Online appendix for the paper

Omission-based Abstraction for Answer Set Programs

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Appendix A Proofs

Proof of Theorem 14

As for membership in (i), we can compute such a set *PB* by an elimination procedure as follows. Starting with $A' = \emptyset$, we repeatedly pick some atom $\alpha \in A \setminus A'$ and test the following condition:

(+) for $A'' = A' \cup \{\alpha\}$, the program $omit(\Pi, A'')$ has no answer set $\widehat{I''}$ such that $\widehat{I''}|_{\overline{A}} = \hat{I}$.

If (+) holds, we set $A' := A''$ and make the next pick from $A \setminus A'$. Upon termination, $PB = A \setminus A'$ is a minimal put-back set. The correctness of this procedure follows from Proposition 8, by which the elimination of spurious answer sets is anti-monotonic in the set *A* of atoms to omit. As for the effort, the test (+) can be done in polynomial time with an **NP** oracle; from this, membership in in $\mathbf{FP}^{\mathbf{NP}}$ follows.

The hardness for $\mathbf{FP}^{\mathbf{NP}}_{\parallel}$ is shown by a reduction from computing, given normal logic programs $\Pi_1, \dots, \Pi_n$ on disjoint sets $X_1, \dots, X_n$ of atoms, the answers $q_1, \dots, q_n$ to whether Π_i has some answer set ($q_i = 1$) or not ($q_i = 0$).[1]

To this end, we use fresh atoms a_i and b_i and construct

$$
\begin{aligned}
\Pi'_i \quad = \{ \quad & a_i \leftarrow not\ b_i \\
& b_i \leftarrow not\ a_i \\
& \bot \leftarrow not\ b_i \\
& H(r) \leftarrow B(r), a_i \qquad r \in \Pi_i \\
& y \leftarrow x, not\ x \qquad x, y \in X_i \\
& a_i \leftarrow x, not\ x \qquad x \in X_i \\
& b_i \leftarrow x, not\ x \qquad x \in X_i \quad \}
\end{aligned}
$$

Clearly, $\{a_i\}$ is an answer set of $omit(\Pi', X_i \cup \{b_i\})$, as the rule $a_i \leftarrow not\ b_i$ is turned into a choice; it is spurious, as only this rule in Π can derive a_i. However, this violates the constraint $\bot \leftarrow not\ b_i$.

Assuming w.l.o.g. that Π_i includes no constraints, for every set *PB* of atoms such that $X_i \not\subseteq PB$, the program $omit(\Pi'_i, (X_i \cup \{b_i\}) \setminus PB)$ has some answer set containing a_i, thanks to the abstraction of the rules with $x, not\ x$ in the body; thus $PB = X_i$ is the minimal candidate for

[1] We are indebted to a reviewer pointing out an error in the original reduction, which we replace by an elegant one suggested by the reviewer.

$$
\begin{array}{lll}
x_i. \qquad \overline{x_i}. & i=1\ldots,n & \text{(A2)}\\
sat \leftarrow x_i, not\ x_i, \overline{x_i}, not\ \overline{x_i}. & i=1\ldots,n & \text{(A3)}\\
z_i \leftarrow not\ \overline{z_i}, not\ \overline{x_i}. & i=1\ldots,n & \text{(A4)}\\
\overline{z_i} \leftarrow not\ z_i, not\ x_i. & i=1\ldots,n & \text{(A5)}\\
y_j \leftarrow not\ \overline{y_j}, not\ sat. & j=1,\ldots,m & \text{(A6)}\\
\overline{y_j} \leftarrow not\ y_j, not\ sat. & j=1,\ldots,m & \text{(A7)}\\
sat \leftarrow l^{\circ}_{i_1}, \ldots l^{\circ}_{i_{n_i}}. & i=1,\ldots,k & \text{(A8)}\\
sat \leftarrow y_j, not\ y_j. & j=1,\ldots,m & \text{(A9)}\\
sat \leftarrow \overline{y_j}, not\ \overline{y_j}. & j=1,\ldots,m & \text{(A10)}\\
sat \leftarrow z_i, not\ z_i. & i=1\ldots,n & \text{(A11)}\\
sat \leftarrow \overline{z_i}, not\ \overline{z_i}. & i=1\ldots,n & \text{(A12)}
\end{array}
$$

Fig. A 1. Program rules for the proof of Theorem 14-(ii), first part

being a put-back set. Furthermore, if Π_i has no answer set, then $\emptyset$ is the single answer set of $omit(\Pi'_i, \{b_i\})$ while if Π_i has some answer set S, then $omit(\Pi'_i, \{b_i\})$ has the answer set $S \cup \{a_i\}$. That is, X_i is the (unique) $\subseteq$-minimal put-back set iff Π_i has no answer set.

We construct the final program as $\Pi' = \bigcup_{i=1}^{n} \Pi'_i$. Then, $\hat{I} = \{a_1, \ldots, a_n\}$ is a spurious answer set of $omit(\Pi', \bigcup_{i=1}^{n} X_i \cup \{b_i\})$, and every minimal put-back set PB for $\hat{I}$ satisfies $b_i \in PB$ iff Π_i is satisfiable; this proves $\mathbf{FP}^{\mathbf{NP}}_{\parallel}$-hardness.

As for (ii), the membership in $\mathbf{FP}^{\Sigma_2^P}[log, wit]$ holds as we can decide the problem by a binary search for a put-back set of bounded size using a Σ_2^p witness oracle, where the finally obtained put-back set is output.

The $\mathbf{FP}^{\Sigma_2^P}[log, wit]$ hardness is shown by a reduction from the following problem. Given a QBF $\Phi = \exists X \forall Y\, E(X,Y)$, compute a smallest size truth assignment σ to X such that $\forall Y\, E(\sigma(X), Y)$ evaluates to true, knowing that some σ with this property exists, where the size of σ is the number of atoms set to true.

More specifically, we assume similar as in the proof of Theorem 12 that $E(X,Y) = \bigvee_{i=1}^{k} D_i$ is a DNF where every $D_i = l_{i_1} \wedge \cdots \wedge l_{i_{n_i}}$ is a conjunction of literals over $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ that contains some literal over Y; moreover, we assume that $E(X,Y)$ is a tautology if all literals over X are removed from it. To verify the latter assumption, we may rewrite Φ to

$$
\exists X \forall Y \bigvee_{x_i \in X} (x_i \wedge \neg x_i \wedge y_j) \vee (x_i \wedge \neg x_i \wedge \neg y_j) \vee E(X,Y), \tag{A1}
$$

for an arbitrary $y_j \in Y$, which has the desired property.

We set up a program Π with rules shown in Figure Appendix A, where $\overline{X} = \{\bar{x}_i \mid x_i \in X\}$, $Z = \{z_1, \ldots, z_n\}$ and $\overline{Z} = \{\bar{z}_i \mid z_i \in Z\}$ are copies of X and $\overline{Y} = \{\bar{y}_j \mid y_j \in Y\}$ is a copy of Y, and l° maps a literal l over $X \cup Y$ to default literals over $Y \cup \overline{Y} \cup Z \cup \overline{Z}$ as follows:

$$
l^{\circ} = \begin{cases} not\ z_i, & \text{if } l = \neg x_i, \\ not\ \overline{z_i}, & \text{if } l = x_i, \\ y_j, & \text{if } l = y_j, \\ \bar{y}_j & \text{if } l = \neg y_j. \end{cases}
$$

We note that Π has no answer set: due to the facts x_i and $\bar{x}_i$, none of the rules (A3)–(A5) is applicable and $z_i, \bar{z}_i$ must be false in every answer set of Π. This in turn implies that in (A8) all *not* z_i, *not* $\bar{z}_i$ literals are true. Now if we assume that *sat* would be true in an answer set of Π, then no rule in (A6) or (A7) would be applicable to derive y_j resp. $\bar{y}_j$, and then by the assumption on $E(X,Y)$ no rule (A8) is applicable; this means that *sat* is not reproducible and thus not in the answer set, which is a contradiction. If on the other hand *sat* would be false in an answer set, then the rules (A6) and (A7) would guess a truth assignment to Y; by the tautology assumption on $E(X,Y)$, some rule (A8) is applicable and derives that *sat* is true, which is again a contradiction.

We then set $A = \mathscr{A}$ and $\hat{I} = \emptyset$; clearly $\hat{I}$ is a spurious answer set of $omit(\Pi, A) = \emptyset$.

The idea behind this construction is as follows. As long as we do not put back *sat*, the abstraction program $omit(\Pi, A')$ will have some answer set. Furthermore, if we do not put back (a) either x_i or $\overline{x_i}$, for all $i = 1, \ldots, n$, (b) both z_i and $\overline{z_i}$ for all $i = 1, \ldots, n$ and (c) all y_j, $\overline{y_j}$, for $j = 1, \ldots, m$, then we can guess by (A3) resp. (A9)–(A12) that *sat* is true, which again means that some answer set exists. The rules (A4)–(A5) serve then to provide with z_i and $\overline{z_i}$ access to x_i and its negation $\neg x_i$, respectively. More in detail, if we put back x_i but not $\overline{x_i}$, then $omit(\Pi, A')$ contains the guessing rule $r_i : \{z_i\} \leftarrow not\ \overline{z_i}$ and the rule $\overline{r_i} : \overline{z_i} \leftarrow not\ z_i, not\ x_i$ resulting from (A4) and (A5), respectively. As in $omit(\Pi, A')$ the rule $\overline{r_i}$ is inapplicable and no other rule has $\overline{z_i}$ in the head, the atom $\overline{z_i}$ must be false; hence the rule r_i amounts to a guess $\{z_i\}$. If z_i is guessed to be true, then *not* z_i and *not* $\overline{z_i}$ faithfully represent the value of the literals $\neg x_i$ and x_i (where x_i is true); this is injected into the rules (A8). On the other hand, if z_i is guessed false, then both *not* z_i and *not* $\overline{z_i}$ are true, which represents that both $\neg x_i$ and x_i are true; if guessing z_i false leads to a (spurious) answer set of the abstract program $omit(\Pi, A')$ (in which *sat* must be necessarily false), no rule (A8) in which z_i or $\overline{z_i}$ occurs can fire. As z_i and $\overline{z_i}$ occur only negated in the rules (A8), guessing z_i true (where z_i and $\overline{z_i}$ faithfully represent x_i and $\neg x_i$, respectively) leads then also to an answer set of $omit(\Pi, A')$. Thus, with respect to answer set existence, z_i and $\overline{z_i}$ serve to access x_i and $\neg x_i$. The case of putting back $\overline{x_i}$ but not x_i is symmetric.

The rules (A6)–(A7) serve to guess an assignment μ to Y (but this only works if *sat* is false). The rules (A8) check whether upon a combined assignment $\sigma \cup \mu$, the formula $E(\sigma(X), \mu(Y))$ evaluates to true; if this is the case, *sat* is concluded which then however blocks the guessing in (A6)–(A7), and thus no answer set exists. Consequently, $E(\sigma(X), \mu(Y))$ evaluates to true for all assignments $\mu(Y)$, i.e., $\forall Y E(\sigma(X), Y)$ is true iff *sat* can be concluded for each guess on y_i and $\overline{y_i}$, i.e., no answer set is possible for it.

In conclusion, it holds that some put-back set of size $s = |X| + 2|X| + 2|Y| + 1$, which is the smallest possible here, exists iff Φ evaluates to true. Note that if we put back a single further atom, for some $x_i \in X$ we have that $\overline{x_i}$ is also a fact in $omit(\Pi, A')$, and thus by the special form of $E(X,Y)$ in (A1), regardless of how one guesses on y_j and $\overline{y_i}$, one can derive *sat* again. Thus the closest put-back set has either size s or $s+1$.

In order to discriminate among different $\sigma(X)$ and select the smallest, we add further rules:

$$sat \leftarrow not\ \overline{z_i}, c_i \qquad i = 1, \ldots, n \tag{A13}$$

$$sat \leftarrow not\ \overline{z_i}, not\ z_i, c_1, \ldots, c_l \tag{A14}$$

where all c_i are fresh atoms; we fix l below.[2] Intuitively, when x_i is put back, then $\neg z_i$ evaluates to true and c_i must be put back as well in order to avoid guessing on *sat*. Furthermore, if both x_i and $\overline{x_i}$ are put back, which means that *not* z_i and *not* $\overline{z_i}$ are true in every answer set, then all

[2] Alternatively, for (A14) rules $sat \leftarrow not\ \overline{z_i}, not\ z_i, c_j$, $j = 1, \ldots, l$ may be used.

$c_1,\ldots,c_l$ must be put back as well. If exactly one of x_i and $\overline{x_i}$, for all $i=1,\ldots,n$ is put back and the corresponding assignment $\sigma(X)$ makes $\forall Y E(\sigma(X),Y)$ true, then the closest put-back set has size $s+1+|\sigma|$; if we let l be large enough, then putting both x_i and $\overline{x_i}$ back is more expensive than putting back a proper assignment and the associated c_i atoms; in fact $l=n$ is sufficient. As the final program Π is constructible in polynomial time from Φ, and the desired smallest $\sigma(X)$ is easily obtained from any smallest put-back set PB for $\hat{I}$ the claimed result follows. □

Proof of Theorem 19

1. Assume towards a contradiction that $X' = X \cup \{ko(n_r) \mid r \in \Pi_A^c\} \cup \{ap(n_r) \mid r \in \Pi^X\} \cup \{bl(n_r) \mid r \in \Pi \setminus \Pi^X\}$ is not answer set of $\Pi' \cup Q_{\hat{I}}^{\overline{A}}$, where $\Pi' = \mathscr{T}_{meta}[\Pi] \cup \mathscr{T}_P[\Pi] \cup \mathscr{T}_C[\Pi,\mathscr{A}] \cup \mathscr{T}_A[\mathscr{A}]$. This means that either (i) X' is not a model of $(\Pi' \cup Q_{\hat{I}}^{\overline{A}})^{X'}$, or (ii) X' is not a minimal model of $(\Pi' \cup Q_{\hat{I}}^{\overline{A}})^{X'}$.

(i) There is some rule $r \in (\Pi' \cup Q_{\hat{I}}^{\overline{A}})^{X'}$ such that $X' \models B(r)$, but $X' \not\models H(r)$. We know that X is an answer set of $\Pi \cup Q_{\hat{I}}^{\overline{A}}$, and thus $X \in AS(\Pi)$. By Theorem 17, we know that $X \cup \{ap(n_r) \mid r \in \Pi^X\} \cup \{bl(n_r) \mid r \in \Pi \setminus \Pi^X\}$ is an answer set of $\mathscr{T}_{meta}[\Pi]$. As X' contains no *ab* atoms, r cannot be in $\mathscr{T}_P[\Pi] \cup \mathscr{T}_C[\Pi,\mathscr{A}] \cup \mathscr{T}_A[\mathscr{A}]$. So r must be in $Q_{\hat{I}}^{\overline{A}}$.
The rule r can be in two forms: (a) $\bot \leftarrow not\ \alpha.$ for some $\alpha \in \hat{I}$, or (b) $\bot \leftarrow \alpha.$ for some $\alpha \in \overline{A} \setminus \hat{I}$.

  (a) As $X' \models B(r)$, then $\alpha \notin X'$ which means $\alpha \notin X$. However having $r \in (\Pi \cup Q_{\hat{I}}^{\overline{A}})^X$ contradicts that X is an answer set of $\Pi \cup Q_{\hat{I}}^{\overline{A}}$.

  (b) Similarly as (a), we reach a contradiction.

(ii) Let $Y' \subset X'$ be a model of $(\Pi' \cup Q_{\hat{I}}^{\overline{A}})^{X'}$, for some $Y' = Y \cup \{ko(n_r) \mid r \in \Pi_A^c\} \cup \{ap(n_r) \mid r \in \Pi^X\} \cup \{bl(n_r) \mid r \in \Pi \setminus \Pi^X\}$. As the auxiliary atoms are fixed, $Y \subset Y'$ must hold. We claim that Y is then a model of $(\Pi \cup Q_{\hat{I}}^{\overline{A}})^X$, which is a contradiction. Assume Y is not such a model. Then there is a rule $r \in (\Pi \cup Q_{\hat{I}}^{\overline{A}})^X$ such that $Y \models B(r)$ but $Y \not\models H(r)$. There are two cases: (a) $r \in \Pi$, or (b) $r \in Q_{\hat{I}}^{\overline{A}}$.

  (a) By definition of Y', this means that $Y' \models B(r)$ and $Y' \not\models H(r)$. However, this contradicts that Y' is a smaller model of $(\Pi' \cup Q_{\hat{I}}^{\overline{A}})^{X'}$ than X' since $H(r)' \in Y'$.

  (b) In both versions of r in $Q_{\hat{I}}^{\overline{A}}$, we get that $r \in (\Pi' \cup Q_{\hat{I}}^{\overline{A}})^{X'}$ which contradicts that Y' is a model of $(\Pi' \cup Q_{\hat{I}}^{\overline{A}})^{X'}$.

2. Assume towards a contradiction that $(Y \cap \mathscr{A})$ is not an answer set of $\Pi \cup Q_{\hat{I}}^{\overline{A}}$. This means that either (i) $(Y \cap \mathscr{A})$ is not a model of $(\Pi \cup Q_{\hat{I}}^{\overline{A}})^{(Y \cap \mathscr{A})}$, or (ii) $(Y \cap \mathscr{A})$ is not a minimal model of $(\Pi \cup Q_{\hat{I}}^{\overline{A}})^{(Y \cap \mathscr{A})}$.

(i) There is some rule $r \in (\Pi \cup Q_{\hat{I}}^{\overline{A}})^{(Y \cap \mathscr{A})}$ such that $(Y \cap \mathscr{A}) \models B(r)$ but $(Y \cap \mathscr{A}) \not\models H(r)$. As we have $(Y \cap \mathscr{A}^+) \in AS(\mathscr{T}_{meta}[\Pi])$, by Theorem 17, we get $(Y \cap \mathscr{A}) \in AS(\Pi)$, thus r cannot be in Π. However, $r \in Q_{\hat{I}}^{\overline{A}}$ also cannot hold, since then r will be in $(Q_{\hat{I}}^{\overline{A}})^Y$ and we know that $Y \models Q_{\hat{I}}^{\overline{A}}$. Thus $(Y \cap \mathscr{A})$ must be a model of $(\Pi \cup Q_{\hat{I}}^{\overline{A}})^{(Y \cap \mathscr{A})}$.

(ii) Assume there exists some $Z \subset (Y \cap \mathscr{A})$ such that $Z \models (\Pi \cup Q_{\hat{I}}^{\overline{A}})^{(Y \cap \mathscr{A})}$. We claim that then $Z' = Z \cup \{ko(n_r) \mid r \in \Pi_A^c\} \cup \{ap(n_r) \mid r \in \Pi'^Y\} \cup \{bl(n_r) \mid r \in \Pi' \setminus \Pi'^Y\}$ is a model of $(\Pi' \cup Q_{\hat{I}}^{\overline{A}})^Y$, which achieves a contradiction. Now let us assume that this is not the case. Then there is some rule $r \in (\Pi' \cup Q_{\hat{I}}^{\overline{A}})^Y$ such that $Z' \models B(r)$ and $Z' \not\models H(r)$. The rule r cannot be in $(Q_{\hat{I}}^{\bar{A}})^Y$, since it contradicts that $Y \models (Q_{\hat{I}}^{\bar{A}})^Y$. The rest of the cases for r also results in a contradiction.

(a) If $r \in \mathscr{T}_{meta}[\Pi]^Y$, then r can only be of form $H(r) \leftarrow ap(n_r), not\ ko(n_r)$, where $H(r) \neq \bot$. So we have $ap(n_r) \in Z'$, $ko(n_r) \notin Z'$ and $H(r) \notin Z'$. For rule r, rules of form 1 in Definition 9 are created in $\mathscr{T}_P[\Pi]$. However, since having $H(r) \notin Y$ causes to have the rule $ab_p(n_r) \leftarrow ap(n_r), not\ H(r)$ in $\mathscr{T}_P[\Pi]^Y$, $H(r) \in Y \setminus Z'$ should hold, which however contradicts that $Z \subset (Y \cap \mathscr{A})$, as then $H(r)' \in Z'$ would hold.

(b) If $r \in \mathscr{T}_P[\Pi]^Y$, then r can only be of form $H(r) \leftarrow ap(n_r)$. As $Z' \not\models H(r)$ we have $H(r)' \in Z'$ which contradicts that $Z \subset (Y \cap \mathscr{A})$. A similar contradiction is reached if $r \in \mathscr{T}_C[\Pi, \mathscr{A}]^Y$, since that means $\alpha \in Z'$ while $\alpha \notin Y$.

(c) Having $r \in \mathscr{T}_A[\mathscr{A}]^Y$ means that $Z' \not\models ab_l(\alpha)'$ for some $\alpha \in \mathscr{A}$, i.e., $ab_l(\alpha) \in Z'$, which contradicts $Y \cap AB_A(\Pi) = \emptyset$. □