Online appendix for the paper

Program Completion in the Input Language of GRINGO

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AMELIA HARRISON, VLADIMIR LIFSCHITZ

DHANANJAY RAJU

University of Texas

(*e-mail:* ameliaj,vl,draju@cs.utexas.edu)

Appendix B Proofs

B.1 Relationship between ϕ and τ

To prove Theorems 1 and 2, we need to investigate the relationship between the operator ϕ used in the definition of completion (Section 5) and the operator τ that the semantics of programs is based on (Section A.2).

If $\mathbf{C}$ is a conjunction of ground literals and ground comparisons then the formula $\tau\mathbf{C}$ is finite, and we can ask whether it is equivalent to $\phi\mathbf{C}$ in the sense of Section 3. The answer to this question is yes:

Lemma 1
For any conjunction $\mathbf{C}$ of ground literals and ground comparisons, $\tau\mathbf{C}$ is equivalent to $\phi\mathbf{C}$.

Proof
It is sufficient to prove this assertion assuming that $\mathbf{C}$ is a single ground literal or a single ground comparison.

Case 1: $\mathbf{C}$ is a ground atom $p(t_1, \ldots, t_n)$. Then $\phi\mathbf{C}$ is

$$\exists x_1 \ldots x_n(x_1 \in t_1 \wedge \cdots \wedge x_n \in t_n \wedge p(x_1, \ldots, x_n)).$$

In view of Observation 1, this formula is equivalent to

$$\exists x_1 \ldots x_n \left(\left(\bigvee_{r_1 \in [t_1]} x_1 = r_1 \right) \wedge \cdots \wedge \left(\bigvee_{r_n \in [t_n]} x_n = r_n \right) \wedge p(x_1, \ldots, x_n) \right),$$

and consequently to

$$\bigvee_{r_1 \in [t_1], \ldots, r_n \in [t_n]} p(r_1, \ldots, r_n).$$

The last formula is $\tau\mathbf{C}$.

Case 2: $\mathbf{C}$ is a negative ground literal $\neg p(t_1, \ldots, t_n)$. The proof is similar.

Case 3: $\mathbf{C}$ is a ground comparison $t_1 \prec t_2$. Then Then $\phi\mathbf{C}$ is

$$\exists x_1 x_2(x_1 \in t_1 \wedge x_2 \in t_2 \wedge x_1 \prec x_2).$$

In view of Observation 1, this formula is equivalent to

$$\exists x_1 x_2 \left(\left(\bigvee_{r_1 \in [t_1]} x_1 = r_1 \right) \wedge \left(\bigvee_{r_2 \in [t_2]} x_2 = r_2 \right) \wedge x_1 \prec x_2 \right),$$

and consequently to

$$\bigvee_{r_1 \in [t_1], r_2 \in [t_2]} r_1 \prec r_2.$$

If the relation $\prec$ holds between some terms r_1, r_2 such that $r_1 \in [t_1]$ and $r_2 \in [t_2]$ then one of the disjunctive terms in the last formula is $\top$, and the formula is equivalent to $\top$; otherwise each disjunctive term is $\bot$, and the formula is equivalent to $\bot$. In both cases, it is equivalent to $\tau\mathbf{C}$. □

Lemma 2

For any closed aggregate expression E and any list $\mathbf{X}$ of distinct variables containing all variables that occur in E, the infinitary formula τE is satisfied by the same interpretations of the vocabulary of E as the EG formula $\phi^{\mathbf{X}} E$.

Proof

Let E be a closed aggregate expression (23). Without loss of generality we can assume that the list $\mathbf{X}$ contains only variables occurring in E. As defined in Section A.2, τE is the conjunction of formulas (A1), where A stands for the set of tuples of precomputed terms of the same length as $\mathbf{X}$, over the subsets Δ of A that do not justify E.

Note first that τE is classically equivalent to the disjunction of formulas

$$\bigwedge_{\mathbf{r} \in \Delta} \tau(\mathbf{C}^{\mathbf{X}}_{\mathbf{r}}) \wedge \bigwedge_{\mathbf{r} \in A \setminus \Delta} \neg\tau(\mathbf{C}^{\mathbf{X}}_{\mathbf{r}}) \tag{1}$$

over the subsets Δ of A that justify E. Indeed, call this disjunction D^+, and let D^- be the disjunction of formulas (1) over all other subsets Δ of A. It is clear that D^- is classically equivalent to $\neg D^+$; on the other hand, $\neg D^-$ is classically equivalent to the conjunction τE.

Consider now an interpretation $\mathcal{I}$ of the vocabulary of E. Set A has exactly one subset Δ for which $\mathcal{I}$ satisfies (1): the set of all tuples $\mathbf{r}$ for which $\mathcal{I} \models \tau(\mathbf{C}^{\mathbf{X}}_{\mathbf{r}})$. Consequently $\mathcal{I}$ satisfies τE iff this subset Δ justifies E. In other words, $\mathcal{I}$ satisfies τE iff

$$\widehat{\alpha} \left(\bigcup_{\mathbf{r} \,:\, \mathcal{I} \models \tau(\mathbf{C}^{\mathbf{X}}_{\mathbf{r}})} [\mathbf{t}^{\mathbf{X}}_{\mathbf{r}}] \right) \prec s. \tag{2}$$

By Lemma 1, the condition $\mathcal{I} \models \tau(\mathbf{C}^{\mathbf{X}}_{\mathbf{r}})$ in this expression can be equivalently replaced by $\mathcal{I} \models \phi(\mathbf{C}^{\mathbf{X}}_{\mathbf{r}})$, and consequently by $\mathcal{I} \models (\phi\mathbf{C})^{\mathbf{X}}_{\mathbf{r}}$. Hence (2) holds iff

$$\widehat{\alpha}\{\mathbf{q} \,:\, \text{there exists } \mathbf{r} \text{ such that } \mathbf{q} \in [\mathbf{t}^{\mathbf{X}}_{\mathbf{r}}] \text{ and } \mathcal{I} \models (\phi\mathbf{C})^{\mathbf{X}}_{\mathbf{r}}\} \prec s. \tag{3}$$

On the other hand, $\phi^{\mathbf{X}} E$ is

$$\exists Y(\alpha\{\mathbf{Z} \mid \exists \mathbf{X}(\mathbf{Z} \in \mathbf{t} \wedge \phi\mathbf{C})\} \prec Y \wedge Y \in s),$$

and $\mathcal{I}$ satisfies this formula iff

$$\mathcal{I} \models \alpha\{\mathbf{Z} \mid \exists \mathbf{X}(\mathbf{Z} \in \mathbf{t} \wedge \phi\mathbf{C})\} \prec s.$$

This condition can be rewritten as

$$\widehat{\alpha}\{\mathbf{q} : \mathcal{I} \models \exists \mathbf{X}(\mathbf{q} \in \mathbf{t} \wedge \phi\mathbf{C})\} \prec s,$$

which is equivalent to (3). □

From Lemmas 1 and 2 we conclude:

Lemma 3
For any conjunction $\mathbf{C}$ of ground literals, ground comparisons, and closed aggregate expressions, and for any list $\mathbf{X}$ of distinct variables containing all variables that occur in $\mathbf{C}$, the infinitary formula $\tau\mathbf{C}$ is satisfied by the same interpretations of the vocabulary of $\mathbf{C}$ as the formula $\phi^{\mathbf{X}}\mathbf{C}$.

B.2 Relation to Infinitary Programs

An *infinitary rule* is an implication $F \rightarrow A$ such that F is an infinitary formula and A is an atom. An *infinitary program* is a conjunction of (possibly infinitely many) infinitary rules. We will prove Theorems 1 and 2 using properties of infinitary programs proved by Lifschitz and Yang (2013). The result of applying transformation τ to an EG program is, generally, not an infinitary program, and the following definitions will be useful.

For any EG program Γ, by $\tau_1\Gamma$ we denote the conjunction of

- the infinitary rules
$$\tau(\mathit{Body}) \rightarrow p(\mathbf{r}) \tag{4}$$
for all instances (3) of the basic rules of Γ and all **r** in $[\mathbf{t}]$, and
- the infinitary rules
$$\tau(\mathit{Body}) \wedge \neg\neg p(\mathbf{r}) \rightarrow p(\mathbf{r}) \tag{5}$$
for all instances (7) of the choice rules of Γ and all **r** in $[\mathbf{t}]$.

By $\tau_2\Gamma$ we denote the conjunction of the infinitary formulas $\neg\tau\mathbf{C}$ for all instances $\leftarrow \mathbf{C}$ of the constraints of Γ.

Lemma 4
Stable models of an EG program Γ can be characterized as the stable models of the infinitary program $\tau_1\Gamma$ that satisfy $\tau_2\Gamma$.

Proof
The infinitary formula obtained by applying τ to a closed basic rule (3) is strongly equivalent to the conjunction of the infinitary rules (4) for all **r** in $[\mathbf{t}]$, because these two formulas are equivalent in the deductive system HT^{∞} (Harrison et al. 2015, Section 6). Similarly, the infinitary formula obtained by applying τ to a closed choice rule (7) is strongly equivalent to the conjunction of the infinitary rules (5) for all **r** in $[\mathbf{t}]$. It follows that Γ has the same stable models as $\tau_1\Gamma \cup \tau_2\Gamma$. We know, on the other hand, that for any infinitary formula F and any conjunction G of infinitary formulas that begin with negation, stable models of $F \wedge G$ can be characterized as the stable models of F that satisfy G. (This is a straightforward extension of Proposition 4 from Ferraris and Lifschitz (2005) to infinitary formulas.) It remains to apply this general fact to $\tau_1\Gamma$ as F and $\tau_2\Gamma$ as G. □

For any infinitary program Π and any atom A, by $\Pi|_A$ we denote the set of formulas F such that $F \rightarrow A$ is a rule of Π. The *completion* of Π is the conjunction of the formulas $A \leftrightarrow (\Pi|_A)^{\vee}$ for all atoms A in the underlying signature.

Lemma 5

For any finite EG program Γ, the completion of the infinitary program $\tau_1\Gamma$ is satisfied by the same interpretations of the vocabulary of Γ as the set of completed definitions of the predicate symbols occurring in Γ.

Proof

We will show, for every predicate symbol p/n occurring in Γ, that its completed definition (11) is satisfied by the same interpretations of the vocabulary of Γ as the conjunction of the formulas

$$p(\mathbf{r}) \leftrightarrow (\tau_1\Gamma|_{p(\mathbf{r})})^\vee$$

over all tuples $\mathbf{r}$ of precomputed terms of length n. An interpretation satisfies (11) iff it satisfies the formulas

$$p(\mathbf{r}) \leftrightarrow \bigvee_{i=1}^{k} \exists \mathbf{U}_i (F_i)^{\mathbf{V}}_{\mathbf{r}}$$

for all tuples $\mathbf{r}$ of precomputed terms of length n. Consequently it is sufficient to check that for every such tuple $\mathbf{r}$, the infinitary formula

$$(\tau_1\Gamma|_{p(\mathbf{r})})^\vee \tag{6}$$

and the EG formula

$$\bigvee_{i=1}^{k} \exists \mathbf{U}_i (F_i)^{\mathbf{V}}_{\mathbf{r}} \tag{7}$$

are satisfied by the same interpretations.

The rules of $\tau_1\Gamma$ with the consequent $p(\mathbf{r})$ are obtained as described in the definition of τ_1 above from instances of the rules $R_1, \ldots, R_k$ that define p/n in Γ. If R_i is a basic rule

$$p(\mathbf{t}_i) \leftarrow \mathit{Body}_i \tag{8}$$

then its instances have the form

$$p\left((\mathbf{t}_i)^{\mathbf{U}_i}_{\mathbf{s}}\right) \leftarrow (\mathit{Body}_i)^{\mathbf{U}_i}_{\mathbf{s}}$$

where $\mathbf{s}$ is a tuple of precomputed terms of the same length as $\mathbf{U}_i$. The infinitary rules with the consequent $p(\mathbf{r})$ contributed by this instance to $\tau_1\Gamma$ have the form

$$\tau\left((\mathit{Body}_i)^{\mathbf{U}_i}_{\mathbf{s}}\right) \rightarrow p(\mathbf{r})$$

where $\mathbf{s}$ satisfies the condition $\mathbf{r} \in [(\mathbf{t}_i)^{\mathbf{U}_i}_{\mathbf{s}}]$. If R_i is a choice rule

$$\{p(\mathbf{t}_i)\} \leftarrow \mathit{Body}_i \tag{9}$$

then its instances have the form

$$\{p\left((\mathbf{t}_i)^{\mathbf{U}_i}_{\mathbf{s}}\right)\} \leftarrow (\mathit{Body}_i)^{\mathbf{U}_i}_{\mathbf{s}}$$

and the corresponding rules of $\tau_1\Gamma$ with the consequent $p(\mathbf{r})$ have the form

$$\tau\left((\mathit{Body}_i)^{\mathbf{U}_i}_{\mathbf{s}}\right) \wedge \neg\neg p(\mathbf{r}) \rightarrow p(\mathbf{r}).$$

Let G_i stand for $\tau(\mathit{Body}_i)$ if R_i is a basic rule (8), and for $\tau(\mathit{Body}_i) \wedge \neg\neg p(\mathbf{r})$ if R_i is a choice rule (9). Using this notation, we can represent formula (6) as

$$\bigvee_{i=1}^{k} \bigvee_{\mathbf{s}\,:\,\mathbf{r}\in[(\mathbf{t}_i)^{\mathbf{U}_i}_{\mathbf{s}}]} (G_i)^{\mathbf{U}_i}_{\mathbf{s}}.$$

An interpretation $\mathcal{I}$ satisfies this formula iff

$$\text{for some } i \in \{1,\dots,k\} \text{ and some } \mathbf{s} \text{ such that } \mathbf{r} \in [(\mathbf{t}_i)^{\mathbf{U}_i}_{\mathbf{s}}],\ \mathcal{I} \models (G_i)^{\mathbf{U}_i}_{\mathbf{s}}. \quad (10)$$

On the other hand, F_i in disjunction (7) is

$$\mathbf{V} \in \mathbf{t_i} \wedge \phi^{\mathbf{X}_i}(Body_i)$$

if R_i is a basic rule (8), and

$$\mathbf{V} \in \mathbf{t_i} \wedge \phi^{\mathbf{X}_i}(Body_i) \wedge p(\mathbf{V})$$

if R_i is a choice rule (9), where $\mathbf{X}_i$ is the list of local variables of rule R_i. Let H_i stand for $\phi^{\mathbf{X}_i}(Body_i)$ if R_i is (8), and for $\phi^{\mathbf{X}_i}(Body_i) \wedge p(\mathbf{r})$ if R_i is (9). Formula (7) can be written as

$$\bigvee_{i=1}^{k} \exists \mathbf{U}_i(\mathbf{r} \in \mathbf{t}_i \wedge H_i).$$

An intepretation $\mathcal{I}$ satisfies this formula iff

$$\text{for some } i \in \{1,\dots,k\} \text{ and some } \mathbf{s},\quad \mathbf{r} \in [(\mathbf{t}_i)^{\mathbf{U}_i}_{\mathbf{s}}] \text{ and } \mathcal{I} \models (H_i)^{\mathbf{U}_i}_{\mathbf{s}}. \quad (11)$$

Lemma 3 shows that formulas $(G_i)^{\mathbf{U}_i}_{\mathbf{s}}$ and $(H_i)^{\mathbf{U}_i}_{\mathbf{s}}$ are satisfied by the same interpretations. Consequently condition (11) is equivalent to condition (10). □

Lemma 6

For any EG program Γ, the infinitary formula $\tau_2\Gamma$ is satisfied by the same interpretations of the vocabulary of Γ as the conjunction of the universal closures of the formula representations of the constraints of Γ.

Proof

We will show, for every constraint $\leftarrow$ *Body* from Γ, that the universal closure of its formula representation $\phi(\leftarrow Body)$ is satisfied by the same interpretations of the vocabulary of Γ as the conjunction of the formulas

$$\neg\tau(Body^{\mathbf{U}}_{\mathbf{r}}) \quad (12)$$

for all tuples $\mathbf{r}$ of precomputed terms of the same length as the tuple $\mathbf{U}$ of the global variables of $\leftarrow$ *Body*. Recall that $\phi(\leftarrow Body)$ is defined as $\neg\phi^{\mathbf{X}}(Body)$, where $\mathbf{X}$ is the list of local variables of $\leftarrow$ *Body*. An interpretation $\mathcal{I}$ satisfies the universal closure of this formula iff it satisfies the formulas

$$\neg\phi^{\mathbf{X}}(Body^{\mathbf{U}}_{\mathbf{r}}) \quad (13)$$

for all tuples $\mathbf{r}$ of precomputed terms of the same length as $\mathbf{U}$. By Lemma 3, formulas (12) and (13) are satisfied by the same interpretations. □

B.3 Proof of Theorem 1

An interpretation $\mathcal{I}$ is *supported* by an infinitary program Π if for each atom A in $\mathcal{I}$ there exists an infinitary formula F such that $F \rightarrow A$ is a rule of Π and $\mathcal{I}$ satisfies F. Every stable model of an infinitary program is supported by it (Lifschitz and Yang 2013, Lemma B).[1] It is easy to see that an interpretation $\mathcal{I}$ satisfies the completion of Π iff $\mathcal{I}$ satisfies Π and is supported by Π. We conclude:

Lemma 7
Every stable model of an infinitary program satisfies its completion.

To prove Theorem 1, assume that $\mathcal{I}$ is a stable model of an EG program Γ. Then $\mathcal{I}$ is a stable model of $\tau_1\Gamma$, and $\mathcal{I}$ satisfies $\tau_2\Gamma$ (Lemma 4). Consequently $\mathcal{I}$ satisfies the completion of $\tau_1\Gamma$ (Lemma 7). It follows that $\mathcal{I}$ satisfies the completed definitions of all predicate symbols occurring in Γ (Lemma 5). On the other hand, since $\mathcal{I}$ satisfies $\tau_2\Gamma$, it satisfies also the universal closures of the formula representations of the constraints of Γ (Lemma 6). □

B.4 Proof of Theorem 2

The proof of Theorem 2 below refers to the concept of a tight infinitary program (Lifschitz and Yang 2013). We first define the set $\text{Pnn}(F)$ of *positive nonnegated atoms* of an infinitary formula F and the set $\text{Nnn}(F)$ of *negative nonnegated atoms* of F:

- $\text{Pnn}(\bot) = \emptyset$.
- For any atom A, $\text{Pnn}(A) = \{A\}$.
- $\text{Pnn}(\mathcal{H}^\wedge) = \text{Pnn}(\mathcal{H}^\vee) = \bigcup_{H\in\mathcal{H}} \text{Pnn}(H)$.
- $\text{Pnn}(G \rightarrow H) = \begin{cases} \emptyset & \text{if } H = \bot, \\ \text{Nnn}(G) \cup \text{Pnn}(H) & \text{otherwise.} \end{cases}$
- $\text{Nnn}(\bot) = \emptyset$.
- For any atom A, $\text{Nnn}(A) = \emptyset$.
- $\text{Nnn}(\mathcal{H}^\wedge) = \text{Nnn}(\mathcal{H}^\vee) = \bigcup_{H\in\mathcal{H}} \text{Nnn}(H)$.
- $\text{Nnn}(G \rightarrow H) = \begin{cases} \emptyset & \text{if } H = \bot, \\ \text{Pnn}(G) \cup \text{Nnn}(H) & \text{otherwise.} \end{cases}$

Let Π be an infinitary program, and $\mathcal{I}$ an interpretation of its signature. About atoms $A, B \in \mathcal{I}$ we say that B is a *parent of A relative to Π and $\mathcal{I}$* if there exists a formula F such that $F \rightarrow A$ is a rule of Π, $\mathcal{I}$ satisfies F, and B is a positive nonnegated atom of F. We say that Π is *tight on $\mathcal{I}$* if there is no infinite sequence $A_0, A_1, \ldots$ of elements of $\mathcal{I}$ such that for every i, A_{i+1} is a parent of A_i relative to Π and $\mathcal{I}$.

If an infinitary program Π is tight on an interpretation $\mathcal{I}$ that satisfies Π and is supported by Π then $\mathcal{I}$ is a stable model of Π (Lifschitz and Yang 2013, Lemma 2). We conclude:

Lemma 8
If an infinitary program Π is tight on an interpretation $\mathcal{I}$ that satisfies the completion of Π then $\mathcal{I}$ is a stable model of Π.

[1] See the long version of the paper, `http://www.cs.utexas.edu/users/ai-lab/?ltc`.

Lemma 9
For any conjunction **C** of ground literals, ground comparisons, and closed aggregate expressions, if $p(t_1,\dots,t_n)$ is a positive nonnegated atom of $\tau\mathbf{C}$ then p/n occurs in a positive literal or in an aggregate expression in **C**.

Proof
Consider the conjunctive term C of **C** such that $p(t_1,\dots,t_n)$ is a positive nonnegated atom of τC. It is clear from the definition of τ that p/n occurs in C. On the other hand, the formulas obtained by applying τ to negative literals and comparisons have no positive nonnegated atoms. Consequently C is either a positive literal or an aggregate expression. □

Lemma 10
If an EG program Γ is tight then $\tau_1\Gamma$ is tight on all interpretations.

Proof
Assume that $\tau_1\Gamma$ is not tight on an interpretation $\mathcal{I}$, and consider an infinite sequence

$$p_0(\mathbf{t}_0), p_1(\mathbf{t}_1), \dots$$

of atoms such that for every i, $p_{i+1}(\mathbf{t}_{i+1})$ is a parent of $p_i(\mathbf{t}_i)$ relative to $\tau_1\Gamma$ and $\mathcal{I}$. We will show that for every i, the graph $G_{\tau_1\Gamma}$ has an edge from p_i/n_i to p_{i+1}/n_{i+1}, where n_i is the length of $\mathbf{t}_i$. The the assertion of the lemma will follow, because an infinite path $p_0/n_0, p_1/n_1, \dots$ in the finite graph $G_{\tau_1\Gamma}$ is impossible if that graph is acyclic.

Consider a rule $F_i \to p_i(\mathbf{t}_i)$ of $\tau_1\Gamma$ such that $p_{i+1}(\mathbf{t}_{i+1})$ is a positive nonnegated atom of F_i. This rule has either the form (4) or the form (5). In both cases, $p_{i+1}(\mathbf{t}_{i+1})$ is a positive nonnegated atom of $\tau(\mathit{Body})$, and we can conclude, by Lemma 9, that p_{i+1}/n_{i+1} occurs in a positive literal or in an aggregate expression in *Body*. It remains to observe that *Body* is the body of an instance of a rule of $\tau_1\Gamma$ that contains t_i/n_i in the head. □

Proof of Theorem 2 Let Γ be a finite tight EG program. Given Theorem 1, we only need to establish the "if" direction of Theorem 2: if an interpretation of the vocabulary of Γ satisifies the completion of Γ then it is a stable model of Γ.

Let $\mathcal{I}$ be an interpretation of the vocabulary of Γ that satisfies the completion of Γ. Then $\mathcal{I}$ satisfies the completion of $\tau_1\Gamma$ (Lemma 5). But $\tau_1\Gamma$ is tight on $\mathcal{I}$ (Lemma 10); consequently $\mathcal{I}$ is a stable model of $\tau_1\Gamma$ (Lemma 8). On the other hand, $\mathcal{I}$ satisfies $\tau_2\Gamma$ (Lemma 6). It follows that $\mathcal{I}$ is a stable model of Γ (Lemma 4). □