Online appendix for the paper

Abstract Solvers for Computing Cautious Consequences of ASP programs

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Appendix A Proofs

A.1 Correctness of the Oracle

Definitions. For a program Π and a type of model $w \in \{cla, sta\}$, we say that M is a *w-model of Π* when either w is sta and M is a stable model of Π or w is cla and M is a classical model of Π. We define $M_{cla} = atoms(\Pi)$ and $M_{sta} = atoms(\Pi)$. Also $T_{cla} = backbone(\Pi)$ and $T_{sta} = cautious(\Pi)$. We say that (Π, w, S, G) is a *suitable quadruple* when Π is a program, $w \in \{cla, sta\}$, $S \subseteq \{un, ov, ch\}$, and $G = (V_{atoms(\Pi)}, \{Oracle\} \cup \bigcup_{x \in S} x)$.

Lemma 1

Let (Π, w, S, G) be a suitable quadruple, and let $L_{O,U,A}$ be a reachable state from $\emptyset_{M_w,\emptyset,B}$ in G, where $B \in \{over, under_{\emptyset}, chunk\}$. There is a path in G from $\emptyset_{O,U,A}$ to $L_{O,U,A}$ that does not contain any control state.

Proof

Let $L_{O,U,A}$ be a state reachable from $\emptyset_{M_w,\emptyset,B}$ in G, where $B \in \{over, under_{\emptyset}, chunk\}$. Assume it is reachable without going through any control state; in this case $A = B$, $U = \emptyset$ and $O = M_w$ as the $Oracle$ rule does not modify these. Otherwise a path H leading to $L_{O,U,A}$ goes through some control state; and after the last control state in this path, a rule among $\{UnderApprox$, $OverApprox$, $Chunk\}$ has been applied, which involves that the state occurring right after applying this rule was $\emptyset_{O',U',A'}$ for some O', U' and A'. The $Oracle$ rule does not modify these components of oracle states, and additionally, by the choice of the last control state in H as the predecessor of $\emptyset_{O',U',A'}$, there is no control state in the part of H from $\emptyset_{O',U',A'}$ to $L_{O,U,A}$. So necessarily $O' = O$, $U' = U$ and $A' = A$. Hence, in any case there is a path from $\emptyset_{O,U,A}$ to $L_{O,U,A}$ that does not contain any control state. □

Lemma 2
Let $(\mathit{\Pi}, w, S, G)$ be a suitable quadruple, and let $L_{O,U,A}$ be a reachable state from $\emptyset_{M_w,\emptyset,B}$ in G, where $B \in \{\mathit{over}, \mathit{under}_\emptyset, \mathit{chunk}\}$. If the rule Fail_A applies to $L_{O,U,A}$ in G, then $\mathit{\Pi}_{O,U,A}$ has no w-model; and, if the rule Find applies, then L is a w-model of $\mathit{\Pi}_{O,U,A}$.

Proof
By Lemma 1, there is a path from $\emptyset_{O,U,A}$ to $L_{O,U,A}$ that does not contain any control state. Hence, this path is justified exclusively by the Oracle rule.

First, assume that $w = \mathit{cla}$. Applying the results from **?**), the lemma holds in this case. If the rule Fail_A applies to $L_{O,U,A}$ in G, then $\mathit{\Pi}_{O,U,A}$ has no classical model, and if the rule Find applies then L is a classical model of $\mathit{\Pi}_{O,U,A}$.

Second, assume that $w = \mathit{sta}$. Then, by the results of **?**) the Lemma also holds in this case. Indeed, if the rule Fail_A applies to $L_{O,U,A}$ in G, then $\mathit{\Pi}_{O,U,A}$ has no stable model; and if the rule Find applies, then L is a stable model of $\mathit{\Pi}_{O,U,A}$. $\square$

A.2 Correctness of the Structure

Lemma 3
Let $(\mathit{\Pi}, w, S, G)$ be a suitable quadruple, and if a state $L_{O,U,A}$ or $\mathit{Cont}(O,U)$ is reachable from $\emptyset_{M_w,\emptyset,B}$ in G, where $B \in \{\mathit{over}, \mathit{under}_\emptyset, \mathit{chunk}\}$, then $U \subseteq T_w \subseteq O$.

Proof
We prove this lemma by induction on the path leading from $\emptyset_{M_w,\emptyset,B}$ to $L_{O,U,A}$ or $\mathit{Cont}(O,U)$. So as to initialize this induction, we simply note that $\emptyset_{M_w,\emptyset,B}$ is such that $\emptyset \subseteq T_w \subseteq M_w$. Now, assume that a state is reachable from $\emptyset_{M_w,\emptyset,B}$ in G and that for any state on the path the lemma holds, in particular on its predecessor. We are going to prove that for this state the lemma holds.

First case: assume that the state is a core state $L_{O,U,A}$. If its predecessor is a core state, then the predecessor is $L'_{O,U,A}$ for some L', since the Oracle rule does not modify these O, U and A. By the induction hypothesis, the lemma holds. If its predecessor is a control state then note that the control rules that may link this predecessor to $L_{O,U,A}$ are $\mathit{OverApprox}$, $\mathit{UnderApprox}$ and Chunk, of which none modifies the over-approximation and under-approximation; hence, the predecessor is $\mathit{Cont}(O,U)$ and by the induction hypothesis the lemma holds.

Second case: when the state is a control state. Then, its predecessor is a core state $L_{O,U,A}$. By the induction hypothesis, $U \subseteq T_w \subseteq O$. The rule applied is a return rule.

- If the rule is $\mathit{Terminal}$, then the state is $\mathit{Cont}(O \cap L, U)$. By Lemma 2, L is a w-model of $\mathit{\Pi}_{O,U,A}$. So no element of $M_w \setminus L$ belongs to T_w, and no element of L can be part of T_w. Hence, $U \subseteq T_w \subseteq O \cap L$.
- If the rule is Fail_{under}, then the state is $\mathit{Cont}(O, U \cup \{a\})$. By Lemma 2, $\mathit{\Pi}_{O,U,A}$ has no w-model. So no w-model of $\mathit{\Pi}$ satisfies a. So a belongs to T_w. Hence, $U \cup \{a\} \subseteq T_w \subseteq O$.

In all cases the lemma holds, which ends the proof by induction. $\square$

Lemma 4
Let $(\mathit{\Pi}, w, S, G)$ be a suitable quadruple, and let $L_{O,U,A}$ be a reachable state from $\emptyset_{M_w,\emptyset,B}$ in G, where $B \in \{\mathit{over}, \mathit{under}_\emptyset, \mathit{chunk}\}$. If Fail_{over} applies to $L_{O,U,A}$, then $T_w = O$.

Proof
Assume that Fail_{over} applies to some state $L_{O,U,A}$ reachable from $\emptyset_{M_w,\emptyset,B}$. Then $A = \mathit{over}$. The path has to go through at least one control state so that $A \neq B$, and hence the rule Find has to have been applied; so Π has at least one w-model and T_w is well defined. Also, by Lemma 2, $\Pi_{O,U,over}$ has no w-model. In other words, $\Pi \cup \{\leftarrow O\}$ has no w-model. As the constraint added to Π is monotonic, Π has no w-model satisfying $\leftarrow O$. In other words, all the w-models of Π satisfy O, so $O \subseteq T_w$. Since, by Lemma 3, $T_w \subseteq O$, also $T_w = O$. $\square$

Lemma 5
Let (Π, w, S, G) be a suitable quadruple, and let $L_{O,U,A}$ be a reachable state from $\emptyset_{M_w,\emptyset,B}$ in G, where $B \in \{\mathit{over}, \mathit{under}_{\emptyset}, \mathit{chunk}\}$. If there is a transition in G from $L_{O,U,A}$ to $\mathit{Cont}(O', U')$ and $A \neq B$, then $O' \setminus U' \subset O \setminus U$.

Proof
Assume that there is a transition in G from $L_{O,U,A}$ to $\mathit{Cont}(O, U)$ and $A \neq B$.

If this transition is justified by Fail_{chunk} or Fail_{under}, then A is chunk_N or under_N for some N. Also $O' = O$ and $U' = U \cup N$, so $O' \setminus U' \subseteq (O \setminus U) \setminus N$. The last control rule applied was necessarily Chunk, so that $N \subseteq O \setminus U$ and $N \neq \emptyset$. Then $(O \setminus U) \setminus N \subset O \setminus U$, so $O' \setminus U' \subset O \setminus U$.

If this transition is justified by Find, we first prove that $O \cap L \neq O$ and $U \subseteq L$. First, assume $A = \mathit{over}$. Then, by Lemma 2, L is a w-model of $\Pi_{O,U,over} = \Pi \cup \{\leftarrow O\}$. Therefore, L is a w-model of Π and a classical model of $\{\leftarrow O\}$. Since it is a w-model of Π and $U \subseteq T_w$, by definition of T_w also $U \subseteq L$. Since L is a classical model of $\{\leftarrow O\}$, also $\overline{O} \cap L \neq \emptyset$. Hence, $O \cap L \neq O$. Now, assume $A = \mathit{chunk}_N$. The last control rule applied was necessarily Chunk, so that $N \subseteq O \setminus U$ and hence $N \subseteq O$. Also, by Lemma 2, L is a w-model of $\Pi_{O,U,chunk_N} = \Pi \cup \{\leftarrow N\}$, so L is a w-model of Π and a classical model of $\{\leftarrow N\}$, and $\overline{N} \cap L \neq \emptyset$. Since it is a w-model of Π and $U \subseteq T_w$, by definition of T_w also $U \subseteq L$. Since L is a classical model of $\{\leftarrow N\}$, also $\overline{O} \cap L \neq \emptyset$, and hence $O \cap L \neq O$. The proof in the case of under_N is identical to the case of chunk_N. So in any case $O \cap L \neq O$ and $U \subseteq L$. So $O' \setminus U' = (O \cap L) \setminus U$ is a strict subset of $O \setminus U$. $\square$

A.3 Finiteness and Lack of Reachable Cycles

Lemma 6
Let Π be a program, and let $S \subseteq \{un, ov, ch\}$. Then, the graph $(V_{atoms(\Pi)}, \{\mathit{Oracle}\} \cup \bigcup_{x \in S} x)$ is finite.

Proof
Any core state relative to $atoms(\Pi)$ is made of a record relative to $atoms(\Pi)$, two sets of literals relative to $atoms(\Pi)$, and one action relative to $atoms(\Pi)$. The set $lit(atoms(\Pi))$ of literals relative to $atoms(\Pi)$ is finite, and so is its powerset; hence there is only a finite amount of possibilities for the two sets of literals relative to $atoms(\Pi)$. Also, since an action can only be over, chunk_M, or under_M for M a set of literals relative to $atoms(\Pi)$, there is only a finite amount of possible actions. Finally, since the set of literals relative to $atoms(\Pi)$ is finite, and so is its powerset; so there are only a finite amount of possible records relative to $atoms(\Pi)$ since repetitions are not allowed in records. So there is a only finite amount of core states relative to

$V_{atoms(\Pi)}$. Since the other types of states are only made of a portion of what makes a core state, there is also a finite amount of them. As a consequence, $V_{atoms(\Pi)}$ is finite, and hence the graph $(V_{atoms(\Pi)}, \{Oracle\} \cup \bigcup_{x \in S} x)$ is finite. $\square$

Lemma 7
Let (Π, w, S, G) be a suitable quadruple. Then, there is no cycle in G reachable from the initial state $\emptyset_{M_w,\emptyset,B}$, where $B \in \{over, under_\emptyset, chunk\}$.

Proof
We are going to define a partial order on $V_{atoms(\Pi)}$.

First, we define an order on records as follows. For any record L, we consider the strings $L_1, \ldots, L_i$ such that each L_k, $1 \leq k \leq i$, contains the literals assigned at level i. We define the order $<$ on string of integers as the lexicographic order on strings on integers. For any core state $L_{O,U,A}$ we define $v(L_{O,U,A})$ as the string $2, v(L)$ if $A \neq B$, and $0, v(L)$ if $A = B$. We consider that any control state $Cont(O,U)$ is such that $v(Cont(O,U)) = 1$, and any state s that is a terminal state is such that $v(s) = 3$.

We then define an order on the gap between over-approximation and under-approximation, which in general is $O \setminus U$. We define the functions ove and und. For any state s, if s is $L_{O,U,A}$ or $Cont(O,U)$ then $ove(s) = O$ and $und(s) = U$, otherwise $ove(s) = \emptyset$ and $und(s) = lit(atoms(\Pi))$. For two sets of literals M and M', we say that $M < M'$ if $M' \subseteq M$.

We write $\leq_{lex}$ to denote the lexicographic composition of orders. Then we define our order on states as follows. For any two states, $s < s'$ iff $(ove(s) \setminus und(s), v(s)) \leq_{lex} (ove(s') \setminus und(s'), v(s'))$. The relations on $v(s)$ and $ove(s) \setminus und(s)$ are clearly partial orders. Hence the obtained lexicographic order is also a partial order. We are now going to show that any edge (s, s') in $\{Oracle\} \cup \bigcup_{x \in S} x$ such that s is reachable from the initial state is such that $s < s'$ and $s \neq s'$. Assume that a state s is reachable from the initial state and the rule $Find$, $Fail_{under}$ or $Fail_{chunk}$ applies to s so as to create the edge (s, s'). Then by Lemma 5, $s < s'$ and $s \neq s'$. So, indeed, any edge (s, s') in $\{Oracle\} \cup \bigcup_{x \in S} x$ such that s is reachable from the initial state is also such that $s < s'$ and $s \neq s'$. As a consequence, since the relation $<$ on states is a partial order and there is only a finite amount of ordered elements, there is no infinite path, and hence no cycle among the reachable elements of $(V_{atoms(\Pi)}, \{Oracle\} \cup \bigcup_{x \in S} x)$. $\square$

***A.4 Proof of Theorem* ??**

By Lemmas 6 and 7, the graph $G = (V_{atoms(\Pi)}, \{Oracle\} \cup \bigcup_{x \in S} x)$ is finite and no cycle is reachable from the initial state. Assume a state $L_{O,U,A}$ is terminal in G; this is impossible since if no other rule applies then $Find$ applies. Similarly, assume a state $Cont(O,U)$ is reachable and terminal in G. Either $O = U$ and $Terminal$ applies, or $O \neq U$ and, by Lemma 3, $U \subset O$ so one of the rules of the nonempty set $\{OverApprox, UnderApprox, Chunk\} \cap \bigcup_{x \in S} x$ applies. In both cases a rule applies, which is a contradiction.

Therefore, the terminal state is $Ok(L)$ for some L. Hence, as to end the proof of the theorem we now study the type of state that can actually be terminal. Assume that $Ok(M)$ is the terminal state reachable from the initial state. Either it was reached by a transition justified by $Fail_{over}$ and, by Lemma 4, in any state $L_{M,U,over}$ from which this transition may have originated holds $T_w = M$, or it was reached by a transition justified by $Terminal$ and, by Lemma 3, in any state $Cont(M,M)$ from which this transition may have originated holds $M \subseteq T_w \subseteq M$, hence $T_w = M$.