d) = h$, iff $rank(\iota(d),h) \in S$.

In particular, z_C has rank h if $rank(aux_C,h) \in S$ and $w^{A\sqsubseteq\exists R.C}$ has rank h if $rank\ (aux^{A\sqsubseteq\exists R.C},h) \in S$. The rank function $k_{\mathscr{M}}([c])$ is well defined. In fact, there is exactly one h such that $rank(\iota(d),h) \in S$ for each $\iota(d)$ (rules (36) and (37)). It is easy to see by Lemma 1 and Lemma 2 that, when $aux_C \approx a$ $(a \in N_I)$, i.e., $auc_C \in [a]$, we have $k_{\mathscr{M}}([a]) = h$ iff $rank(aux_C,h) \in S$. As a consequence, all the concepts C such that $\mathbf{T}(C)$ occurs in K (or in Q) have that same rank in $\mathscr{M}$ and in S.

To conclude the proof of Proposition 2 it suffices to prove that $\mathscr{M}$ is a model of KB, i.e. it satisfies all the axioms in KB. The proof is as in (Krötzsch 2010b) (see Lemma 2), except that we have to consider the additional axioms $A \sqsubseteq \mathbf{T}(B)$ and $\mathbf{T}(B) \sqsubseteq C$.

For $A \sqsubseteq \mathbf{T}(B)$ in KB, we have $supTyp(A,B) \in S$. Let us assume that $d \in A^I$. We want to prove that $d \in (\mathbf{T}(B))^I$. By construction $inst(\iota(d),A) \in S$. By rule (30), $typ(\iota(d),B) \in S$. By rule (47), $inst(\iota(d),B) \in S$, i.e., $d \in B^I$. Let $rank(\iota(d),h) \in S$, i.e. $k_{\mathscr{M}}(d) = h$.

To show that d is a typical B, we have to show that, for all the domain elements e with rank $j < h$, $e \notin B^I$. Given that $typ(\iota(d),B)$ and $rank(\iota(d),h)$ are in S, from rule (49), $box\_neg(h,B) \in S$. From the repeated application of rule (41), $box\_neg(j,B) \in S$, for all $j < h$. Hence, from rule (42), for all $e \in \Delta$ such that $rank(\iota(e),j) \in S$ (i.e., $k_{\mathscr{M}}(e) = j < h$) $-inst(\iota(e),B) \in S$ and therefore, $inst(\iota(e),B) \notin S$. Thus, for all $e \in \Delta$ such that $k_{\mathscr{M}}(e) = j < h$, $e \notin B^I$. So, $d \in (\mathbf{T}(B))^I$.

For $\mathbf{T}(B) \sqsubseteq C$ in KB, we have $subTyp(B,C) \in S$. Let $d \in (\mathbf{T}(B))^I$. We have to prove that $d \in A^I$. Assume that $k_{\mathscr{M}}(d) = h$, i.e., $rank(\iota(d),h) \in S$. As $d \in (\mathbf{T}(B))^I$, $d \in B^I$ and, for all $e \in \Delta$ such that $k_{\mathscr{M}}(e) = j < h$, $e \notin B^I$ (and hence, by construction, $inst(\iota(e),B) \notin S$). From $d \in B^I$, by the definition of $\mathscr{M}$, $inst(\iota(d),B) \in S$.

Consider also the rank of aux_B. Let $rank(aux_B,j) \in S$. By rule (51) it must be that $inst(aux_B,B)$ is in S. Either $j = h$ or $j \neq h$. If $j = h$, then from $rank(aux_B,h) \in S$, we conclude by rule (50) that $box\_neg(h,B) \in S$, and, given that $inst(\iota(d),B)$ and $rank(\iota(d),h)$ are in S, by rule (48), $typ(\iota(d),B) \in S$. Thus, by rule (31), $inst(\iota(d),C) \in S$.

We can exclude the case $j \neq h$, as both the hypothesis $j < h$ and the hypothesis $j > h$ lead to a contradiction. For $j < h$: the fact that $inst(aux_B,B) \in S$ contradicts the fact that, for all $e \in \Delta$ such that $k_{\mathscr{M}}(e) = j < h$, $inst(\iota(e),B) \notin S$. For $j > h$: from $rank(aux_B,j) \in S$, we can conclude by (50) that $box\_neg(j,B) \in S$, which would imply, by (41) and (42), that $\neg inst(\iota(d),B) \in S$ (from the fact that $rank(\iota(d),h) \in S$ and $h < j$). Again a contradiction.

Hence, $\mathscr{M}$ is a model of KB. For $Q = C(a)$, from the hypothesis $-inst(a,C) \in S$, hence $inst(a,C) \notin S$ and, by construction, $a^I \notin C^I$ in $\mathscr{M}$. For $Q = \mathbf{T}(C)(a)$, from the hypothesis $-typ(a,C) \in S$, hence $typ(a,C) \notin S$. If $inst(a,C) \notin S$ then, by construction of $\mathscr{M}$, $a^I \notin C^I$ and, clearly, $a^I \notin (\mathbf{T}(C))^I$. Instead, if $inst(a,C) \in S$, as $typ(a,C) \notin S$, it must be that, for $rank(a,h)$ and $rank(aux_C,j)$ in S, $h \neq j$ (otherwise, by rules (48) and (50), would conclude $typ(a,C) \in S$). Also, it can be seen that the hypothesis $h < j$ leads to a contradiction. Hence, $h > j$ and, by construction, $k_{\mathscr{M}}(a) > k_{\mathscr{M}}(C) = j$, so that $a^I \notin (\mathbf{T}(C))^I$.

This completes the proof of Proposition 2.

D.2 Proof of Proposition 3

Proposition 3. For a SROEL$(\sqcap,\times)^{\mathbf{R}}\mathbf{T}$ *knowledge base K in normal form and a query Q, if* $\mathscr{M} =$

$(\Delta, <, \cdot^I)$ is a model of K falsifying a query Q, then there exists an answer set S of the ASP program $\Pi(K) \cup \{-\pi_Q\}$.

Proof
Let Q be a query $C(a)$ (respectively, $\mathbf{T}(C)(a)$). We show that such an answer set S can be constructed from the model $\mathscr{M}$ such that $inst(a,C) \in S$ (respectively, $typ(a,C) \in S$). Without loss of generality, we can assume that $\mathscr{M}$ has no more than $max_K + 1$ different rank values (from 0 to max_K) and that the rank values have been made contiguous, according to Theorem 1. In the ASP program we let the upper bound n to be equal to max_K and, in the following, we let h_{max} be the maximum rank of domain elements in $\mathscr{M}$ (observe that $h_{max} \leq max_K$). We exploit $\mathscr{M}$ to construct the answer set S by assigning the ranks to the constants in N_I and to the auxiliary constants $aux^{A \sqsubseteq \exists R.C}$ and aux_C according to the ranks of the elements in $\mathscr{M}$.

Let S_0 contain the following facts:

0. $nom(c)$ for $c \in N_I$; $auxsupex(c)$ for $c = aux^{A \sqsubseteq \exists R.C}$; $auxtc(aux_B, B)$ for all $\mathbf{T}(B)$ in K or Q;
1. $ind(c)$ for all $c \in N_I$ and for all c auxiliary constants;
2. $rank(c,h)$, if $k_{\mathscr{M}}(c^I) = h$, for each $c \in N_I$;
3. $rank(aux_B, h)$, if there exists $x \in (\mathbf{T}(B))^I$ and $k_{\mathscr{M}}(x) = h$;
4. $rank(aux_B, h_{max})$ if $B^I = \emptyset$;
5. $rank(aux^{A \sqsubseteq \exists R.C}, h)$ if $A^I \neq \emptyset$ and $h = min\{k_{\mathscr{M}}(x) \mid x \in (C \sqcap \exists R^-.A))^I\}$;
6. $rank(aux^{A \sqsubseteq \exists R.C}, h_{max})$ if $A^I = \emptyset$;
7. $inst(aux_B, B) \in S$, if $B^I \neq \emptyset$, for $B \in N_C$ and $\mathbf{T}(B)$ occurring in K; otherwise, let $-inst(aux_B, B) \in S$.
8. $-inst(a,C) \in S$, if $Q = C(a)$;
9. $-typ(a,C) \in S$, if $Q = \mathbf{T}(C)(a)$;
10. $L \in S$, for any $L \in \Pi_K$, where L is the ASP literal representing a rule in K (according to the input translation in Section 4 (Part 1) and in Appendix C).
11. $upperbound(max_K), poss\_rank(0), \ldots, poss\_rank(max_K), some\_at(0), \ldots, some\_at(h_{max})$

The rank of $c \in N_I$ is equal to the rank of c^I in $\mathscr{M}$. The rank of aux_B is equal to the rank of any typical B element in $\mathscr{M}$, if any (as all the typical B elements have the same rank in $\mathscr{M}$). $aux^{A \sqsubseteq \exists R.C}$ is given the rank h_{max}, when $A^I = \emptyset$, otherwise it is given a minimal rank of the elements in the $(C \sqcap \exists R^-.A)^I$ concept interpretation[3]. Also, by item 5, aux_B is set to be an instance of concept B if and only if B has some instance in $\mathscr{M}$.

As in the proof of soundness of the materialization calculus in (Krötzsch 2010b) (see Lemma 2), we assign a concept expression $\kappa(c)$ to each constant occurring in the ASP program $\Pi(K) \cup \{-\pi_Q\}$:

- if $c \in N_I$, then $\kappa(c) = \{c\}$;
- if $c = aux^{A \sqsubseteq \exists R.C}$, then $\kappa(c) = C \sqcap \exists R^-.A$;
- if $c = aux_B$, then $\kappa(c) = \mathbf{T}(B)$.

We say that a set of literals S *is satisfied in the model* $\mathscr{M}$, if the following conditions hold:
- for $B \in N_C$, if $inst(c,B) \in S$, then $\mathscr{M} \models \kappa(c) \sqsubseteq B$ and $\kappa(c)^I \neq \emptyset$
- for $d \in N_I$, if $inst(c,d) \in S$, then $\mathscr{M} \models \kappa(c) \sqsubseteq \{d\}$ and $\kappa(c)^I \neq \emptyset$
- for $B \in N_C$, if $typ(c,B) \in S$, then $\mathscr{M} \models \kappa(c) \sqsubseteq \mathbf{T}(B)$ and $\kappa(c)^I \neq \emptyset$
- for $R \in N_R$, if $triple(c,R,d) \in S$, then $\mathscr{M} \models \kappa(c) \sqsubseteq \exists R.\kappa(d)$ and $\kappa(c)^I \neq \emptyset$
- for $R \in N_R$, if $self(c,R) \in S$, then $\mathscr{M} \models \kappa(c) \sqsubseteq \exists R.Self$ and $\kappa(c)^I \neq \emptyset$

[3] Notice that, although inverse roles are not in the language of $SROEL(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$, at the semantic level the set of domain elements in $(C \sqcap \exists R^-.A)^I$ is well defined, according to the usual semantics of inverse roles (Horrocks et al. 2000), i.e., $(\exists R^-.A)^I = \{x \in \Delta \mid \text{exists } y \in A^I \text{ such that } (y,x) \in R^I\}$.

- if $rank(c,h) \in S$ and $\kappa(c)^I \neq \emptyset$, then $k_{\mathscr{M}}(\kappa(c)) = h$
- if $box\_neg(h,A) \in S$ then, for all $x \in \Delta$ such that $k_{\mathscr{M}}(x) = h$, $x \in (\Box\neg A)^I$
- if $-box\_neg(h,A) \in S$ then, for all $x \in \Delta$ s.t. $k_{\mathscr{M}}(x) = h$, $x \notin (\Box\neg A)^I$
- for $B \in N_C$, if $-inst(c,B) \in S$ and $\kappa(c)^I \neq \emptyset$, then $\mathscr{M} \not\models \kappa(c) \sqsubseteq B$
- for $B \in N_C$, if $-typ(c,B) \in S$ and $\kappa(c)^I \neq \emptyset$, then $\mathscr{M} \not\models \kappa(c) \sqsubseteq \mathbf{T}(B)$
- for $B \in N_C$, if $bot(B) \in S$, then $\mathscr{M} \models B \sqsubseteq \bot$
- for $B \in N_C$, if $top(B) \in S$, then $\mathscr{M} \models \top \sqsubseteq B$

Notice that, from the previous conditions it is not the case that $bot(B)$ and $inst(a,B)$ are both in S, for some $B \in N_C$, otherwise, we would have (from $inst(a,B) \in S$) $\mathscr{M} \models \kappa(a) \sqsubseteq B$ with $\kappa(a)^I \neq \emptyset$ and (from $bot(B) \in S$) that $\mathscr{M} \models B \sqsubseteq \bot$.

Let us consider the portion P_0 the ASP program $\Pi(K) \cup \{-\pi_Q\}$ containing Π_K, plus the rules (32)-(39), the rules (52), (53) and the fact $-\pi_Q$. Once a unique rank is assigned to each constant c in N_I and to auxiliary constants, and the rank values are all contiguous and start from 0 (as required by rules (38) and (39)), and in particular the rank of the typical B elements (if any) have been fixed (as in $\mathscr{M}$) by introducing $rank(aux_B,h)$ in S, for some h, and $inst(aux_B,B)$ if $B^I \neq \emptyset$, the set S_0 satisfies the ASP rules in P_0 and is supported, that is, S_0 is an answer set of the program P_0.

All the other rules in the program do not involve default negation and their application uniquely determines an answer set, if it exists. So if there is an answer set of the ASP program $\Pi(K) \cup \{-\pi_Q\}$ it can be obtained by repeatedly applying the rules in P_1 containing all the rules Π_{IR} (Part 2) and the rules (40)-(51), (54) in Π_T (Part 3).

We can show that the application of the rule of the program preserves the property that S is satisfied in the model $\mathscr{M}$. Starting from S_0, which is an answer set of the portion P_0 of the program we show that the iterative application of the remaining ASP rules (those in P_1) gives a new set S of literals that is satisfied in $\mathscr{M}$.

The proof can be done by induction on the number of applications of the rules used to add a given literal in S.

Let S be the set of literals obtained after the exhaustive application of all the rules in P_1 starting from S_0. S is satisfied by the model $\mathscr{M}$ of KB. Hence, S cannot contain complementary literals such as $inst(b,A)$ and $-inst(b,A)$, otherwise S would not be satisfied in $\mathscr{M}$. Also, $inst(a,C)$ and $bot(C)$ cannot be in S for any a and C. Therefore, S is a consistent set of literals, and satisfies all the rules in P_1 as well as in P_0. Moreover, any literal in S is supported in S because it either belongs to S_0 (and is supported in P_0), or it is derived from S_0 by a sequence of rule applications. Hence, S is an answer set of $\Pi(K) \cup \{-\pi_Q\}$. By construction, $-inst(a,C) \in S$ (resp., $-typ(a,C) \in S$). $\Box$

Appendix E Proofs for Section 5

Proposition 5

Given a normalized knowledge base K and a query Q, if there is a model $\mathscr{M} = (\Delta, <, \cdot^I)$ of K which is $\mathbf{T}$-minimal wrt K, Q and falsifies Q, then there is an answer set S of the ASP program $\Pi(K)$, which is $\mathbf{T}$-minimal wrt K, Q and such that $\pi_Q \notin S$; and vice-versa.

Proof

Let $\mathscr{M} = (\Delta, <, \cdot^I)$ of K which is $\mathbf{T}$-minimal wrt K, Q and falsifies Q. By Proposition 3, there exists an answer set S of the ASP program $\Pi(K) \cup \{-\pi_Q\}$. As $\mathscr{M}$ is $\mathbf{T}$-complete, by construction,

S is also **T**-complete. Also, by construction, the ranks of the concepts $C \in \mathscr{T}_{K,Q}$ are the same in $\mathscr{M}$ as in S (i.e., $k_{\mathscr{M}}(C) = h < \infty$ iff $rank(aux_C, h), inst(aux_C, C) \in S$). We have to show that S is **T**-minimal wrt K, Q. Suppose, by absurdum, that S is not **T**-minimal. Hence, there is a **T**-complete answer set S' of $\Pi(K)$ such that $S' \preceq_{\mathbf{T}} S$. By Proposition 2, from S' we can build a model $\mathscr{M}'$ of K such that the ranks of the concepts $C \in \mathscr{T}_{K,Q}$ are the same in $\mathscr{M}'$ as in S' (see the construction in Appendix D, Section D.1). By construction, $\mathscr{M}'$ is also **T**-complete. Hence, there is a **T**-complete model $\mathscr{M}'$ of K such that $\mathscr{M}' \preceq_{\mathbf{T}} \mathscr{M}$, which contradicts the hypothesis that $\mathscr{M}$ is **T**-minimal.

Vice-versa, let S be an answer set of the ASP program $\Pi(K)$, which is **T**-minimal wrt K, Q and such that $inst(a, C) \notin S$. By Proposition 2, from S we can build a model $\mathscr{M}$ of K such that the ranks of the concepts $C \in \mathscr{T}_{K,Q}$ are the same in $\mathscr{M}$ as in S. By construction $\mathscr{M}$ is **T**-complete (as S is **T**-complete). We have to show that $\mathscr{M}$ is a **T**-minimal model of K. Suppose by absurdum that $\mathscr{M}$ is not **T**-minimal. Then, there is another **T**-complete model $\mathscr{M}'$ of K such that $\mathscr{M}' \preceq_{\mathbf{T}} \mathscr{M}$. By Proposition 3, there exists an answer set S' of the ASP program $\Pi(K) \cup \{-\pi_Q\}$. By construction, S' is **T**-complete and assigns to the concepts $C \in \mathscr{T}_{K,Q}$ the same ranks as $\mathscr{M}'$ (see the construction in Appendix D, Section D.2). Hence, it must be that $S' \preceq_{\mathbf{T}} S$, which contradicts the hypothesis that S is **T**-minimal. $\square$

Proposition 6
The problem of deciding the existence of a **T** minimal answer set of $\Pi(K)$ falsifying π_Q is in Σ_2^P.

Proof
This problem can be solved by nondeterministically guessing a set S of literals of polynomial size in the size of K and then verifying that:
(1) S is an answer set of $\Pi(K)$;
(2) S is **T**-complete wrt K, Q;
(3) $\pi_Q \notin S$;
(4) S is **T**-minimal wrt K, Q among the **T**-complete answer sets of $\Pi(K)$.

Verification of (1), (2) and (3) requires polynomial time in the size of K. In particular, for (1) the Gelfond and Lifschitz' transform of $\Pi(K)$ wrt S, $\Pi(K)^S$ (which has polynomial size and does not contain default negation), can be computed in polynomial time as well as its logical consequences. For (2), **T**-completeness can be verified by checking if $inst(aux_C, C)$ is in S, for all the $aux_C \in Aux_{K,Q}$ such that $satisfiable(C)$ holds (using the definition of predicate *satisfiable* in Section 5 based on the polynomial encoding of K in (Giordano and Theseider Dupré 2016)). (4) can be checked by calling an NP oracle which verifies that S is **T**-minimal among the **T**-complete answer sets of K. In fact, the verification that S is not a **T**-minimal answer set of K can be done by an NP algorithm which nondeterministically generates a set of literals S' (of polynomial size in the size of K) such that $S' \preceq_{\mathbf{T}} S$ ($S' \preceq_{\mathbf{T}} S$ can be checked in polynomial time). Hence, the problem of deciding existence of **T** minimal answer set of $\Pi(K)$ falsifying π_Q is in NP^{NP}. $\square$

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