Online appendix for the paper

The Intricacies of Three-Valued Extensional Semantics for Higher-Order Logic Programs

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Appendix A Proof of Theorem 1

Recall that propositional programs consist of clauses of the form $\mathsf{p} \leftarrow \mathsf{L}_1, \ldots, \mathsf{L}_n$, where each L_i is either a propositional variable or the negation of a propositional variable; the L_i are called *literals*, *negative* if they have negation and *positive* otherwise. As hinted in Section 3, we need to consider propositional programs with a possibly countably infinite number of clauses, as this is the case with the ground instantiation of a higher-order program. Moreover, we must allow a positive literal L_i to also be one of the constants true and false. The reason for this is that in the ground instantiation of an $\mathcal{H}$ program, there may exist ground expressions of the form $(\mathsf{E}_1 \approx \mathsf{E}_2)$ in the bodies of clauses. These have specific meanings under the semantics of Section 5 and can not be treated as propositional variables. In the case where the two expressions E_1 and E_2 are syntactically identical, the expression $(\mathsf{E}_1 \approx \mathsf{E}_2)$ will be treated as the constant true (i.e., it is assumed that $I((\mathsf{E}_1 \approx \mathsf{E}_2)) = \mathit{true}$ for every interpretation I of the ground instantiation), and otherwise as the constant false.

We use, throughout all sections of the Appendix, the standard representation of partial interpretations of propositional programs by $\langle T, F\rangle$, where T and F are disjoint subsets of the Herbrand base B_{P} of a propositional program P (i.e., the set of propositional variables appearing in P) denoting the sets of propositional variables considered to be true and false, respectively, in the interpretation. Naturally, $\langle T, F\rangle$ is a total interpretation if $T \cup F = B_{\mathsf{P}}$.

The truth ordering $\leq$ and Fitting ordering $\preceq$ of interpretations can be defined in two equivalent ways:

Definition 20
If $I = \langle T, F\rangle$ and $I' = \langle T', F'\rangle$ are two partial interpretations of a propositional program P then we say that $I \leq I'$ if $T \subseteq T'$ and $F' \subseteq F$, or, equivalently, if $I(\mathsf{p}) \leq I'(\mathsf{p})$ for every propositional variable p of P. Moreover, we say that $I \preceq I'$ if $T \subseteq T'$ and $F \subseteq F'$, or, equivalently, if $I(\mathsf{p}) \preceq I'(\mathsf{p})$ for every propositional variable p of P.

A model M of a propositional program is as usual considered to be $\leq$-minimal (respectively, $\preceq$-minimal) if there does not exist a different model N of P, such that $N \leq M$ (respectively, $N \preceq M$).

Theorem 1
Let P be a program and let $\mathsf{Gr}(\mathsf{P})$ be its ground instantiation. Also let M be a partial interpretation of $\mathsf{Gr}(\mathsf{P})$ and let $\mathcal{M}$ be the Herbrand interpretation of P, such that $v_{\mathcal{M}}(\mathsf{A}) = M(\mathsf{A})$ for every $\mathsf{A} \in U_{\mathsf{P},o}$. Then, $\mathcal{M}$ is a Herbrand model of P if and only if M is a model of $\mathsf{Gr}(\mathsf{P})$. Moreover, $\mathcal{M}$ is $\leq$-minimal (respectively, $\preceq$-minimal) if and only if M is $\leq$-minimal (respectively, $\preceq$-minimal).

Proof

Step 1 M is a model of $\mathsf{Gr}(\mathsf{P}) \Rightarrow \mathcal{M}$ is a Herbrand model of P: For every Herbrand state s of P there exists a ground substitution θ such that $\theta(\mathsf{V}) = s(\mathsf{V})$, and therefore $s(\mathsf{V}) = [\![\theta(\mathsf{V})]\!]_{\mathcal{M},s'}$, for all states s' and variables V in P. Also, for every clause $\mathsf{A} \leftarrow \mathsf{L}_1, \dots, \mathsf{L}_m$ in P there exists a respective ground instance $\mathsf{A}\theta \leftarrow \mathsf{L}_1\theta, \dots, \mathsf{L}_m\theta$ in $\mathsf{Gr}(\mathsf{P})$. As M is a model of $\mathsf{Gr}(\mathsf{P})$, $M(\mathsf{A}\theta) \geq min\{M(\mathsf{L}_1\theta), \dots, M(\mathsf{L}_m\theta)\}$. By assumption, $v_{\mathcal{M}}(\mathsf{A}\theta) = M(\mathsf{A}\theta)$ and $v_{\mathcal{M}}(\mathsf{L}_i\theta) = M(\mathsf{L}_i\theta)$ for all $i \leq m$. Moreover, it is easy to see (by a trivial induction on the structure of the expression) that $\mathsf{A}\theta = [\![\mathsf{A}]\!]_{\mathcal{M},s}$, which implies that $v_{\mathcal{M}}(\mathsf{A}\theta) = v_{\mathcal{M}}([\![\mathsf{A}]\!]_{\mathcal{M},s})$. Similarly, $v_{\mathcal{M}}(\mathsf{L}_i\theta) = v_{\mathcal{M}}([\![\mathsf{L}_i]\!]_{\mathcal{M},s})$, for all $i \leq m$. It follows immediately that $v_{\mathcal{M}}([\![\mathsf{A}]\!]_{\mathcal{M},s}) \geq min\{v_{\mathcal{M}}([\![\mathsf{L}_1]\!]_{\mathcal{M},s}), \dots, v_{\mathcal{M}}([\![\mathsf{L}_m]\!]_{\mathcal{M},s})\}$, which implies that $\mathcal{M}$ is a model of P.

Step 2 $\mathcal{M}$ is a Herbrand model of $\mathsf{P} \Rightarrow M$ is a model of $\mathsf{Gr}(\mathsf{P})$: Every clause in $\mathsf{Gr}(\mathsf{P})$ is a ground instance of a clause $\mathsf{A} \leftarrow \mathsf{L}_1, \dots, \mathsf{L}_m$ in P and is therefore of the form $\mathsf{A}\theta \leftarrow \mathsf{L}_1\theta, \dots, \mathsf{L}_m\theta$ for some ground substitution θ. Consider a Herbrand state s, such that $s(\mathsf{V}) = \theta(\mathsf{V})$ for every variable V in P. Because $\mathcal{M}$ is a model of P, we have that $v_{\mathcal{M}}([\![\mathsf{A}]\!]_{\mathcal{M},s}) \geq min\{v_{\mathcal{M}}([\![\mathsf{L}_1]\!]_{\mathcal{M},s}), \dots, v_{\mathcal{M}}([\![\mathsf{L}_m]\!]_{\mathcal{M},s})\}$. Again, it is easy to see that $\mathsf{A}\theta = [\![\mathsf{A}]\!]_{\mathcal{M},s}$ and therefore $v_{\mathcal{M}}(\mathsf{A}\theta) = v_{\mathcal{M}}([\![\mathsf{A}]\!]_{\mathcal{M},s})$. Similarly, $v_{\mathcal{M}}(\mathsf{L}_i\theta) = v_{\mathcal{M}}([\![\mathsf{L}_i]\!]_{\mathcal{M},s})$ for all $i \leq m$. Additionally, $v_{\mathcal{M}}(\mathsf{A}\theta) = M(\mathsf{A}\theta)$ and $v_{\mathcal{M}}(\mathsf{L}_i\theta) = M(\mathsf{L}_i\theta)$ for all $i \leq m$, so $M(\mathsf{A}\theta) \geq min\{M(\mathsf{L}_1\theta), \dots, M(\mathsf{L}_m\theta)\}$, which implies that M is a model of $\mathsf{Gr}(\mathsf{P})$.

Step 3 M is minimal $\Rightarrow \mathcal{M}$ is minimal: Assume there exists a model $\mathcal{N}$ of P, distinct from $\mathcal{M}$, such that $\mathcal{N} \leq \mathcal{M}$ (respectively, $\mathcal{N} \preceq \mathcal{M}$). Then we can construct an interpretation N for $\mathsf{Gr}(\mathsf{P})$ such that for every ground atom A, $N(\mathsf{A}) = v_{\mathcal{N}}(\mathsf{A})$. It is obvious that $N \leq M$ (respectively, $N \preceq M$), since $N(\mathsf{A}) = v_{\mathcal{N}}(\mathsf{A}) \leq v_{\mathcal{M}}(\mathsf{A}) = M(\mathsf{A})$ (respectively, $v_{\mathcal{N}}(\mathsf{A}) \preceq v_{\mathcal{M}}(\mathsf{A})$). Also, N is distinct from M, since $N(\mathsf{B}) = v_{\mathcal{N}}(\mathsf{B}) \neq v_{\mathcal{M}}(\mathsf{B}) = M(\mathsf{B})$ for at least one ground atom B. As we showed in Step 2, the fact that $\mathcal{N}$ is a model of P implies that N is a model of $\mathsf{Gr}(\mathsf{P})$, which is of course a contradiction, since M is a $\leq$-minimal (respectively, $\preceq$-minimal) model of $\mathsf{Gr}(\mathsf{P})$. Therefore, $\mathcal{M}$ must be a $\leq$-minimal (respectively, $\preceq$-minimal) model of P.

Step 4 $\mathcal{M}$ is minimal $\Rightarrow M$ is minimal: By the reverse of the argument used in Step 3: Assume there exists a model N of $\mathsf{Gr}(\mathsf{P})$, distinct from M, such that $N \leq M$ (respectively, $N \preceq M$). Then we can construct an interpretation $\mathcal{N}$ for P such that for every ground atom A, $N(\mathsf{A}) = v_{\mathcal{N}}(\mathsf{A})$. It is obvious that $\mathcal{N} \leq \mathcal{M}$ (respectively, $\mathcal{N} \preceq \mathcal{M}$), since $v_{\mathcal{N}}(\mathsf{A}) = N(\mathsf{A}) \leq M(\mathsf{A}) = v_{\mathcal{M}}(\mathsf{A})$ (respectively, $N(\mathsf{A}) \preceq M(\mathsf{A})$). Also, $\mathcal{N}$ is distinct from $\mathcal{M}$, since their valuation functions are distinct. As we showed in Step 1, the fact that N is a model of $\mathsf{Gr}(\mathsf{P})$ implies that $\mathcal{N}$ is a model of P, which is of course a contradiction, since $\mathcal{M}$ is a $\leq$-minimal (respectively, $\preceq$-minimal) model of P. Therefore, M must be a $\leq$-minimal (respectively, $\preceq$-minimal) model of $\mathsf{Gr}(\mathsf{P})$. $\square$

Appendix B Proof of Lemma 1

For the proof of Lemma 1, which we present in this appendix, we rely on the method of (Przymusinska and Przymusinski 1990) for the construction of the well-founded model. We first give the necessary definitions from (Przymusinska and Przymusinski 1990).

Definition 21
Let P be a propositional program and let J be an interpretation of P. The operator $\Theta_J(\cdot)$ on the set of interpretations of P is defined as follows: for every interpretation I and every propositional variable p of P,

$$\Theta_J(I)(\mathsf{p}) = \begin{cases} \mathit{true}, & \text{there exists a clause } \mathsf{p} \leftarrow \mathsf{L}_1, \ldots, \mathsf{L}_n \text{ in } \mathsf{P} \text{ s.t. for all } i \leq n, \\ & \text{either } J(\mathsf{L}_i) = \mathit{true} \text{ or } \mathsf{L}_i \text{ is a positive literal and } I(\mathsf{L}_i) = \mathit{true}; \\ \mathit{false}, & \text{for all clauses } \mathsf{p} \leftarrow \mathsf{L}_1, \ldots, \mathsf{L}_n \text{ in } \mathsf{P} \text{ there exists an } i \leq n, \text{ s.t.} \\ & \text{either } J(\mathsf{L}_i) = \mathit{false} \text{ or } \mathsf{L}_i \text{ is a positive literal and } I(\mathsf{L}_i) = \mathit{false}; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover we define the following sequence of interpretations:

$$\begin{array}{lcl} \Theta_J^{\uparrow 0} & = & \langle T_0, F_0 \rangle = \langle \emptyset, B_{\mathsf{P}} \rangle \\ \Theta_J^{\uparrow (n+1)} & = & \langle T_{n+1}, F_{n+1} \rangle = \Theta_J(\Theta_J^{\uparrow n}) \\ \Theta_J^{\uparrow \omega} & = & \langle T_\omega, F_\omega \rangle = \langle \bigcup_{n<\omega} T_\beta, \bigcap_{n<\omega} F_\beta \rangle \end{array}$$

It is shown in (Przymusinska and Przymusinski 1990) that, for any interpretation J, the operator Θ_J has a unique least fixed-point given by $\Theta_J^{\uparrow \omega}$.

Definition 22
Let P be a propositional program. For every countable ordinal $\alpha \leq \gamma$, we define the interpretation M_α as follows:

$$\begin{array}{lcl} M_0 & = & \langle T_0, F_0 \rangle = \langle \emptyset, \emptyset \rangle \\ M_{\alpha+1} & = & \langle T_{\alpha+1}, F_{\alpha+1} \rangle = \Theta_{M_\alpha}^{\uparrow \omega}, \text{ for a successor ordinal } \alpha + 1 \\ M_\alpha & = & \langle T_\alpha, F_\alpha \rangle = \langle \bigcup_{\beta<\alpha} T_\beta, \bigcup_{\beta<\alpha} F_\beta \rangle, \text{ for a limit ordinal } \alpha \end{array}$$

Again from (Przymusinska and Przymusinski 1990), there exists the least countable ordinal λ, such that $M_\lambda = \Theta_{M_\lambda}^{\uparrow \omega}$ and M_λ coincides with the well-founded model M_{P} of the propositional program P.

We now present the proof of Lemma 1.

Lemma 1
The Herbrand interpretation $\mathcal{M}_{\mathsf{P}}$ of the program of Example 4 is not extensional.

Proof
We repeat here the program of Example 4 for the reader's convenience:

```
s Q ← Q (s Q)
p R ← R
q R ← ∼(w R)
w R ← ∼R
```

Recall that the predicate variable Q is of type $o \to o$ and the predicate variable R is of type o. The ground instantiation of the above program is infinite, as so:

```
s p ← p (s p)
s q ← q (s q)
s w ← w (s w)
p (s p) ← (s p)
p (s q) ← (s q)
p (s w) ← (s w)
q (s p) ←∼(w (s p))
w (s p) ←∼(s p)
q (s q) ←∼(w (s q))
w (s q) ←∼(s q)
q (s w) ←∼(w (s w))
w (s w) ←∼(s w)
...
```

The well-founded model $M_{\mathsf{Gr(P)}}$ of the above (infinite) propositional program is the valuation function of $\mathcal{M}_{\mathsf{P}}$. It has already been argued in Section 6 and Section 8, that $M_{\mathsf{Gr(P)}}(\texttt{s p}) \neq M_{\mathsf{Gr(P)}}(\texttt{s q})$ and this should be obvious to the reader who is familiar with the well-founded model semantics; however, for reasons of completeness, we present a formal argument in the second part of this proof. In the same sections, we claimed that this is despite the fact that $\texttt{p} \cong_{M_{\mathsf{Gr(P)}}, o \to o} \texttt{q}$, which we will immediately proceed to prove. Of course, by Definitions 17 and 18, the facts that $M_{\mathsf{Gr(P)}}(\texttt{s p}) \neq M_{\mathsf{Gr(P)}}(\texttt{s q})$ and $\texttt{p} \cong_{M_{\mathsf{Gr(P)}}, o \to o} \texttt{q}$, render $\mathcal{M}_{\mathsf{P}}$ not extensional.

First, we show that $\texttt{p} \cong_{M_{\mathsf{Gr(P)}}, o \to o} \texttt{q}$, i.e. that for all $\mathsf{A}, \mathsf{A}' \in U_{\mathsf{P},o}$ such that $\mathsf{A} \cong_{M_{\mathsf{Gr(P)}}, o} \mathsf{A}'$, $\texttt{p A} \cong_{M_{\mathsf{Gr(P)}}, o} \texttt{q A}'$ holds. By definition $M_{\mathsf{Gr(P)}}$ is a fixed-point of the operator $\Theta_{M_{\mathsf{Gr(P)}}}(\cdot)$, therefore for any ground atom B we have that $M_{\mathsf{Gr(P)}}(\mathsf{B})$ equals to *true*, if there exists a clause $\mathsf{B} \leftarrow \mathsf{L}_1, \ldots, \mathsf{L}_n$ in $\mathsf{Gr(P)}$, such that $M_{\mathsf{Gr(P)}}(\mathsf{L}_i) = \textit{true}$ for all $i \leq n$; it equals to *false* if for every clause $\mathsf{B} \leftarrow \mathsf{L}_1, \ldots, \mathsf{L}_n$ in $\mathsf{Gr(P)}$, we have that $M_{\mathsf{Gr(P)}}(\mathsf{L}_i) = \textit{false}$ for at least one $i \leq n$; and it equals to 0 otherwise. Observe that there exists only one clause in $\mathsf{Gr(P)}$ such that $\texttt{p A}$ is the head of the clause, in particular it is the ground instance $\texttt{p A} \leftarrow \mathsf{A}$ of the clause $\texttt{p R} \leftarrow \texttt{R}$ of P. This suggests that $M_{\mathsf{Gr(P)}}(\texttt{p A}) = M_{\mathsf{Gr(P)}}(\mathsf{A})$ (1). Similarly, the ground instance $\texttt{q A}' \leftarrow \sim(\texttt{w A}')$ of the clause $\texttt{q R} \leftarrow \sim(\texttt{w R})$ is the only clause in $\mathsf{Gr(P)}$ with $\texttt{q A}'$ as its head atom and from this we can infer that $M_{\mathsf{Gr(P)}}(\texttt{q A}') = M_{\mathsf{Gr(P)}}(\sim(\texttt{w A}')) = \neg M_{\mathsf{Gr(P)}}(\texttt{w A}')$ (2). Finally, the only clause in $\mathsf{Gr(P)}$, such that $\texttt{w A}'$ is the head of the clause, is the ground instance $\texttt{w A}' \leftarrow \sim\mathsf{A}'$ of the clause $\texttt{w R} \leftarrow \sim\texttt{R}$ of P, which implies that $M_{\mathsf{Gr(P)}}(\texttt{w A}') = M_{\mathsf{Gr(P)}}(\sim\mathsf{A}') = \neg M_{\mathsf{Gr(P)}}(\mathsf{A}')$ (3). By (2) and (3) we have $M_{\mathsf{Gr(P)}}(\texttt{q A}') = M_{\mathsf{Gr(P)}}(\mathsf{A}')$, and, in conjunction with (1), that $M_{\mathsf{Gr(P)}}(\texttt{p A}) = M_{\mathsf{Gr(P)}}(\texttt{q A}')$, because $\mathsf{A} \cong_{M_{\mathsf{Gr(P)}}, o} \mathsf{A}'$ implies, by Definition 17, that $M_{\mathsf{Gr(P)}}(\mathsf{A}) = M_{\mathsf{Gr(P)}}(\mathsf{A}')$. Therefore, we also have $\texttt{p A} \cong_{M_{\mathsf{Gr(P)}}, o} \texttt{q A}'$ and, in consequence, $\texttt{p} \cong_{M_{\mathsf{Gr(P)}}, o \to o} \texttt{q}$.

For the second part, we show that $M_{\mathsf{Gr(P)}}(\texttt{s p}) \neq M_{\mathsf{Gr(P)}}(\texttt{s q})$. We do this in two steps; first we show that $M_{\mathsf{Gr(P)}}(\texttt{s p}) = \textit{false}$ and then that $M_{\mathsf{Gr(P)}}(\texttt{s q}) = 0$.

For the first step, it suffices to show that $M_1(\texttt{s p}) = \Theta_{M_0}^{\uparrow\omega}(\texttt{s p}) = \textit{false}$. For this, we prove that $\Theta_{M_0}^{\uparrow n}(\texttt{s p}) = \textit{false}$ and $\Theta_{M_0}^{\uparrow n}(\texttt{p (s p)}) = \textit{false}$, for all $n < \omega$, by an induction on n. The basis case is trivial, as $\Theta_{M_0}^{\uparrow 0} = \langle \emptyset, U_{\mathsf{P},o} \rangle$, assigns the value *false* to every atom. For the induction step, we show the statement for $n+1$ assuming that it holds for n. We see that there exists only one clause in $\mathsf{Gr}(\mathsf{P})$ such that `s p` is the head of the clause; this is the ground instance `s p` $\leftarrow$ `p (s p)` of the clause `s Q` $\leftarrow$ `Q (s Q)` of P. By the induction hypothesis, we have that $\Theta_{M_0}^{\uparrow n}(\texttt{p (s p)}) = \textit{false}$, therefore $\Theta_{M_0}^{\uparrow (n+1)}(\texttt{s p}) = \textit{false}$. Similarly, the only clause in $\mathsf{Gr}(\mathsf{P})$ with `p (s p)` as the head of the clause is the ground instance `p (s p)` $\leftarrow$ `(s p)` of the clause `p R` $\leftarrow$ `R` of P. By the induction hypothesis, we have that $\Theta_{M_0}^{\uparrow n}(\texttt{s p}) = \textit{false}$, therefore $\Theta_{M_0}^{\uparrow (n+1)}(\texttt{p (s p)}) = \textit{false}$.

For the second step, we perform an induction on α, during which we simultaneously show that $M_\alpha(\texttt{s q}) = 0$, $M_\alpha(\texttt{q (s q)}) = 0$ and $M_\alpha(\texttt{w (s q)}) = 0$, for all countable ordinals α. The basis case is trivial, as $M_0 = \langle \emptyset, \emptyset \rangle$ assigns the value 0 to all atoms. For the induction step, we first prove the statement for a successor ordinal $\alpha+1$, assuming that it holds for all countable ordinals up to α. Indeed, there exists exactly one clause in $\mathsf{Gr}(\mathsf{P})$ with `w (s q)` as its head atom, in particular the ground instance `w (s q)` $\leftarrow \sim$`(s q)` of the clause `q R` $\leftarrow \sim$`(w R)`. As $\sim$`(s q)` is a negative literal, for every $n < \omega$ the value of $\Theta_{M_\alpha}^{\uparrow (n+1)}(\texttt{w (s q)})$ is defined by $M_\alpha(\sim\texttt{(s q)})$. By the induction hypothesis, we have $M_\alpha(\texttt{s q}) = M_\alpha(\sim\texttt{(s q)}) = 0$, therefore it follows that $\Theta_{M_\alpha}^{\uparrow (n+1)}(\texttt{w (s q)}) = 0$ and, because this holds for every $n < \omega$, that $M_{\alpha+1}(\texttt{w (s q)}) = 0$. Moreover, the ground instance `q (s q)` $\leftarrow \sim$`(w (s q))` of the clause `q R` $\leftarrow \sim$`(w R)` is the only clause in $\mathsf{Gr}(\mathsf{P})$ with `q (s q)` as its head atom. Again, $\sim$`(w (s q))` is a negative literal and so for every $n < \omega$ the value of $\Theta_{M_\alpha}^{\uparrow (n+1)}(\texttt{q (s q)})$ only depends on $M_\alpha(\sim\texttt{(w (s q))})$. By the induction hypothesis, we have $M_\alpha(\texttt{w (s q)}) = M_\alpha(\sim\texttt{(w (s q))}) = 0$, therefore it follows that $\Theta_{M_\alpha}^{\uparrow (n+1)}(\texttt{q (s q)}) = 0$. Since this holds for every $n < \omega$, we also have that $M_{\alpha+1}(\texttt{q (s q)}) = 0$. Finally, there exists only one clause in $\mathsf{Gr}(\mathsf{P})$ such that `s q` is the head of the clause, in particular the ground instance `s q` $\leftarrow$ `q (s q)` of the clause `s Q` $\leftarrow$ `Q (s Q)` of P. We have already shown that $\Theta_{M_\alpha}^{\uparrow (n+1)}(\texttt{q (s q)}) = 0$ for all $n < \omega$; moreover, by the induction hypothesis, $M_\alpha(\texttt{q (s q)}) = 0$. Consequently, for all $n < \omega$, $\Theta_{M_\alpha}^{\uparrow (n+2)}(\texttt{s q}) = 0$ and thus $M_{\alpha+1}(\texttt{s q}) = 0$. It remains to show $M_\alpha(\texttt{s q}) = 0$, $M_\alpha(\texttt{q (s q)}) = 0$ and $M_\alpha(\texttt{w (s q)}) = 0$ for a limit ordinal α. In this case, we have that $M_\alpha = \langle \bigcup_{\beta<\alpha} T_\beta, \bigcup_{\beta<\alpha} F_\beta \rangle$. By the induction hypothesis, $M_\beta(\texttt{s q}) = 0$ for all $\beta < \alpha$, which means that `s q` $\notin T_\beta$ and `s q` $\notin F_\beta$. In other words, `s q` $\notin \bigcup_{\beta<\alpha} T_\beta$ and `s q` $\notin \bigcup_{\beta<\alpha} F_\beta$, therefore $M_\alpha(\texttt{s q}) = 0$. In the same way we can show that $M_\alpha(\texttt{q (s q)}) = 0$ and $M_\alpha(\texttt{w (s q)}) = 0$. This concludes the induction and so we have proven that $M_{\mathsf{Gr}(\mathsf{P})}(\texttt{s q}) = 0$. $\square$

Appendix C Proof of Theorem 2

Before we proceed with the proof of Theorem 2, we recall some necessary definitions. Note that the proof makes use of a fixed-point characterization of the perfect model semantics given in (Przymusinska and Przymusinski 1990), rather than the more traditional definition of (Przymusinski 1988). The following definition of the local stratification of possibly infinite propositional programs is adapted to allow for the presence of expressions of the form $(\mathsf{E}_1 \approx \mathsf{E}_2)$.

Definition 23
A propositional program P is called *locally stratified* if and only if it is possible to decom-

pose the Herbrand base B_{P} of P into disjoint sets (called *strata*) $S_1, S_2, \ldots, S_\alpha, \ldots, \alpha < \gamma$, where γ is a countable ordinal, such that for every clause $\mathsf{H} \leftarrow \mathsf{A}_1, \ldots, \mathsf{A}_m, {\sim}\mathsf{B}_1, \ldots, {\sim}\mathsf{B}_n$ in P, we have that for every $i \leq m$, $\mathit{stratum}(\mathsf{A}_i) \leq \mathit{stratum}(\mathsf{H})$ and for every $i \leq n$, $\mathit{stratum}(\mathsf{B}_i) < \mathit{stratum}(\mathsf{H})$, where *stratum* is a function such that $\mathit{stratum}(\mathsf{C}) = \beta$, if the propositional variable $\mathsf{C} \in B_{\mathsf{P}}$ belongs to S_β and $\mathit{stratum}(\mathsf{C}) = 0$, if $\mathsf{C} \notin B_{\mathsf{P}}$ and is a constant (equivalently, of the form $(\mathsf{E}_1 \approx \mathsf{E}_2)$). Any decomposition of the described form is called a *local stratification* of P.

In (Przymusinska and Przymusinski 1990), for any interpretation J, the operator Ψ_J is defined and shown to have a unique least fixed-point given by $\Psi_J^{\uparrow\omega}$ of the next definition. This is then used to give an iterated fixed-point characterization of the perfect model of a locally stratified program.

Definition 24
Let P be a propositional program and let J be an interpretation of P. The operator $\Psi_J : 2^{B_{\mathsf{P}}} \to 2^{B_{\mathsf{P}}}$ is defined as follows: for every $I \subseteq B_{\mathsf{P}}$, $\Psi_J(I) = \{\mathsf{p} \in B_{\mathsf{P}} \mid$ there exists a clause $\mathsf{p} \leftarrow \mathsf{L}_1, \ldots, \mathsf{L}_n$ in P such that, for all $i \leq n$, either $J(\mathsf{L}_i) = \mathit{true}$ or $\mathsf{L}_i \in I\}$. Moreover we define the following sequence:

$$\begin{array}{lcl} \Psi_J^{\uparrow 0} & = & \emptyset \\ \Psi_J^{\uparrow (n+1)} & = & \Psi_J(\Psi_J^{\uparrow n}) \\ \Psi_J^{\uparrow \omega} & = & \bigcup_{n<\omega} \Psi_J^{\uparrow n} \end{array}$$

We follow the usual convention of identifying a subset of the Herbrand base with a total (two-valued) interpretation of the program and use the two notions interchangeably. E.g., a set $\Psi_J^{\uparrow n}$ of the above sequence is considered to be equivalent to the interpretation $\langle \Psi_J^{\uparrow n}, B_{\mathsf{P}} - \Psi_J^{\uparrow n} \rangle$ and $\mathsf{p} \in \Psi_J^{\uparrow n}$ is considered to be equivalent to $\Psi_J^{\uparrow n}(\mathsf{p}) = \mathit{true}$.

Given a local stratification $S_1, S_2, \ldots, S_\alpha, \ldots, \alpha < \gamma$, of a propositional program P, we define the sets $\mathcal{B}_\alpha = \bigcup_{\beta<\alpha} S_\beta$ for every countable ordinal $\alpha \leq \gamma$. Clearly, $B_{\mathsf{P}} = \mathcal{B}_\gamma$. Then the perfect model of P can be constructed (Przymusinska and Przymusinski 1990) as the last interpretation N_γ in an $\preceq$-increasing sequence of partial interpretations of P:

Definition 25
Let P be a propositional program and let $S_1, S_2, \ldots, S_\alpha, \ldots, \alpha < \gamma$, where γ is a countable ordinal, be a local stratification of P. For every countable ordinal $\alpha \leq \gamma$, we define the interpretation N_α as follows:

$$\begin{array}{lcl} N_0 & = & \langle T_0, F_0 \rangle = \langle \emptyset, \emptyset \rangle \\ N_{\alpha+1} & = & \langle T_{\alpha+1}, F_{\alpha+1} \rangle = \langle \Psi_{N_\alpha}^{\uparrow\omega}, \mathcal{B}_{\alpha+1} - \Psi_{N_\alpha}^{\uparrow\omega} \rangle, \text{ for a successor ordinal } \alpha+1 \\ N_\alpha & = & \langle T_\alpha, F_\alpha \rangle = \langle \bigcup_{\beta<\alpha} T_\beta, \bigcup_{\beta<\alpha} F_\beta \rangle, \text{ for a limit ordinal } \alpha \end{array}$$

Theorem 3 (Przymusinska and Przymusinski 1990)
Let P be a propositional program. The sequence $N_0, N_1, \ldots, N_\alpha, \ldots, N_\gamma$ is $\preceq$-increasing. Moreover, N_γ coincides with the perfect model N_{P} of P.

The next lemma and its following corollary are the basis for our proof of Theorem 2.

Lemma 2
Let P be an $\mathcal{H}$ program. If P is stratified then the ground instantiation $\mathsf{Gr}(\mathsf{P})$ of P is locally stratified.

Proof
Consider a decomposition $S_1, \ldots, S_r$ of the set of predicate constants of P such that the requirements of Definition 19 are satisfied. This defines a decomposition $S'_1, \ldots, S'_r$ of $U_{\mathsf{P},o}$, which is also the Herbrand base of $\mathsf{Gr}(\mathsf{P})$, as follows:

$$S'_i = \{\mathsf{A} \in U_{\mathsf{P},o} \mid \text{the leftmost predicate constant of } \mathsf{A} \text{ belongs to } S_i\}$$

It is easy to check that $S'_1, \ldots, S'_r$ corresponds to a local stratification of $\mathsf{Gr}(\mathsf{P})$. □

An immediate result of the above lemma is that the model $\mathcal{N}_\mathsf{P}$ can be defined for every stratified program of $\mathcal{H}$:

Corollary 1
Let P be an $\mathcal{H}$ program. If P is stratified, then the perfect model $\mathcal{N}_\mathsf{P}$ of P exists and coincides with its well-founded model $\mathcal{M}_\mathsf{P}$.

Proof
By Lemma 2, if P is stratified then $\mathsf{Gr}(\mathsf{P})$ is locally stratified. Therefore the unique perfect model $N_{\mathsf{Gr}(\mathsf{P})}$ of $\mathsf{Gr}(\mathsf{P})$ exists (Przymusinski 1988) and $\mathcal{N}_\mathsf{P}$ is defined. Moreover, the perfect model of a locally stratified propositional program, which is the valuation function of $\mathcal{N}_\mathsf{P}$, coincides with its well-founded model $M_{\mathsf{Gr}(\mathsf{P})}$ (Przymusinski 1988), i.e. the valuation function of $\mathcal{M}_\mathsf{P}$. In other words, in this case $\mathcal{N}_\mathsf{P}$ and $\mathcal{M}_\mathsf{P}$ coincide, because they have the same valuation function. □

Theorem 2
The well-founded model $\mathcal{M}_\mathsf{P}$ of a stratified program P is extensional.

Proof
By Corollary 1, if P is stratified then $\mathcal{M}_\mathsf{P}$ coincides with $\mathcal{N}_\mathsf{P}$. Therefore, it suffices to show that $\mathcal{N}_\mathsf{P}$ is extensional and for this we rely upon the constructive definition of $N_{\mathsf{Gr}(\mathsf{P})}$ from (Przymusinska and Przymusinski 1990) presented above.

Consider a stratification $S_1, \ldots, S_r$ of the set of predicate constants of P. As argued in the proof of Lemma 2, the following decomposition $S'_1, \ldots, S'_r$ of $U_{\mathsf{P},o}$:

$$S'_i = \{\mathsf{A} \in U_{\mathsf{P},o} \mid \text{the leftmost predicate constant of } \mathsf{A} \text{ belongs to } S_i\}$$

corresponds to a local stratification of $\mathsf{Gr}(\mathsf{P})$. Therefore, whenever a ground atom A begins with a predicate constant p, we will have $stratum(\mathsf{A}) = stratum(\mathsf{p})$. Moreover, by Theorem 3, $N_{\mathsf{Gr}(\mathsf{P})} = N_r$.

Since the valuation function of $\mathcal{N}_\mathsf{P}$ is $N_{\mathsf{Gr}(\mathsf{P})}$, essentially we need to show that $\mathsf{E} \cong_{N_{\mathsf{Gr}(\mathsf{P})},\rho} \mathsf{E}$, for every ground expression E of every argument type ρ. We perform an induction on the structure of ρ. For the base types ι and o the statement holds by definition. For the induction step, we prove the statement for a predicate type $\pi = \rho_1 \to \cdots \to \rho_m \to o$, assuming that it holds for all types simpler than π (i.e., for the types $\rho_1, \ldots, \rho_m, o$ and, recursively, the types that are simpler than $\rho_1, \ldots, \rho_m$). Let A be any atom of the following form: A is headed by a predicate constant p and all variables in $vars(\mathsf{A})$ are of types simpler than π. Let θ, θ' be ground substitutions, such that $vars(\mathsf{A}) \subseteq dom(\theta), dom(\theta')$ and $\theta(\mathsf{V}) \cong_{N_{\mathsf{Gr}(\mathsf{P})},\rho} \theta'(\mathsf{V})$ for any $\mathsf{V} : \rho$ in $vars(\mathsf{A})$. We claim it suffices to show the following two properties $P_1(\alpha)$ and $P_2(\alpha)$, for all finite ordinals (i.e., natural numbers) α:

$P_1(\alpha)$**:** If $N_\alpha(\mathsf{A}\theta) = \mathit{true}$ then $N_{\mathsf{Gr(P)}}(\mathsf{A}\theta') = \mathit{true}$.
$P_2(\alpha)$**:** If $N_\alpha(\mathsf{A}\theta) = \mathit{false}$ then $N_{\mathsf{Gr(P)}}(\mathsf{A}\theta') = \mathit{false}$.

To see why proving the above properties is enough to establish that $\mathsf{E} \cong_{N_{\mathsf{Gr(P)}},\pi} \mathsf{E}$, observe the following: first of all, we assumed that π is of the form $\rho_1 \to \cdots \to \rho_m \to o$, so if $\mathsf{V}_1 : \rho_1, \ldots, \mathsf{V}_m : \rho_m$ are variables, then $\mathsf{E}\,\mathsf{V}_1 \cdots \mathsf{V}_m$ is an atom of the form described above. As $N_{\mathsf{Gr(P)}} = N_r$, if $N_{\mathsf{Gr(P)}}(\mathsf{E}\,\theta(\mathsf{V}_1) \cdots \theta(\mathsf{V}_m)) = N_r(\mathsf{E}\,\theta(\mathsf{V}_1) \cdots \theta(\mathsf{V}_m)) = \mathit{true}$ and property $P_1(r)$ holds, then we can infer that $N_{\mathsf{Gr(P)}}(\mathsf{E}\,\theta'(\mathsf{V}_1) \cdots \theta'(\mathsf{V}_m)) = \mathit{true}$. Because the relations $\cong_{N_{\mathsf{Gr(P)}},\rho_i}$ are symmetric, θ and θ' are interchangeable. Therefore the same argument can be used to infer the reverse implication, i.e. $N_{\mathsf{Gr(P)}}(\mathsf{E}\,\theta'(\mathsf{V}_1) \cdots \theta'(\mathsf{V}_m)) = \mathit{true} \Rightarrow N_{\mathsf{Gr(P)}}(\mathsf{E}\,\theta(\mathsf{V}_1) \cdots \theta(\mathsf{V}_m)) = \mathit{true}$, thus resulting to an equivalence. If $P_2(r)$ holds, the analogous equivalence can be shown for the value *false* in the same way and so it follows that $\mathsf{E} \cong_{N_{\mathsf{Gr(P)}},\pi} \mathsf{E}$. Finally, r is determined by the stratification of the higher-order program and is therefore finite, so we only need to prove properties $P_1(\alpha)$ and $P_2(\alpha)$ for finite ordinals.

We will proceed by a second induction on α.

Second Induction Basis ($\alpha = 0$) We have $N_0 = \langle \emptyset, \emptyset \rangle$. As this interpretation does not assign the value *true* or the value *false* to any atom, both properties $P_1(0)$ and $P_2(0)$ hold vacuously.

Second Induction Step ($\alpha + 1$) We first show $P_1(\alpha + 1)$. We have that $N_{\alpha+1} = \langle \Psi_{N_\alpha}^{\uparrow\omega}, \mathcal{B}_{\alpha+1} - \Psi_{N_\alpha}^{\uparrow\omega} \rangle$; observe that $\Psi_{N_\alpha}^{\uparrow\omega}(\mathsf{A}\theta) = \mathit{true}$ if and only if there exists some $n < \omega$ for which $\Psi_{N_\alpha}^{\uparrow n}(\mathsf{A}\theta) = \mathit{true}$. Therefore, in order to prove $P_1(\alpha + 1)$, we first need to perform a third induction on n and prove the following property:

$P_1'(\alpha + 1, n)$**:** If $\Psi_{N_\alpha}^{\uparrow n}(\mathsf{A}\theta) = \mathit{true}$ then $N_{\mathsf{Gr(P)}}(\mathsf{A}\theta') = \mathit{true}$.

Third Induction Basis ($n = 0$) Property $P_1'(\alpha+1, 0)$ holds vacuously, since $\Psi_{M_\alpha}^{\uparrow 0} = \emptyset$, i.e. it does not assign the value *true* to any atom.

Third Induction Step ($n+1$) We now show property $P_1'(\alpha+1, n+1)$, assuming that $P_1'(\alpha + 1, n)$ holds. If $\Psi_{N_\alpha}^{\uparrow(n+1)}(\mathsf{A}\theta) = \mathit{true}$, then there exists a clause $\mathsf{A}\theta \leftarrow \mathsf{L}_1, \ldots, \mathsf{L}_k$ in $\mathsf{Gr(P)}$ such that, for each $i \le k$, either $N_\alpha(\mathsf{L}_i) = \mathit{true}$ or L_i is an atom and $\Psi_{N_\alpha}^{\uparrow n}(\mathsf{L}_i) = \mathit{true}$. This clause is a ground instance of a clause $\mathsf{p}\,\mathsf{V}_1 \cdots \mathsf{V}_m \leftarrow \mathsf{K}_1, \ldots, \mathsf{K}_k$ in the higher-order program and there exists a substitution θ'', such that $(\mathsf{p}\,\mathsf{V}_1 \cdots \mathsf{V}_m)\theta'' = \mathsf{A}$ and, for any variable $\mathsf{V} \notin \{\mathsf{V}_1, \ldots, \mathsf{V}_m\}$ appearing in the body of the clause, $\theta''(\mathsf{V})$ is an appropriate ground term, so that $\mathsf{L}_i = \mathsf{K}_i\theta''\theta$ for all $i \le k$. Observe that the variables appearing in the clause $(\mathsf{p}\,\mathsf{V}_1 \cdots \mathsf{V}_m)\theta'' \leftarrow \mathsf{K}_1\theta'', \ldots, \mathsf{K}_k\theta''$ are exactly the variables appearing in A and they are all of types simpler than π. We distinguish the following cases for each $\mathsf{K}_i\theta''$, $i \le k$:

1. $\mathsf{K}_i\theta''$ *is of the form* $(\mathsf{E}_1 \approx \mathsf{E}_2)$*:* As remarked in Appendix A, an expression of the form $(\mathsf{E}_1 \approx \mathsf{E}_2)$ has the same value in any interpretation. If $N_\alpha(\mathsf{K}_i\theta''\theta) = \Psi_{N_\alpha}^{\uparrow n}(\mathsf{K}_i\theta''\theta) = \mathit{true}$, by definition we have $\mathsf{E}_1\theta = \mathsf{E}_2\theta$. Since E_1 and E_2 are expressions of type ι, all variables in E_1 and E_2 are also of type ι and, because $\cong_{N_{\mathsf{Gr(P)}},\iota}$ is defined as equality, we will have $\mathsf{E}_1\theta = \mathsf{E}_1\theta'$ and $\mathsf{E}_2\theta = \mathsf{E}_2\theta'$. Therefore $\mathsf{E}_1\theta' = \mathsf{E}_2\theta'$ and $N_{\mathsf{Gr(P)}}(\mathsf{K}_i\theta''\theta') = \mathit{true}$ will also hold.
2. $\mathsf{K}_i\theta''$ *is an atom and starts with a predicate constant:* As we observed, the variables appearing in $\mathsf{K}_i\theta''$ are of types simpler than π. Because $\mathsf{K}_i\theta''\theta$ is an atom, either $N_\alpha(\mathsf{L}_i) = N_\alpha(\mathsf{K}_i\theta''\theta) = \mathit{true}$ or $\Psi_{N_\alpha}^{\uparrow n}(\mathsf{L}_i) = \Psi_{N_\alpha}^{\uparrow n}(\mathsf{K}_i\theta''\theta) = \mathit{true}$ may hold. In the former case,

by the second induction hypothesis we can apply property $P_1(\alpha)$ and it follows that $N_{\mathsf{Gr(P)}}(\mathsf{K}_i\theta''\theta') = \mathit{true}$. Similarly, in the latter case, the same conclusion can be reached by the third induction hypothesis and property $P'_1(\alpha+1, n)$.

3. $\mathsf{K}_i\theta''$ *is an atom and starts with a predicate variable:* As in the previous case, it may be $N_\alpha(\mathsf{L}_i) = N_\alpha(\mathsf{K}_i\theta''\theta) = \mathit{true}$ or $\Psi^{\uparrow n}_{N_\alpha}(\mathsf{L}_i) = \Psi^{\uparrow n}_{N_\alpha}(\mathsf{K}_i\theta''\theta) = \mathit{true}$. Let $\mathsf{K}_i\theta'' = \mathsf{V}\,\mathsf{E}_1\,\cdots\,\mathsf{E}_{m'}$ for some $\mathsf{V} \in \mathit{vars}(\mathsf{A})$. Then $\mathsf{B} = \theta(\mathsf{V})\,\mathsf{E}_1\,\cdots\,\mathsf{E}_{m'}$ is an atom that begins with a predicate constant and, by $\mathit{vars}(\mathsf{K}_i\theta'') \subseteq \mathit{vars}(\mathsf{A})$, all of the variables of B are of types simpler than π. Hence, by the second induction hypothesis, B satisfies property $P_1(\alpha)$ and if $N_\alpha(\mathsf{K}_i\theta''\theta) = N_\alpha(\mathsf{B}\theta) = \mathit{true}$ then it follows that $N_{\mathsf{Gr(P)}}(\mathsf{B}\theta') = \mathit{true}$ (1). Similarly, by the third induction hypothesis, B also satisfies property $P'_1(\alpha+1, n)$, so if $\Psi^{\uparrow n}_{N_\alpha}(\mathsf{K}_i\theta''\theta) = \Psi^{\uparrow n}_{N_\alpha}(\mathsf{B}\theta) = \mathit{true}$, then the same conclusion, that $N_{\mathsf{Gr(P)}}(\mathsf{B}\theta') = \mathit{true}$ (1), is reached again. Observe that the types of all arguments of $\theta(\mathsf{V})$, i.e. the types of $\mathsf{E}_j\theta'$ for all $j \leq m'$, are simpler than the type of V and consequently, since $\mathsf{V} \in \mathit{vars}(\mathsf{A})$, simpler than π. For each $j \leq m'$, let ρ_j be the type of E_j and let ρ be the type of V; by the first induction hypothesis, $\mathsf{E}_j\theta' \cong_{N_{\mathsf{Gr(P)}},\rho_j} \mathsf{E}_j\theta'$. Moreover, by assumption we have that $\theta(\mathsf{V}) \cong_{N_{\mathsf{Gr(P)}},\rho} \theta'(\mathsf{V})$. Then, by definition and by (1) $N_{\mathsf{Gr(P)}}(\theta(\mathsf{V})\,\mathsf{E}_1\theta'\,\cdots\,\mathsf{E}_{m'}\theta') = N_{\mathsf{Gr(P)}}(\theta'(\mathsf{V})\,\mathsf{E}_1\theta'\,\cdots\,\mathsf{E}_{m'}\theta') = N_{\mathsf{Gr(P)}}(\mathsf{K}_i\theta''\theta') = \mathit{true}$.
4. $\mathsf{K}_i\theta''$ *is a negative literal and its atom starts with a predicate constant:* Let $\mathsf{K}_i\theta''$ be of the form $\sim\mathsf{B}$, where B is an atom that starts with a predicate constant. It is $N_\alpha(\sim\mathsf{B}\theta) = N_\alpha(\mathsf{K}_i\theta''\theta) = N_\alpha(\mathsf{L}_i) = \mathit{true}$ and therefore $N_\alpha(\mathsf{B}\theta) = \mathit{false}$. Moreover, by $\mathit{vars}(\mathsf{K}_i\theta'') \subseteq \mathit{vars}(\mathsf{A})$, all the variables of B are of types simpler than π, so we can apply the second induction hypothesis, in particular property $P_2(\alpha)$, to B and conclude that $N_{\mathsf{Gr(P)}}(\mathsf{B}\theta') = \mathit{false}$. Then $N_{\mathsf{Gr(P)}}(\sim\mathsf{B}\theta') = N_{\mathsf{Gr(P)}}(\mathsf{K}_i\theta''\theta') = \mathit{true}$.
5. $\mathsf{K}_i\theta''$ *is a negative literal and its atom starts with a predicate variable:* Let $\mathsf{K}_i\theta'' = \sim(\mathsf{V}\,\mathsf{E}_1\,\cdots\,\mathsf{E}_{m'})$ for some $\mathsf{V} \in \mathit{vars}(\mathsf{A})$. Then $\mathsf{B} = \theta(\mathsf{V})\,\mathsf{E}_1\,\cdots\,\mathsf{E}_{m'}$ is an atom that begins with a predicate constant and, by $\mathit{vars}(\mathsf{K}_i\theta'') \subseteq \mathit{vars}(\mathsf{A})$, all of the variables of B are of types simpler than π. Also, $N_\alpha(\sim\mathsf{B}\theta) = N_\alpha(\mathsf{K}_i\theta''\theta) = N_\alpha(\mathsf{L}_i) = \mathit{true}$ and therefore $N_\alpha(\mathsf{B}\theta) = \mathit{false}$. Hence, by the second induction hypothesis and in particular property $P_2(\alpha)$, it follows that $N_{\mathsf{Gr(P)}}(\mathsf{B}\theta') = N_{\mathsf{Gr(P)}}(\theta(\mathsf{V})\,\mathsf{E}_1\theta'\,\cdots\,\mathsf{E}_{m'}\theta') = \mathit{false}$ (1). Observe that the types of all arguments of $\theta(\mathsf{V})$, i.e. the types of $\mathsf{E}_j\theta'$ for all $j \leq m'$, are simpler than the type of V and consequently, since $\mathsf{V} \in \mathit{vars}(\mathsf{A})$, simpler than π. For each $j \leq m'$, let ρ_j be the type of E_j and let ρ be the type of V; by the first induction hypothesis, $\mathsf{E}_j\theta' \cong_{N_{\mathsf{Gr(P)}},\rho_j} \mathsf{E}_j\theta'$. Moreover, by assumption we have that $\theta(\mathsf{V}) \cong_{N_{\mathsf{Gr(P)}},\rho} \theta'(\mathsf{V})$. Then, by definition and by (1), $N_{\mathsf{Gr(P)}}(\theta(\mathsf{V})\,\mathsf{E}_1\theta'\,\cdots\,\mathsf{E}_{m'}\theta') = N_{\mathsf{Gr(P)}}(\theta'(\mathsf{V})\,\mathsf{E}_1\theta'\,\cdots\,\mathsf{E}_{m'}\theta') = \mathit{false}$. Obviously, this makes $N_{\mathsf{Gr(P)}}(\sim(\theta'(\mathsf{V})\,\mathsf{E}_1\theta'\,\cdots\,\mathsf{E}_{m'}\theta')) = N_{\mathsf{Gr(P)}}(\mathsf{K}_i\theta''\theta') = \mathit{true}$.

We have shown that, for each $i \leq k$, $N_{\mathsf{Gr(P)}}(\mathsf{K}_i\theta''\theta') = \mathit{true}$. Since the clause $\mathsf{A}\theta' \leftarrow \mathsf{K}_1\theta''\theta', \ldots, \mathsf{K}_k\theta''\theta'$ is in $\mathsf{Gr(P)}$ and $N_{\mathsf{Gr(P)}}$ is a model of $\mathsf{Gr(P)}$, we can conclude that $N_{\mathsf{Gr(P)}}(\mathsf{A}\theta') = \mathit{true}$.

This concludes the proof for $P'_1(\alpha{+}1, n)$. Notice that property $P'_1(\alpha{+}1, n)$ immediately implies property $P_1(\alpha{+}1)$: as mentioned before, $N_{\alpha+1}(\mathsf{A}\theta) = \Psi^{\uparrow\omega}_{N_\alpha}(\mathsf{A}\theta) = \mathit{true}$ if and only if there exists some $n < \omega$ for which $\Psi^{\uparrow n}_{N_\alpha}(\mathsf{A}\theta) = \mathit{true}$ and then $N_{\mathsf{Gr(P)}}(\mathsf{A}\theta') = \mathit{true}$ follows from property $P'_1(\alpha{+}1, n)$. It remains to prove property $P_2(\alpha{+}1)$. Observe that the atoms $\mathsf{A}\theta$ and $\mathsf{A}\theta'$ both start with the same predicate constant p and recall that we have chosen a local stratification for $\mathsf{Gr(P)}$, such that $\mathit{stratum}(\mathsf{A}\theta) = \mathit{stratum}(\mathsf{A}\theta') = \mathit{stratum}(\mathsf{p})$. Moreover, we have that $N_{\alpha+1} = \langle \Psi^{\uparrow\omega}_{N_\alpha}, \mathcal{B}_{\alpha+1} - \Psi^{\uparrow\omega}_{N_\alpha} \rangle$, so if $N_{\alpha+1}(\mathsf{A}\theta) = \mathit{false}$, it follows that $\mathsf{A}\theta \in \mathcal{B}_{\alpha+1}$. Because $stratum(\mathsf{A}\theta) = stratum(\mathsf{A}\theta')$, it must also be $\mathsf{A}\theta' \in \mathcal{B}_{\alpha+1}$, which implies that $N_{\alpha+1}(\mathsf{A}\theta')$ can be either *true* (if $\mathsf{A}\theta' \in \Psi_{N_\alpha}^{\uparrow\omega}$) or *false* (if $\mathsf{A}\theta' \notin \Psi_{N_\alpha}^{\uparrow\omega}$), but not 0. For the sake of contradiction, assume that $N_{\alpha+1}(\mathsf{A}\theta') = true$. As the relations $\cong_{N_{\mathsf{Gr(P)}},\rho_i}$ are symmetric, θ and θ' are interchangeable, so property $P_1(\alpha+1)$ applies and yields $N_{\mathsf{Gr(P)}}(\mathsf{A}\theta) = true$. Because (by Theorem 3) $N_{\alpha+1} \preceq N_{\mathsf{Gr(P)}}$, this contradicts our initial assumption that $N_{\alpha+1}(\mathsf{A}\theta) = false$. Therefore, it must be $N_{\alpha+1}(\mathsf{A}\theta') = false$ and so, again by $N_{\alpha+1} \preceq N_{\mathsf{Gr(P)}}$, it follows that $N_{\mathsf{Gr(P)}}(\mathsf{A}\theta') = false$. $\square$