Online appendix for the paper

Reasoning on Multi-Relational Contextual Hierarchies via Answer Set Programming with Algebraic Measures

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Appendix A Single-relational Example

We also give an example of a single-relational sCKR.

Example 5
We consider a single-relation hierarchy on coverage by reviewing the example from (Bozzato et al. 2019; Bozzato et al. 2018b). Let us consider a sCKR $\mathfrak{K}_{org1} = \langle \mathfrak{C}, \mathrm{K}_{\mathsf{N}} \rangle$ with with $\mathfrak{C} = (\mathsf{N}, \prec_c)$ describing the organization of a corporation. The corporation wants to define different policies with respect to its local branches, represented by the coverage hierarchy in $\mathfrak{C}$. The corporation is active in the fields of Musical instruments (M), Electronics (E) and Robotics (R). A supervisor (S) can be assigned to manage only one of these fields. Defeasible axioms in contexts in K_{N} define the assignment of local supervisors to their field:

$$\mathfrak{C} : \{\mathsf{c}_{br1} \prec_c \mathsf{c}_{world},\ \mathsf{c}_{br2} \prec_c \mathsf{c}_{world},\ \mathsf{c}_{br1} \prec_c \mathsf{c}_{br2},\ \mathsf{c}_{local1} \prec_c \mathsf{c}_{br1},\}$$
$$\mathsf{c}_{world} : \{M \sqcap E \sqsubseteq \bot,\ M \sqcap R \sqsubseteq \bot,\ E \sqcap R \sqsubseteq \bot,\ \mathrm{D}(S \sqsubseteq E)\}$$
$$\mathsf{c}_{br1} : \{\mathrm{D}_c(S \sqsubseteq M)\} \qquad \mathsf{c}_{br2} : \{\mathrm{D}_c(S \sqsubseteq R)\} \qquad \mathsf{c}_{local1} : \{S(i)\}$$

In c_{world} we say that supervisors are assigned to Electronics, while in the sub-context for c_{br2} we contradict this by assigning all local supervisors to the Robotics area and in c_{br1} we further specialize this by assigning supervisors to the Musical instruments area. In the context c_{local1} for a local site we have information about an instance i. Note that different assignments of areas for i are possible by instantiating the defeasible axioms: intuitively, we want to prefer the interpretations that override the higher defeasible axioms in c_{world} and c_{br2}.

Observe that different justified *CAS* models are possible, depending on the different assignments of the individual i in c_{local1} to the alternative areas denoted by defeasible axioms. We have three possible clashing assumptions sets for context c_{local1}:

$$\chi^1_c(\mathsf{c}_{local1}) = \{\langle S \sqsubseteq E, i\rangle, \langle S \sqsubseteq R, i\rangle\} \quad \chi^2_c(\mathsf{c}_{local1}) = \{\langle S \sqsubseteq M, i\rangle, \langle S \sqsubseteq R, i\rangle\}$$
$$\chi^3_c(\mathsf{c}_{local1}) = \{\langle S \sqsubseteq M, i\rangle, \langle S \sqsubseteq E, i\rangle\}$$

By the ordering on clashing assumption sets, in particular $\chi^1_c(\mathsf{c}_{local1}) > \chi^2_c(\mathsf{c}_{local1})$, $\chi^1_c(\mathsf{c}_{local1}) > \chi^3_c(\mathsf{c}_{local1})$ and $\chi^3_c(\mathsf{c}_{local1}) > \chi^2_c(\mathsf{c}_{local1})$. Thus, $\mathfrak{K}_{org1}$ has one preferred model which corresponds to χ^1_c: it corresponds to the intended interpretation in which the defeasible axiom $\mathrm{D}(S \sqsubseteq M)$ associated to c_{br1} wins over the more general rules asserted in c_{br2} and c_{world}. ◊

Table B 1. $\mathcal{SROIQ}$*-RLD normal form for axioms in* $\mathcal{L}_\Sigma$

Strict axioms: for $A, B \in \mathrm{NC}$, $R, S, T \in \mathrm{NR}$, $a, b \in \mathrm{NI}$, $\mathsf{c} \in \mathsf{N}$:

$A(a)$ $\quad R(a,b)$ $\quad a = b$ $\quad a \neq b$

$A \sqsubseteq B$ $\quad \{a\} \sqsubseteq B$ $\quad A \sqcap B \sqsubseteq C$

$\exists R.A \sqsubseteq B$ $\quad A \sqsubseteq \exists R.\{a\}$ $\quad A \sqsubseteq \forall R.B$ $\quad A \sqsubseteq \leqslant 1 R.\top$

$R \sqsubseteq T$ $\quad R \circ S \sqsubseteq T$ $\quad \mathrm{Dis}(R,S)$ $\quad \mathrm{Inv}(R,S)$ $\quad \mathrm{Irr}(R)$

$eval(A, \mathsf{c}) \sqsubseteq B$ $\quad eval(R, \mathsf{c}) \sqsubseteq S$

Defeasible axioms: for $A, B \in \mathrm{NC}$, $R, S \in \mathrm{NR}$, $a \in \mathrm{NI}$, $rel \in \{t, c\}$:

$\mathrm{D}_{rel}(A \sqsubseteq B)$ $\quad \mathrm{D}_{rel}(A \sqcap B \sqsubseteq C)$ $\quad \mathrm{D}_{rel}(\exists R.A \sqsubseteq B)$

$\mathrm{D}_{rel}(A \sqsubseteq \exists R.\{a\})$ $\quad \mathrm{D}_{rel}(A \sqsubseteq \forall R.B)$ $\quad \mathrm{D}_{rel}(A \sqsubseteq \leqslant 1 R.\top)$

$\mathrm{D}_{rel}(R \sqsubseteq S)$ $\quad \mathrm{D}_{rel}(R \circ S \sqsubseteq T)$ $\quad \mathrm{D}_{rel}(\mathrm{Dis}(R,S))$ $\quad \mathrm{D}_{rel}(\mathrm{Inv}(R,S))$ $\quad \mathrm{D}_{rel}(\mathrm{Irr}(R))$

Appendix B ASP Translation and Rule Set Tables

We provide further details on the ASP encoding introduced in Section 5. The ASP translation is defined by adapting the encoding presented in (Bozzato et al. 2019; Bozzato et al. 2018b) (which, in turn, is based on the translation introduced in (Bozzato et al. 2018a)) to the manage the interpretation of multiple relations in simple CKRs.

The ASP translation is defined for $\mathcal{SROIQ}$-RLD multi-relational simple CKRs of the form $\mathfrak{K} = \langle \mathfrak{C}, \mathrm{K}_{\mathsf{N}} \rangle$ with $\mathfrak{C} = (\mathsf{N}, \prec_t, \prec_c)$, i.e. over time and coverage contextual relations.

The language of $\mathcal{SROIQ}$-RLD (Bozzato et al. 2018a) restrict the form of $\mathcal{SROIQ}$-RL expressions in defeasible axioms: in defeasible axioms, $D \sqcap D$ can not appear as a right-side concept and each right-side concept $\forall R.D$ has $D \in \mathrm{NC}$. We consider the $\mathcal{SROIQ}$-RLD normal form transformation proposed in (Bozzato et al. 2018a) for the formulation of the rules (considering axioms that can appear in simple CKRs) and we assume again the Unique Name Assumption. For ease of reference, the form of (strict and defeasible) axioms in normal form is presented in Table B 1. Note that we further simplified the normalization of defeasible class and role assertions and negative assertions as they can be easily represented using defeasible class and role inclusions with auxiliary symbols.

As in the original formulation (inspired by the materialization calculus in (Krötzsch 2010)), the translation includes sets of *input rules* I (which encode DL axioms and signature as facts), *deduction rules* P (normal rules providing instance level inference) and *output rules* O (that encode in terms of a fact the ABox assertion to be proved).

The sets of rules for the proposed translation are presented in tables in the following pages. The input rules I_{rl} and deduction rules P_{rl} for $\mathcal{SROIQ}$-RL axioms are shown in Table B 2. Table B 3 shows input rules I_{glob} and deduction rules P_{glob} for the translation of the contextual structure in $\mathfrak{C}$ local input rules I_{eval} and deduction rules P_{eval} for managing *eval* expressions, and output rules O for encoding the output instance query. Input rules I_{D} in Table B 4 provide the encoding of defeasible axioms. Deduction rules in P_{D} manage the interpretation of defeasible axioms and knowledge propagation. Table B 5 shows rules defining the overriding of axioms. Rules for the inheritance of strict axioms are shown in Table B 6, while rules in Table B 7 define defeasible

Table B 2. *$\mathcal{SROIQ}$-RL input and deduction rules*

$\mathcal{SROIQ}$-RL input translation $I_{rl}(S,c)$

(irl-nom)	$a \in \mathrm{NI} \mapsto \{\mathtt{nom}(a,c)\}$
(irl-cls)	$A \in \mathrm{NC} \mapsto \{\mathtt{cls}(A,c)\}$
(irl-rol)	$R \in \mathrm{NR} \mapsto \{\mathtt{rol}(R,c)\}$
(irl-inst1)	$A(a) \mapsto \{\mathtt{insta}(a,A,c)\}$
(irl-triple)	$R(a,b) \mapsto \{\mathtt{triplea}(a,R,b,c)\}$
(irl-eq)	$a = b \mapsto \{\mathtt{eq}(a,b,c,\mathsf{main})\}$
(irl-neq)	$a \neq b \mapsto \emptyset$
(irl-inst3)	$\{a\} \sqsubseteq B \mapsto \{\mathtt{insta}(a,B,c)\}$
(irl-subc)	$A \sqsubseteq B \mapsto \{\mathtt{subClass}(A,B,c)\}$
(irl-top)	$\top(a) \mapsto \{\mathtt{insta}(a,\mathtt{top},c)\}$
(irl-bot)	$\bot(a) \mapsto \{\mathtt{insta}(a,\mathtt{bot},c)\}$
(irl-subcnj)	$A_1 \sqcap A_2 \sqsubseteq B \mapsto \{\mathtt{subConj}(A_1,A_2,B,c)\}$
(irl-subex)	$\exists R.A \sqsubseteq B \mapsto \{\mathtt{subEx}(R,A,B,c)\}$
(irl-supex)	$A \sqsubseteq \exists R.\{a\} \mapsto \{\mathtt{supEx}(A,R,a,c)\}$
(irl-forall)	$A \sqsubseteq \forall R.B \mapsto \{\mathtt{supForall}(A,R,B,c)\}$
(irl-leqone)	$A \sqsubseteq \leqslant 1R.\top \mapsto \{\mathtt{supLeqOne}(A,R,c)\}$
(irl-subr)	$R \sqsubseteq S \mapsto \{\mathtt{subRole}(R,S,c)\}$
(irl-subrc)	$R \circ S \sqsubseteq T \mapsto \{\mathtt{subRChain}(R,S,T,c)\}$
(irl-dis)	$\mathrm{Dis}(R,S) \mapsto \{\mathtt{dis}(R,S,c)\}$
(irl-inv)	$\mathrm{Inv}(R,S) \mapsto \{\mathtt{inv}(R,S,c)\}$
(irl-irr)	$\mathrm{Irr}(R) \mapsto \{\mathtt{irr}(R,c)\}$

$\mathcal{SROIQ}$-RL deduction rules P_{rl}

(prl-instd)	$\mathtt{instd}(x,z,c,\mathsf{main}) \leftarrow \mathtt{insta}(x,z,c).$
(prl-tripled)	$\mathtt{tripled}(x,r,y,c,\mathsf{main}) \leftarrow \mathtt{triplea}(x,r,y,c).$
(prl-eq)	$\mathtt{unsat}(t) \leftarrow \mathtt{eq}(x,y,c,t).$
(prl-top)	$\mathtt{instd}(x,\mathtt{top},c,\mathsf{main}) \leftarrow \mathtt{nom}(x,c).$
(prl-bot)	$\mathtt{unsat}(t) \leftarrow \mathtt{instd}(x,\mathtt{bot},c,t).$
(prl-subc)	$\mathtt{instd}(x,z,c,t) \leftarrow \mathtt{subClass}(y,z,c), \mathtt{instd}(x,y,c,t).$
(prl-subcnj)	$\mathtt{instd}(x,z,c,t) \leftarrow \mathtt{subConj}(y_1,y_2,z,c), \mathtt{instd}(x,y_1,c,t), \mathtt{instd}(x,y_2,c,t).$
(prl-subex)	$\mathtt{instd}(x,z,c,t) \leftarrow \mathtt{subEx}(v,y,z,c), \mathtt{tripled}(x,v,x',c,t), \mathtt{instd}(x',y,c,t).$
(prl-supex)	$\mathtt{tripled}(x,r,x',c,t) \leftarrow \mathtt{supEx}(y,r,x',c), \mathtt{instd}(x,y,c,t).$
(prl-supforall)	$\mathtt{instd}(y,z',c,t) \leftarrow \mathtt{supForall}(z,r,z',c), \mathtt{instd}(x,z,c,t), \mathtt{tripled}(x,r,y,c,t).$
(prl-leqone)	$\mathtt{unsat}(t) \leftarrow \mathtt{supLeqOne}(z,r,c), \mathtt{instd}(x,z,c,t), \mathtt{tripled}(x,r,x_1,c,t), \mathtt{tripled}(x,r,x_2,c,t).$
(prl-subr)	$\mathtt{tripled}(x,w,x',c,t) \leftarrow \mathtt{subRole}(v,w,c), \mathtt{tripled}(x,v,x',c,t).$
(prl-subrc)	$\mathtt{tripled}(x,w,z,c,t) \leftarrow \mathtt{subRChain}(u,v,w,c), \mathtt{tripled}(x,u,y,c,t), \mathtt{tripled}(y,v,z,c,t).$
(prl-dis)	$\mathtt{unsat}(t) \leftarrow \mathtt{dis}(u,v,c), \mathtt{tripled}(x,u,y,c,t), \mathtt{tripled}(x,v,y,c,t).$
(prl-inv1)	$\mathtt{tripled}(y,v,x,c,t) \leftarrow \mathtt{inv}(u,v,c), \mathtt{tripled}(x,u,y,c,t).$
(prl-inv2)	$\mathtt{tripled}(y,u,x,c,t) \leftarrow \mathtt{inv}(u,v,c), \mathtt{tripled}(x,v,y,c,t).$
(prl-irr)	$\mathtt{unsat}(t) \leftarrow \mathtt{irr}(u,c), \mathtt{tripled}(x,u,x,c,t).$
(prl-sat)	$\leftarrow \mathtt{unsat}(\mathsf{main}).$

inheritance. Table B 8 shows rules for the propagation of defeasible axioms on a relation *rel1* over the other relation. Auxiliary test rules in P_{D} are shown in Table B 9. Finally, rules and directives in P_{pref} define the asprin preference: the definition of asprin local and global preferences is shown in Table B 11, while rules in Table B 10 provide auxiliary rules.

Given a multi-relational sCKR $\mathfrak{K} = \langle \mathfrak{C}, \mathrm{K_N} \rangle$ in $\mathcal{SROIQ}$-RLD normal form with $\mathfrak{C} = (\mathsf{N}, \prec_t, \prec_c)$, a program $PK(\mathfrak{K})$ that encodes $\mathfrak{K}$ is obtained as follows:

1. the *global program* for $\mathfrak{C}$ is built as: $PG(\mathfrak{C}) = I_{glob}(\mathfrak{C}) \cup P_{glob}$
2. for each $\mathsf{c} \in \mathsf{N}$, we define each local program for context c as: $PC(\mathsf{c},\mathfrak{K}) = I_{loc}(\mathrm{K_c},\mathsf{c}) \cup P_{loc}$, where $I_{loc}(\mathrm{K_c},\mathsf{c}) = I_{rl}(\mathrm{K_c},\mathsf{c}) \cup I_{eval}(\mathrm{K_c},\mathsf{c}) \cup I_{\mathrm{D}}(\mathrm{K_c},\mathsf{c})$ and $P_{loc} = P_{rl} \cup P_{eval} \cup P_{\mathrm{D}}$
3. The *CKR program* $PK(\mathfrak{K})$ is defined as: $PK(\mathfrak{K}) = PG(\mathfrak{C}) \cup \bigcup_{\mathsf{c} \in \mathsf{N}} PC(\mathsf{c},\mathfrak{K})$

Query answering $\mathfrak{K} \models \mathsf{c} : \alpha$ is then obtained by testing whether the instance query, translated to ASP by $O(\alpha,\mathsf{c})$, is a consequence of the *preferred* models of $PK(\mathfrak{K})$, i.e., whether $PK(\mathfrak{K}) \cup P_{pref} \models O(\alpha,\mathsf{c})$ holds. This can be extended to conjunctive queries Q by applying the output rules to its atoms and checking if $PK(\mathfrak{K}) \cup P_{pref} \models O(Q)$ holds.

Table B 3. *Global, local and output rules*

Global input rules $I_{glob}(\mathfrak{C})$

(igl-ctx)	$\mathsf{c} \in \mathsf{N} \mapsto \{\texttt{context}(\mathsf{c})\}$
(igl-rel-t)	$\prec_t \in \mathfrak{C} \mapsto \{\texttt{relation}(\mathsf{time})\}$
(igl-rel-c)	$\prec_c \in \mathfrak{C} \mapsto \{\texttt{relation}(\mathsf{covers})\}$
(igl-covers-t)	$\mathsf{c}_1 \prec_t \mathsf{c}_2 \mapsto \{\texttt{prec}(\mathsf{c}_1, \mathsf{c}_2, \mathsf{time})\}$
(igl-covers-c)	$\mathsf{c}_1 \prec_c \mathsf{c}_2 \mapsto \{\texttt{prec}(\mathsf{c}_1, \mathsf{c}_2, \mathsf{covers})\}$

Global deduction rules P_{glob}

(pgl-preceq1)	$\texttt{preceq}(c_1, c_2, rel) \leftarrow \texttt{prec}(c_1, c_2, rel).$
(pgl-preceq2)	$\texttt{preceq}(c_1, c_1, rel) \leftarrow \texttt{context}(c_1), \texttt{relation}(rel).$
(pgl-preceqexc1)	$\texttt{preceq\_except}(c_1, c_2, rel) \leftarrow \texttt{relation}(rel), \texttt{preceq}(c_1, c_3, rel_1), \texttt{preceq}(c_3, c_2, rel_2), rel \neq rel_1, rel \neq rel_2.$
(pgl-preceqexc2)	$\texttt{preceq\_except}(c_1, c_2, rel) \leftarrow \texttt{relation}(rel), \texttt{preceq}(c_1, c_2, rel_1), rel \neq rel_1.$

Local *eval* input rules $I_{eval}(S, c)$

(ilc-subevalat)	$eval(A, \mathsf{c}_1) \sqsubseteq B \mapsto \{\texttt{subEval}(A, \mathsf{c}_1, B, \mathsf{c})\}$
(ilc-subevalr)	$eval(R, \mathsf{c}_1) \sqsubseteq T \mapsto \{\texttt{subEvalR}(R, \mathsf{c}_1, T, \mathsf{c})\}$

Local *eval* deduction rules P_{eval}

(plc-subevalat)	$\texttt{instd}(x, b, c, t) \leftarrow \texttt{subEval}(a, c_1, b, c), \texttt{instd}(x, a, c_1, t).$
(plc-subevalr)	$\texttt{tripled}(x, s, y, c, t) \leftarrow \texttt{subEvalR}(r, c_1, s, c), \texttt{tripled}(x, r, y, c_1, t).$
(plc-subevalatp)	$\texttt{instd}(x, b, c, t) \leftarrow \texttt{subEval}(a, c_1, b, c_2), \texttt{instd}(x, a, c_1, t), \texttt{prec}(c, c_3, rel1), \texttt{preceq}(c_3, c_2, rel2), rel1 \neq rel2.$
(plc-subevalrp)	$\texttt{tripled}(x, s, y, c, t) \leftarrow \texttt{subEvalR}(r, c_1, s, c_2), \texttt{tripled}(x, r, y, c_1, t), \texttt{prec}(c, c_3, rel1), \texttt{preceq}(c_3, c_2, rel2), rel1 \neq rel2.$

Output translation $O(\alpha, \mathsf{c})$

(o-concept)	$A(a) \mapsto \{\texttt{instd}(a, A, \mathsf{c}, \mathsf{main})\}$
(o-role)	$R(a, b) \mapsto \{\texttt{tripled}(a, R, b, \mathsf{c}, \mathsf{main})\}$

Table B 4. *Input rules $I_{\mathrm{D}}(S, c)$ for defeasible axioms*

(id-subc)	$\mathrm{D}_{rel}(A \sqsubseteq B) \mapsto \{\,\texttt{def\_subclass}(A, B, c, rel).\,\}$
(id-subcnj)	$\mathrm{D}_{rel}(A_1 \sqcap A_2 \sqsubseteq B) \mapsto \{\,\texttt{def\_subcnj}(A_1, A_2, B, c, rel).\,\}$
(id-subex)	$\mathrm{D}_{rel}(\exists R.A \sqsubseteq B) \mapsto \{\,\texttt{def\_subex}(R, A, B, c, rel).\,\}$
(id-supex)	$\mathrm{D}_{rel}(A \sqsubseteq \exists R.\{a\}) \mapsto \{\,\texttt{def\_supex}(A, R, a, c, rel).\,\}$
(id-forall)	$\mathrm{D}_{rel}(A \sqsubseteq \forall R.B) \mapsto \{\,\texttt{def\_supforall}(A, R, B, c, rel).\,\}$
(id-leqone)	$\mathrm{D}_{rel}(A \sqsubseteq \leqslant 1 R.\top) \mapsto \{\,\texttt{def\_supleqone}(A, R, c, rel).\,\}$
(id-subr)	$\mathrm{D}_{rel}(R \sqsubseteq S) \mapsto \{\,\texttt{def\_subr}(R, S, c, rel).\,\}$
(id-subrc)	$\mathrm{D}_{rel}(R \circ S \sqsubseteq T) \mapsto \{\,\texttt{def\_subrc}(A_1, A_2, B, c, rel).\,\}$
(id-dis)	$\mathrm{D}_{rel}(\mathrm{Dis}(R, S)) \mapsto \{\,\texttt{def\_dis}(R, S, c, rel).\,\}$
(id-inv)	$\mathrm{D}_{rel}(\mathrm{Inv}(R, S)) \mapsto \{\,\texttt{def\_inv}(R, S, c, rel).\,\}$
(id-irr)	$\mathrm{D}_{rel}(\mathrm{Irr}(R)) \mapsto \{\,\texttt{def\_irr}(R, c, rel).\,\}$

Appendix C Translation correctness: more details

Let us consider a CAS-interpretation $\mathfrak{I}_{CAS} = \langle \mathfrak{I}, \overline{\chi} \rangle$ with $\overline{\chi} = (\chi_t, \chi_c)$. We construct its set of atoms corresponding to its overriding assumptions as:

$$OVR(\mathfrak{I}_{CAS}) = \{\, \texttt{ovr}(p(\mathbf{e}), \mathsf{c}, rel) \mid \langle \alpha, \mathbf{e} \rangle \in \chi_{rel}(\mathsf{c}), I_{rl}(\alpha, \mathsf{c}_1) = p \,\}.$$

Table B 5. *Deduction rules P_{D} for defeasible axioms: overriding rules*

(ovr-subc)	$\mathtt{ovr}(\mathtt{subClass}, x, y, z, c_1, c, rel1) \leftarrow$	$\mathtt{def\_subclass}(y, z, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{instd}(x, y, c, \mathsf{main}), \mathtt{not\,test\_fails}(\mathtt{nlit}(x, z, c)).$
(ovr-cnj)	$\mathtt{ovr}(\mathtt{subConj}, x, y_1, y_2, z, c_1, c, rel1) \leftarrow$	$\mathtt{def\_subcnj}(y_1, y_2, z, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{instd}(x, y_1, c, \mathsf{main}), \mathtt{instd}(x, y_2, c, \mathsf{main}),$ $\mathtt{not\,test\_fails}(\mathtt{nlit}(x, z, c)).$
(ovr-subex)	$\mathtt{ovr}(\mathtt{subEx}, x, r, y, z, c_1, c, rel1) \leftarrow$	$\mathtt{def\_subex}(r, y, z, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{tripled}(x, r, w, c, \mathsf{main}), \mathtt{instd}(w, y, c, \mathsf{main}),$ $\mathtt{not\,test\_fails}(\mathtt{nlit}(x, z, c)).$
(ovr-supex)	$\mathtt{ovr}(\mathtt{supEx}, x, y, r, w, c_1, c, rel1) \leftarrow$	$\mathtt{def\_supex}(y, r, w, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{instd}(x, y, c, \mathsf{main}), \mathtt{not\,test\_fails}(\mathtt{nrel}(x, r, w, c)).$
(ovr-forall)	$\mathtt{ovr}(\mathtt{supForall}, x, y, z, r, w, c_1, c, rel1) \leftarrow$	$\mathtt{def\_supforall}(z, r, w, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{instd}(x, z, c, \mathsf{main}), \mathtt{tripled}(x, r, y, c, \mathsf{main}),$ $\mathtt{not\,test\_fails}(\mathtt{nlit}(y, w, c)).$
(ovr-leqone)	$\mathtt{ovr}(\mathtt{supLeqOne}, x, x_1, x_2, z, r, c_1, c, rel1) \leftarrow$	$\mathtt{def\_supleqone}(z, r, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{instd}(x, z, c, \mathsf{main}), \mathtt{tripled}(x, r, x_1, c, \mathsf{main}),$ $\mathtt{tripled}(x, r, x_2, c, \mathsf{main}),$
(ovr-subr)	$\mathtt{ovr}(\mathtt{subRole}, x, y, r, s, c_1, c, rel1) \leftarrow$	$\mathtt{def\_subr}(r, s, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{tripled}(x, r, y, c, \mathsf{main}), \mathtt{not\,test\_fails}(\mathtt{nrel}(x, s, y, c)).$
(ovr-subrc)	$\mathtt{ovr}(\mathtt{subRChain}, x, y, z, r, s, t, c_1, c, rel1) \leftarrow$	$\mathtt{def\_subrc}(r, s, t, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{tripled}(x, r, y, c, \mathsf{main}), \mathtt{tripled}(y, s, z, c, \mathsf{main}),$ $\mathtt{not\,test\_fails}(\mathtt{nrel}(x, t, z, c)).$
(ovr-dis)	$\mathtt{ovr}(\mathtt{dis}, x, y, r, s, c_1, c, rel1) \leftarrow$	$\mathtt{def\_dis}(r, s, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{tripled}(x, r, y, c, \mathsf{main}), \mathtt{tripled}(x, s, y, c, \mathsf{main}).$
(ovr-inv1)	$\mathtt{ovr}(\mathtt{inv}, x, y, r, s, c_1, c, rel1) \leftarrow$	$\mathtt{def\_inv}(r, s, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{tripled}(x, r, y, c, \mathsf{main}), \mathtt{not\,test\_fails}(\mathtt{nrel}(x, s, y, c)).$
(ovr-inv2)	$\mathtt{ovr}(\mathtt{inv}, x, y, r, s, c_1, c, rel1) \leftarrow$	$\mathtt{def\_inv}(r, s, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{tripled}(y, s, x, c, \mathsf{main}), \mathtt{not\,test\_fails}(\mathtt{nrel}(x, r, y, c)).$
(ovr-irr)	$\mathtt{ovr}(\mathtt{irr}, x, r, c_1, c, rel1) \leftarrow$	$\mathtt{def\_irr}(r, c_1, rel1),$ $\mathtt{prec}(c, c_2, rel1), \mathtt{preceq}(c_2, c_1, rel2), rel1 \neq rel2,$ $\mathtt{tripled}(x, r, x, c, \mathsf{main}).$

We can build[1] from its components a corresponding Herbrand interpretation $I(\mathfrak{I}_{CAS})$ of the program $PK(\mathfrak{K})$ as the smallest set of literals containing:

– all facts of $PK(\mathfrak{K})$;
– $\mathtt{instd}(a, A, \mathsf{c}, \mathsf{main})$, if $\mathcal{I}(\mathsf{c}) \models A(a)$;
– $\mathtt{tripled}(a, R, b, \mathsf{c}, \mathsf{main})$, if $\mathcal{I}(\mathsf{c}) \models R(a, b)$;
– each $\mathtt{ovr}$-literal from $OVR(\mathfrak{I}_{CAS})$;

[1] See similar construction in (Bozzato et al. 2018a, Section A.5.2) for further details.

Table B 6. *Deduction rules P_D for defeasible axioms: strict inheritance rules*

(props-inst)	$\texttt{instd}(x,z,c,\mathsf{main}) \leftarrow \texttt{insta}(x,z,c_1),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-triple)	$\texttt{tripled}(x,r,y,c,\mathsf{main}) \leftarrow \texttt{triplea}(x,r,y,c_1),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-subc)	$\texttt{instd}(x,z,c,t) \leftarrow \texttt{subClass}(y,z,c_1),\texttt{instd}(x,y,c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-cnj)	$\texttt{instd}(x,z,c,t) \leftarrow \texttt{subConj}(y_1,y_2,z,c_1),\texttt{instd}(x,y_1,c,t),\texttt{instd}(x,y_2,c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-subex)	$\texttt{instd}(x,z,c,t) \leftarrow \texttt{subEx}(v,y,z,c_1),\texttt{tripled}(x,v,x',c,t),\texttt{instd}(x',y,c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-supex)	$\texttt{tripled}(x,r,x',c,t) \leftarrow \texttt{supEx}(y,r,x',c_1),\texttt{instd}(x,y,c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-forall)	$\texttt{instd}(y,z',c,t) \leftarrow \texttt{supForall}(z,r,z',c_1),\texttt{instd}(x,z,c,t),\texttt{tripled}(x,r,y,c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-leqone)	$\texttt{unsat}(t) \leftarrow \texttt{supLeqOne}(z,r,c_1),\texttt{instd}(x,z,c,t),$ $\texttt{tripled}(x,r,x_1,c,t),\texttt{tripled}(x,r,x_2,c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-subr)	$\texttt{tripled}(x,w,x',c,t) \leftarrow \texttt{subRole}(v,w,c_1),\texttt{tripled}(x,v,x',c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-subrc)	$\texttt{tripled}(x,w,z,c,t) \leftarrow \texttt{subRChain}(u,v,w,c_1),\texttt{tripled}(x,u,y,c,t),\texttt{tripled}(y,v,z,c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-dis)	$\texttt{unsat}(t) \leftarrow \texttt{dis}(u,v,c_1),\texttt{tripled}(x,u,y,c,t),\texttt{tripled}(x,v,y,c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-inv1)	$\texttt{tripled}(y,v,x,c,t) \leftarrow \texttt{inv}(u,v,c_1),\texttt{tripled}(x,u,y,c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-inv2)	$\texttt{tripled}(x,u,y,c,t) \leftarrow \texttt{inv}(u,v,c_1),\texttt{tripled}(y,v,x,c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$
(props-irr)	$\texttt{unsat}(t) \leftarrow \texttt{irr}(u,c_1),\texttt{tripled}(x,u,x,c,t),$ $\texttt{prec}(c,c_2,rel1),\texttt{preceq}(c_2,c_1,rel2),rel1 \neq rel2.$

- each literal l with environment $t \neq \mathsf{main}$, if $\texttt{test}(t) \in I(\mathfrak{I}_{CAS})$ and l is in the head of a rule $r \in grnd(PK(\mathfrak{K}))$ with $B(r) \subseteq I(\mathfrak{I}_{CAS})$;
- $\texttt{test}(t)$, if $\texttt{test\_fails}(t)$ appears in the body of an overriding rule r in $grnd(PK(\mathfrak{K}))$ and the head of r is an ovr literal in $OVR(\mathfrak{I}_{CAS})$;
- $\texttt{unsat}(t) \in I(\mathfrak{I}_{CAS})$, if adding the literal corresponding to t to the local interpretation of its context c violates some axiom of the local knowledge K_{c};
- $\texttt{test\_fails}(t)$, if $\texttt{unsat}(t) \notin I(\mathfrak{I}_{CAS})$.

Note that $\texttt{unsat}(\mathsf{main})$ is not included in $I(\mathfrak{I}_{CAS})$. Moreover, as all the facts of $PK(\mathfrak{K})$ are included in the set, also the atoms prec and preceq defining the contextual relations of $\mathfrak{K}$ are included in $I(\mathfrak{I}_{CAS})$.

The correctness result provided by Theorem 2 in Section 5 is a consequence of the following Lemma 1, showing the correspondence between the minimal justified CKR-models of $\mathfrak{K}$ and the answer sets of $PK(\mathfrak{K})$, and Lemma 2, proving the correspondence between preferred models and answer sets selected by the asprin preference in P_{pref}.

Lemma 1

Let $\mathfrak{K}$ be a multi-relational sCKR in $\mathcal{SROIQ}$-RLD normal form, then:

Table B 7. *Deduction rules P_D for defeasible axioms: defeasible inheritance rules*

(propd-subc)	$\texttt{instd}(x,z,c,t) \leftarrow \texttt{def\_subclass}(y,z,c_1,rel1), \texttt{instd}(x,y,c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{subClass},x,y,z,c_1,c,rel1).$
(propd-cnj)	$\texttt{instd}(x,z,c,t) \leftarrow \texttt{def\_subcnj}(y_1,y_2,z,c_1,rel1), \texttt{instd}(x,y_1,c,t), \texttt{instd}(x,y_2,c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{subConj},x,y_1,y_2,z,c_1,c,rel1).$
(propd-subex)	$\texttt{instd}(x,z,c,t) \leftarrow \texttt{def\_subex}(v,y,z,c_1,rel1), \texttt{tripled}(x,v,x',c,t), \texttt{instd}(x',y,c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{subEx},x,v,y,z,c_1,c,rel1).$
(propd-supex)	$\texttt{tripled}(x,r,x',c,t) \leftarrow \texttt{def\_supex}(y,r,x',c_1,rel1), \texttt{instd}(x,y,c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{supEx},x,y,r,x',c_1,c,rel1).$
(propd-forall)	$\texttt{instd}(y,z',c,t) \leftarrow \texttt{def\_supforall}(z,r,z',c_1,rel1), \texttt{instd}(x,z,c,t), \texttt{tripled}(x,r,y,c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{supForall},x,y,z,r,z',c_1,c,rel1).$
(propd-leqone)	$\texttt{unsat}(t) \leftarrow \texttt{def\_supleqone}(z,r,c_1,rel1), \texttt{instd}(x,z,c,t),$ $\texttt{tripled}(x,r,x_1,c,t), \texttt{tripled}(x,r,x_2,c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{supLeqOne},x,x_1,x_2,z,r,c_1,c,rel1).$
(propd-subr)	$\texttt{tripled}(x,w,x',c,t) \leftarrow \texttt{def\_subr}(v,w,c_1,rel1), \texttt{tripled}(x,v,x',c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{subRole},x,y,v,w,c_1,c,rel1).$
(propd-subrc)	$\texttt{tripled}(x,w,z,c,t) \leftarrow \texttt{def\_subrc}(u,v,w,c_1,rel1), \texttt{tripled}(x,u,y,c,t), \texttt{tripled}(y,v,z,c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{subRChain},x,y,z,u,v,w,c_1,c,rel1).$
(propd-dis)	$\texttt{unsat}(t) \leftarrow \texttt{def\_dis}(u,v,c_1,rel1), \texttt{tripled}(x,u,y,c,t), \texttt{tripled}(x,v,y,c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{dis},x,y,u,v,c_1,c,rel1).$
(propd-inv1)	$\texttt{tripled}(y,v,x,c,t) \leftarrow \texttt{def\_inv}(u,v,c_1,rel1), \texttt{tripled}(x,u,y,c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{inv},x,y,u,v,c_1,c,rel1).$
(propd-inv2)	$\texttt{tripled}(x,u,y,c,t) \leftarrow \texttt{def\_inv}(u,v,c_1,rel1), \texttt{tripled}(y,v,x,c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{inv},x,y,u,v,c_1,c,rel1).$
(propd-irr)	$\texttt{unsat}(t) \leftarrow \texttt{def\_irr}(u,c_1,rel1), \texttt{tripled}(x,u,x,c,t),$ $\texttt{prec}(c,c_2,rel1), \texttt{preceq}(c_2,c_1,rel2), rel1 \neq rel2,$ $\texttt{not ovr}(\texttt{irr},x,u,c_1,c,rel1).$

(i). for every (named) justified clashing assumption $\overline{\chi}$, the interpretation $S = I(\hat{\mathfrak{I}}(\overline{\chi}))$ is an answer set of $PK(\mathfrak{K})$;

(ii). every answer set S of $PK(\mathfrak{K})$ is of the form $S = I(\hat{\mathfrak{I}}(\overline{\chi}))$ with $\overline{\chi}$ a (named) justified clashing assumption for $\mathfrak{K}$.

Proof (Sketch)

Intuitively, we are interested in computing the correspondence with all (not necessarily preferred) answer sets of $PK(\mathfrak{K})$: we can show that the new form of rules for managing multiple contextual relations do not influence the construction of such answer sets, thus the result can be proved similarly to Lemma 6 in (Bozzato et al. 2018a) and its extension to hierarchies in (Bozzato et al. 2018b, Lemma 1).

Let us consider the interpretation $S = I(\hat{\mathfrak{I}}(\overline{\chi}))$ as defined above and the reduct $G_S(PK(\mathfrak{K}))$

Table B 8. *Deduction rules P_D for defeasible axioms: parallel inheritance rules*

(propp-subc)	$\texttt{instd}(x,z,c,t)$	$\leftarrow \texttt{def\_subclass}(y,z,c_1,rel1), \texttt{instd}(x,y,c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$
(propp-cnj)	$\texttt{instd}(x,z,c,t)$	$\leftarrow \texttt{def\_subcnj}(y_1,y_2,z,c_1,rel1), \texttt{instd}(x,y_1,c,t), \texttt{instd}(x,y_2,c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$
(propp-subex)	$\texttt{instd}(x,z,c,t)$	$\leftarrow \texttt{def\_subex}(v,y,z,c_1,rel1), \texttt{tripled}(x,v,x',c,t), \texttt{instd}(x',y,c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$
(propp-supex)	$\texttt{tripled}(x,r,x',c,t)$	$\leftarrow \texttt{def\_supex}(y,r,x',c_1,rel1), \texttt{instd}(x,y,c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$
(propp-forall)	$\texttt{instd}(y,z',c,t)$	$\leftarrow \texttt{def\_supforall}(z,r,z',c_1,rel1), \texttt{instd}(x,z,c,t), \texttt{tripled}(x,r,y,c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$
(propp-leqone)	$\texttt{unsat}(t)$	$\leftarrow \texttt{def\_supleqone}(z,r,c_1,rel1), \texttt{instd}(x,z,c,t),$ $\texttt{tripled}(x,r,x_1,c,t), \texttt{tripled}(x,r,x_2,c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$
(propp-subr)	$\texttt{tripled}(x,w,x',c,t)$	$\leftarrow \texttt{def\_subr}(v,w,c_1,rel1), \texttt{tripled}(x,v,x',c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$
(propp-subrc)	$\texttt{tripled}(x,w,z,c,t)$	$\leftarrow \texttt{def\_subrc}(u,v,w,c_1,rel1), \texttt{tripled}(x,u,y,c,t), \texttt{tripled}(y,v,z,c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$
(propp-dis)	$\texttt{unsat}(t)$	$\leftarrow \texttt{def\_dis}(u,v,c_1,rel1), \texttt{tripled}(x,u,y,c,t), \texttt{tripled}(x,v,y,c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$
(propp-inv1)	$\texttt{tripled}(y,v,x,c,t)$	$\leftarrow \texttt{def\_inv}(u,v,c_1,rel1), \texttt{tripled}(x,u,y,c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$
(propp-inv2)	$\texttt{tripled}(x,u,y,c,t)$	$\leftarrow \texttt{def\_inv}(u,v,c_1,rel1), \texttt{tripled}(y,v,x,c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$
(propp-irr)	$\texttt{unsat}(t)$	$\leftarrow \texttt{def\_irr}(u,c_1,rel1), \texttt{tripled}(x,u,x,c,t),$ $\texttt{preceq}(c,c_1,rel2), rel1 \neq rel2.$

of $PK(\mathfrak{K})$ with respect S. The lemma can then be proved by showing that the answer sets of $PK(\mathfrak{K})$ coincide with the sets $S = I(\hat{\mathfrak{I}}(\overline{\chi}))$ where $\overline{\chi} = (\chi_t, \chi_c)$ is composed by justified clashing assumptions of $\mathfrak{K}$.

(i). Assuming that $\overline{\chi} = (\chi_t, \chi_c)$ is justified, we show that $S = I(\hat{\mathfrak{I}}(\overline{\chi}))$ is an answer set of $PK(\mathfrak{K})$.

We first prove that $S \models G_S(PK(\mathfrak{K}))$, that is for every rule instance $r \in G_S(PK(\mathfrak{K}))$ it holds that $S \models r$. This is proved by examining the possible rule forms that occur in $G_S(PK(\mathfrak{K}))$. Here we show some representative cases (see also (Bozzato et al. 2018a)):

- **(prl-instd):** then $\texttt{insta}(a,A,\mathsf{c}) \in I(\hat{\mathfrak{I}}(\overline{\chi}))$ and, by definition of the translation, $A(a) \in \mathrm{K}_{\mathsf{c}}$. This implies that $\mathcal{I}(\mathsf{c}) \models A(a)$ and thus $\texttt{instd}(a,A,\mathsf{c},\mathsf{main})$ is added to $I(\hat{\mathfrak{I}}(\overline{\chi}))$.
- **(prl-subc):** then $\{\texttt{subClass}(A,B,\mathsf{c}), \texttt{instd}(a,A,\mathsf{c},t)\} \subseteq I(\hat{\mathfrak{I}}(\overline{\chi}))$. By definition of the translation, we have $A \sqsubseteq B \in \mathrm{K}_{\mathsf{c}}$. For the construction of $I(\hat{\mathfrak{I}}(\overline{\chi}))$, if $t = \mathsf{main}$ then $\mathcal{I}(\mathsf{c}) \models A(a)$. This implies that $\mathcal{I}(\mathsf{c}) \models B(a)$ and $\texttt{instd}(a,B,\mathsf{c},\mathsf{main})$ is added to $I(\hat{\mathfrak{I}}(\overline{\chi}))$. Otherwise, if $t \neq \mathsf{main}$ then $\texttt{instd}(a,B,\mathsf{c},t)$ is directly added to $I(\hat{\mathfrak{I}}(\chi))$ by its construction.
- **(ovr-subc):** then $\{\texttt{def\_subclass}(A,B,\mathsf{c}_1,rel1),\ \texttt{prec}(\mathsf{c},\mathsf{c}_2,rel1),\ \texttt{preceq}(\mathsf{c}_2,\mathsf{c}_1,rel2),$ $\texttt{instd}(a,A,\mathsf{c},\mathsf{main})\} \subseteq I(\hat{\mathfrak{I}}(\overline{\chi}))$. Since $r \in G_S(PK(\mathfrak{K}))$, then $\texttt{test\_fails}(\texttt{nlit}(a,B,\mathsf{c})) \notin I(\hat{\mathfrak{I}}(\overline{\chi}))$. By construction of $I(\hat{\mathfrak{I}}(\overline{\chi}))$, this implies that $\texttt{unsat}(\texttt{nlit}(a,B,\mathsf{c})) \in I(\hat{\mathfrak{I}}(\overline{\chi}))$, meaning that $\mathcal{I}(\mathsf{c}) \models \neg B(a)$. Thus, $\mathcal{I}(\mathsf{c})$ satisfies the clashing set $\{A(a), \neg B(a)\}$ for the clashing assumption $\langle A \sqsubseteq B, a\rangle$ for $rel1$ in context c. Consequently, $\langle A \sqsubseteq B, a\rangle \in \chi_{rel1}(\mathsf{c})$ and by construction $\texttt{ovr}(\texttt{subClass}, a, A, B, \mathsf{c})$ is added to $I(\hat{\mathfrak{I}}(\overline{\chi}))$.
- **(props-subc):** then $\{\texttt{subClass}(A,B,\mathsf{c}_1),\ \texttt{instd}(a,A,\mathsf{c},t),\ \texttt{preceq}(\mathsf{c}_2,\mathsf{c}_1,rel2),$

Table B 9. *Deduction rules P_{D} for defeasible axioms: test rules*

(test-subc)	$\mathtt{test}(\mathtt{nlit}(x,z,c))$	$\leftarrow \mathtt{def\_subclass}(y,z,c_1,rel1), \mathtt{instd}(x,y,c,\mathsf{main}),$ $\mathtt{prec}(c,c_2,rel1), \mathtt{preceq}(c_2,c_1,rel2), rel1 \neq rel2.$
(constr-subc)		$\leftarrow \mathtt{test\_fails}(\mathtt{nlit}(x,z,c)), \mathtt{ovr}(\mathtt{subClass},x,y,z,c_1,c,rel).$
(test-subcnj)	$\mathtt{test}(\mathtt{nlit}(x,z,c))$	$\leftarrow \mathtt{def\_subcnj}(y_1,y_2,z,c_1,rel1),$ $\mathtt{instd}(x,y_1,c,\mathsf{main}), \mathtt{instd}(x,y_2,c,\mathsf{main}),$ $\mathtt{prec}(c,c_2,rel1), \mathtt{preceq}(c_2,c_1,rel2), rel1 \neq rel2.$
(constr-subcnj)		$\leftarrow \mathtt{test\_fails}(\mathtt{nlit}(x,z,c)), \mathtt{ovr}(\mathtt{subConj},x,y_1,y_2,z,c_1,c,rel).$
(test-subex)	$\mathtt{test}(\mathtt{nlit}(x,z,c))$	$\leftarrow \mathtt{def\_subex}(r,y,z,c_1,rel1),$ $\mathtt{tripled}(x,r,w,c,\mathsf{main}), \mathtt{instd}(w,y,c,\mathsf{main}),$ $\mathtt{prec}(c,c_2,rel1), \mathtt{preceq}(c_2,c_1,rel2), rel1 \neq rel2.$
(constr-subex)		$\leftarrow \mathtt{test\_fails}(\mathtt{nlit}(x,z,c)), \mathtt{ovr}(\mathtt{subEx},x,r,y,z,c_1,c,rel).$
(test-supex)	$\mathtt{test}(\mathtt{nrel}(x,r,w,c))$	$\leftarrow \mathtt{def\_supex}(y,r,w,c_1,rel1), \mathtt{instd}(x,y,c,\mathsf{main}),$ $\mathtt{prec}(c,c_2,rel1), \mathtt{preceq}(c_2,c_1,rel2), rel1 \neq rel2.$
(constr-supex)		$\leftarrow \mathtt{test\_fails}(\mathtt{nrel}(x,r,w,c)), \mathtt{ovr}(\mathtt{supEx},x,r,y,w,c_1,c,rel).$
(test-supforall)	$\mathtt{test}(\mathtt{nlit}(y,w,c))$	$\leftarrow \mathtt{def\_supforall}(z,r,w,c_1,rel1),$ $\mathtt{instd}(x,z,c,\mathsf{main}), \mathtt{tripled}(x,r,y,c,\mathsf{main}),$ $\mathtt{prec}(c,c_2,rel1), \mathtt{preceq}(c_2,c_1,rel2), rel1 \neq rel2.$
(constr-supforall)		$\leftarrow \mathtt{test\_fails}(\mathtt{nlit}(y,w,c)), \mathtt{ovr}(\mathtt{supForall},x,y,z,r,w,c_1,c,rel).$
(test-subr)	$\mathtt{test}(\mathtt{nrel}(x,s,y,c))$	$\leftarrow \mathtt{def\_subr}(r,s,c_1,rel1), \mathtt{tripled}(x,r,y,c,\mathsf{main}),$ $\mathtt{prec}(c,c_2,rel1), \mathtt{preceq}(c_2,c_1,rel2), rel1 \neq rel2.$
(constr-subr)		$\leftarrow \mathtt{test\_fails}(\mathtt{nrel}(x,s,y,c)), \mathtt{ovr}(\mathtt{subRole},x,r,y,s,c_1,c,rel).$
(test-subrc)	$\mathtt{test}(\mathtt{nrel}(x,t,z,c))$	$\leftarrow \mathtt{def\_subrc}(r,s,t,c_1,rel1),$ $\mathtt{tripled}(x,r,y,c,\mathsf{main}), \mathtt{tripled}(y,s,z,c,\mathsf{main}),$ $\mathtt{prec}(c,c_2,rel1), \mathtt{preceq}(c_2,c_1,rel2), rel1 \neq rel2.$
(constr-subrc)		$\leftarrow \mathtt{test\_fails}(\mathtt{nrel}(x,t,z,c)), \mathtt{ovr}(\mathtt{subRChain},x,y,z,r,s,t,c_1,c,rel).$
(test-inv1)	$\mathtt{test}(\mathtt{nrel}(x,s,y,c))$	$\leftarrow \mathtt{def\_inv}(r,s,c_1,rel1), \mathtt{tripled}(x,r,y,c,\mathsf{main}),$ $\mathtt{prec}(c,c_2,rel1), \mathtt{preceq}(c_2,c_1,rel2), rel1 \neq rel2.$
(test-inv2)	$\mathtt{test}(\mathtt{nrel}(y,r,x,c))$	$\leftarrow \mathtt{def\_inv}(r,s,c_1,rel), \mathtt{tripled}(x,s,y,c,\mathsf{main}),$ $\mathtt{prec}(c,c_2,rel1), \mathtt{preceq}(c_2,c_1,rel2), rel1 \neq rel2.$
(constr-inv1)		$\leftarrow \mathtt{not}\, \mathtt{test\_fails}(\mathtt{nrel}(x,s,y,c)), \mathtt{ovr}(\mathtt{inv},x,y,r,s,c_1,c,rel).$
(constr-inv2)		$\leftarrow \mathtt{not}\, \mathtt{test\_fails}(\mathtt{nrel}(y,r,x,c)), \mathtt{ovr}(\mathtt{inv},x,y,r,s,c_1,c,rel).$
(test-fails1)	$\mathtt{test\_fails}(\mathtt{nlit}(x,z,c))$	$\leftarrow \mathtt{instd}(x,z,c,\mathtt{nlit}(x,z,c)), \mathtt{not}\, \mathtt{unsat}(\mathtt{nlit}(x,z,c)).$
(test-fails2)	$\mathtt{test\_fails}(\mathtt{nrel}(x,r,y,c))$	$\leftarrow \mathtt{tripled}(x,r,y,c,\mathtt{nrel}(x,r,y,c)), \mathtt{not}\, \mathtt{unsat}(\mathtt{nrel}(x,r,y,c)).$
(test-add1)	$\mathtt{instd}(x,z,c,\mathtt{nlit}(x,z,c))$	$\leftarrow \mathtt{test}(\mathtt{nlit}(x,z,c)).$
(test-add2)	$\mathtt{tripled}(x,r,y,c,\mathtt{nrel}(x,r,y,c))$	$\leftarrow \mathtt{test}(\mathtt{nrel}(x,r,y,c)).$
(test-copy1)	$\mathtt{instd}(x_1,y_1,c,t)$	$\leftarrow \mathtt{instd}(x_1,y_1,c,\mathsf{main}), \mathtt{test}(t).$
(test-copy2)	$\mathtt{tripled}(x_1,r,y_1,c,t)$	$\leftarrow \mathtt{tripled}(x_1,r,y_1,c,\mathsf{main}), \mathtt{test}(t).$

$\mathtt{prec}(\mathsf{c},\mathsf{c}_2,rel1)\} \subseteq I(\hat{\mathfrak{I}}(\overline{\chi}))$. By definition, $A \sqsubseteq B \in \mathrm{K}_{\mathsf{c}_1}$ and, if $t = \mathsf{main}$, $\mathcal{I}(\mathsf{c}) \models A(a)$. Thus, for the definition of *CAS*-model (condition (i) on strict axioms propagation), $\mathtt{instd}(a,B,\mathsf{c},t)$ is added to $I(\hat{\mathfrak{I}}(\overline{\chi}))$. If $t \neq \mathsf{main}$, then $\mathtt{instd}(a,B,\mathsf{c},t)$ is added to $I(\hat{\mathfrak{I}}(\chi))$ by construction.

- **(propd-subc):** then $\{\mathtt{def\_subclass}(A,B,\mathsf{c}_1,rel1), \mathtt{instd}(a,A,\mathsf{c},t), \mathtt{prec}(\mathsf{c},\mathsf{c}_2,rel1), \mathtt{preceq}(\mathsf{c}_2,\mathsf{c}_1,rel2)\} \subseteq I(\hat{\mathfrak{I}}(\overline{\chi}))$. Since $r \in G_S(PK(\mathfrak{K}))$, $\mathtt{ovr}(\mathtt{subClass},a,A,B,\mathsf{c}_1,\mathsf{c},rel1) \notin OVR(\hat{\mathfrak{I}}(\overline{\chi}))$ and hence $\langle A \sqsubseteq B, a\rangle \notin \chi_{rel1}(\mathsf{c})$. By definition, $D(A \sqsubseteq B) \in \mathrm{K}_{\mathsf{c}_1}$ and, if $t = \mathsf{main}$, $\mathcal{I}(\mathsf{c}) \models A(a)$. Thus, for the definition of *CAS*-model (condition (iii) on defeasible axioms propagation), $\mathtt{instd}(a,B,\mathsf{c},t)$ is added to $I(\hat{\mathfrak{I}}(\overline{\chi}))$. If $t \neq \mathsf{main}$, then $\mathtt{instd}(a,B,\mathsf{c},t)$ is added to $I(\hat{\mathfrak{I}}(\chi))$ by construction.
- **(propp-subc):** then $\{\mathtt{def\_subclass}(A,B,\mathsf{c}_1,rel1), \mathtt{instd}(a,A,\mathsf{c},t), \mathtt{preceq}(\mathsf{c},\mathsf{c}_1,rel2)\} \subseteq I(\hat{\mathfrak{I}}(\overline{\chi}))$. By definition, $D(A \sqsubseteq B) \in \mathrm{K}_{\mathsf{c}_1}$ and, if $t = \mathsf{main}$, $\mathcal{I}(\mathsf{c}) \models A(a)$ with $\mathsf{c} \prec_{rel2} \mathsf{c}_1$. Thus,

Table B 10. *Rules in P_{pref} for preference definitions: preparation rules*

(prep-indiv)	$\mathtt{ind}(x)$	$\leftarrow$	$\mathtt{nom}(x,c)$.
(prep-ovr-subs)	$\mathtt{p\_ovr}(\mathtt{subClass}(x,y,z),c,rel)$	$\leftarrow$	$\mathtt{def\_subclass}(y,z,c,rel)$, $\mathtt{ind}(x)$.
(prep-ovr-subc)	$\mathtt{p\_ovr}(\mathtt{subConj}(x,y1,y2,z),c,rel)$	$\leftarrow$	$\mathtt{def\_subcnj}(y1,y2,z,c,rel)$, $\mathtt{ind}(x)$.
(prep-ovr-subex)	$\mathtt{p\_ovr}(\mathtt{subEx}(x,r,y,z),c,rel)$	$\leftarrow$	$\mathtt{def\_subex}(r,y,z,c,rel)$, $\mathtt{ind}(x)$.
(prep-ovr-supex)	$\mathtt{p\_ovr}(\mathtt{supEx}(x,y,r,w),c,rel)$	$\leftarrow$	$\mathtt{def\_supex}(y,r,w,c,rel)$, $\mathtt{ind}(x)$.
(prep-ovr-supfa)	$\mathtt{p\_ovr}(\mathtt{supForall}(x,y,z,r,w),c,rel)$	$\leftarrow$	$\mathtt{def\_supforall}(z,r,w,c,rel)$, $\mathtt{ind}(x),\mathtt{ind}(y)$.
(prep-ovr-suble)	$\mathtt{p\_ovr}(\mathtt{supLeqOne}(x,x1,x2,z,r),c,rel)$	$\leftarrow$	$\mathtt{def\_supleqone}(z,r,c,rel)$, $\mathtt{ind}(x),\mathtt{ind}(x1),\mathtt{ind}(x2)$.
(prep-ovr-subr)	$\mathtt{p\_ovr}(\mathtt{subRole}(x,y,r,s),c,rel)$	$\leftarrow$	$\mathtt{def\_subr}(r,s,c,rel)$, $\mathtt{ind}(x),\mathtt{ind}(y)$.
(prep-ovr-subrc)	$\mathtt{p\_ovr}(\mathtt{subRChain}(x,y,z,r,s,t),c,rel)$	$\leftarrow$	$\mathtt{def\_subrc}(r,s,t,c,rel)$, $\mathtt{ind}(x),\mathtt{ind}(y),\mathtt{ind}(z)$.
(prep-ovr-dis)	$\mathtt{p\_ovr}(\mathtt{dis}(x,y,r,s),c,rel)$	$\leftarrow$	$\mathtt{def\_dis}(r,s,c,rel)$, $\mathtt{ind}(x),\mathtt{ind}(y)$.
(prep-ovr-inv)	$\mathtt{p\_ovr}(\mathtt{inv}(x,y,r,s),c,rel)$	$\leftarrow$	$\mathtt{def\_inv}(r,s,c,rel)$, $\mathtt{ind}(x),\mathtt{ind}(y)$.
(prep-ovr-irr)	$\mathtt{p\_ovr}(\mathtt{irr}(x,r),c,rel)$	$\leftarrow$	$\mathtt{def\_irr}(r,c,rel)$, $\mathtt{ind}(x)$.
(act-ovr-subs)	$\mathtt{ovr}(\mathtt{subClass}(x,y,z),c_1,c,rel)$	$\leftarrow$	$\mathtt{ovr}(\mathtt{subClass},x,y,z,c_1,c,rel)$
(act-ovr-subc)	$\mathtt{ovr}(\mathtt{subConj}(x,y1,y2,z),c_1,c,rel)$	$\leftarrow$	$\mathtt{ovr}(\mathtt{subConj},x,y1,y2,z,c_1,c,rel)$.
(act-ovr-subex)	$\mathtt{ovr}(\mathtt{subEx}(x,r,y,z),c_1,c,rel)$	$\leftarrow$	$\mathtt{ovr}(\mathtt{subEx},x,r,y,z,c_1,c,rel)$.
(act-ovr-supex)	$\mathtt{ovr}(\mathtt{supEx}(x,y,r,w),c_1,c,rel)$	$\leftarrow$	$\mathtt{ovr}(\mathtt{supEx},x,y,r,w,c_1,c,rel)$.
(act-ovr-supfa)	$\mathtt{ovr}(\mathtt{supForall}(x,y,z,r,w),c_1,c,rel)$	$\leftarrow$	$\mathtt{ovr}(\mathtt{supForall},x,y,z,r,w,c_1,c,rel)$.
(act-ovr-suble)	$\mathtt{ovr}(\mathtt{supLeqOne}(x,x1,x2,z,r),c_1,c,rel)$	$\leftarrow$	$\mathtt{ovr}(\mathtt{supLeqOne},x,x1,x2,z,r,c_1,c,rel)$.
(act-ovr-subr)	$\mathtt{ovr}(\mathtt{subRole}(x,y,r,s),c_1,c,rel)$	$\leftarrow$	$\mathtt{ovr}(\mathtt{subRole},x,y,r,s,c_1,c,rel)$.
(act-ovr-subrc)	$\mathtt{ovr}(\mathtt{subRChain}(x,y,z,r,s,t),c_1,c,rel)$	$\leftarrow$	$\mathtt{ovr}(\mathtt{subRChain},x,y,z,r,s,t,c_1,c,rel)$.
(act-ovr-dis)	$\mathtt{ovr}(\mathtt{dis}(x,y,r,s),c_1,c,rel)$	$\leftarrow$	$\mathtt{ovr}(\mathtt{dis},x,y,r,s,c_1,c,rel)$.
(act-ovr-inv)	$\mathtt{ovr}(\mathtt{inv}(x,y,r,s),c_1,c,rel)$	$\leftarrow$	$\mathtt{ovr}(\mathtt{inv},x,y,r,s,c_1,c,rel)$.
(act-ovr-irr)	$\mathtt{ovr}(\mathtt{irr}(x,r),c_1,c,rel)$	$\leftarrow$	$\mathtt{ovr}(\mathtt{irr},x,r,c_1,c,rel)$.

for the definition of *CAS*-model (condition (ii) on propagation of defeasible axioms over other relations), $\mathtt{instd}(a,B,\mathsf{c},t)$ is added to $I(\hat{\mathfrak{I}}(\overline{\chi}))$. If $t \neq \mathsf{main}$, then $\mathtt{instd}(a,B,\mathsf{c},t)$ is added to $I(\hat{\mathfrak{I}}(\chi))$ by construction.

– **(test-subc):** then $\{\mathtt{def\_subclass}(A,B,\mathsf{c}_1,rel1), \mathtt{instd}(a,A,\mathsf{c},\mathsf{main}), \mathtt{prec}(\mathsf{c},\mathsf{c}_2,rel1), \mathtt{preceq}(\mathsf{c}_2,\mathsf{c}_1,rel2)\} \subseteq I(\hat{\mathfrak{I}}(\chi))$. Thus $\mathrm{D}(A \sqsubseteq B) \in \mathrm{K}_{\mathsf{c}_1}$ and $\mathcal{I}(\mathsf{c}) \models A(a)$ with $\mathsf{c} \prec_{rel1} \mathsf{c}_2 \preceq_{rel2} \mathsf{c}_1$. By the construction of $I(\hat{\mathfrak{I}}(\overline{\chi}))$ we have that $\mathtt{test}(\mathtt{nlit}(a,B,\mathsf{c})) \in I(\hat{\mathfrak{I}}(\chi))$.

Minimality of $S = I(\hat{\mathfrak{I}}(\chi))$ w.r.t. the positive deduction rules of $G_S(PK(\mathfrak{K}))$ can then be motivated as in the original proof in (Bozzato et al. 2018a): thus, $I(\hat{\mathfrak{I}}(\overline{\chi}))$ is an answer set of $PK(\mathfrak{K})$.

(ii). Let S be an answer set of $PK(\mathfrak{K})$. We show that there is some justified clashing assumption $\overline{\chi}$ for $\mathfrak{K}$ such that $S = I(\hat{\mathfrak{I}}(\overline{\chi}))$ holds.

Note that as S is an answer set for the CKR program, all literals on $\mathtt{ovr}$ and $\mathtt{test\_fails}$ in S are derivable from the reduct $G_S(PK(\mathfrak{K}))$. By the definition of $I(\hat{\mathfrak{I}}(\overline{\chi}))$ we can easily build

Table B 11. *asprin program P_{pref} for preference definitions*

(pref-local)	`#preference(LocPref(C, REL), poset){`
	`    ¬ovr(A,Cp, C,REL) >> ovr(A,Cp,C,REL);`
	`    ¬ovr(A1,C1,C,REL) >> ¬ovr(A2,C2,C,REL) :`
	`        preceq_except(C1b,C1,REL), preceq_except(C2b,C2,REL),`
	`        prec(C,C2b,REL), prec(C2b,C1b,REL),`
	`        p_ovr(A1,C1,REL), p_ovr(A2,C2,REL).`
	`} : context(C), relation(REL).`

(pref-rel_local)	`#preference(RelPref(REL), pareto){`
	`        **LocPref(C, REL) : context(C)`
	`} : relation(REL).`

(pref-global)	`#preference(GlobPref, lexico){`
	`    W::**RelPref(REL) : relation_weight(REL, W)`
	`}.`

(pref-optima)	`#optimize(GlobPref).`

a model $\mathfrak{I}_S = \langle \mathfrak{I}_S, \overline{\chi}^S \rangle$ from the answer set S as follows: for every $c \in \mathsf{N}$, we build the local interpretation $\mathcal{I}_S(c) = \langle \Delta_c, \cdot^{\mathcal{I}(c)} \rangle$ as follows:

- $\Delta_c = \{d \mid d \in \mathrm{NI}\}$;
- $a^{\mathcal{I}(c)} = a$, for every $a \in \mathrm{NI}$;
- $A^{\mathcal{I}(c)} = \{d \in \Delta_c \mid S \models \mathtt{instd}(d, A, c, \mathsf{main})\}$, for every $A \in \mathrm{NC}$;
- $R^{\mathcal{I}(c)} = \{(d, d') \in \Delta_c \times \Delta_c \,|\, S \models \mathtt{tripled}(d, R, d', c, \mathsf{main})\}$ for $R \in \mathrm{NR}$;

Finally, $\overline{\chi}^S = (\chi_t^S, \chi_c^S)$ where $\chi_{rel}^S(c) = \{\langle \alpha, \mathbf{e} \rangle \mid I_{rl}(\alpha, c') = p, \mathtt{ovr}(p(\mathbf{e}), c, rel) \in S\}$. We have to show that $\mathfrak{I}_S$ meets the definition of a least justifed CAS-model for a multi-relational $\mathfrak{K}$, that is:

(i) for every $\alpha \in \mathrm{K}_{\mathsf{c}}$ (strict axiom), and $\mathsf{c}' \preceq_* \mathsf{c}$, $\mathcal{I}_S(\mathsf{c}') \models \alpha$;
(ii) for every $\mathrm{D}_i(\alpha) \in \mathrm{K}_{\mathsf{c}}$ and $\mathsf{c}' \preceq_{-i} \mathsf{c}$, $\mathcal{I}_S(\mathsf{c}') \models \alpha$;
(iii) for every $\mathrm{D}_i(\alpha) \in \mathrm{K}_{\mathsf{c}}$ and $\mathsf{c}'' \prec_i \mathsf{c}' \preceq_{-i} \mathsf{c}$, if $\langle \alpha, \mathbf{d} \rangle \notin \chi_i(\mathsf{c}'')$, then $\mathcal{I}_S(\mathsf{c}'') \models \phi_\alpha(\mathbf{d})$.

Note that, since we are considering multi-relational CKRs based only on two relations (time and coverage), the relational closure $\mathsf{c}' \preceq_{-i} \mathsf{c}$ can be read as $\mathsf{c}' \preceq_j \mathsf{c}$ with $j \neq i$: this corresponds to the conditions $\mathtt{preceq}(\mathsf{c}', \mathsf{c}, rel2)$ with $rel1 \neq rel2$ in the formulation of the rules.

Item (i) should be proved in the local case where $\mathsf{c}' = \mathsf{c}$ and in the "strict propagation" case where $\mathsf{c}' \prec_* \mathsf{c}$. The second case can be shown similarly to the local case, considering strict propagation rules in Table B 6. Thus, considering $\mathsf{c}' = \mathsf{c}$, we verify the condition by showing that, for every K_{c}, we have $\mathcal{I}_S(\mathsf{c}) \models \mathrm{K}_{\mathsf{c}}$. This can be shown by cases considering the form of all of the (strict) axioms $\beta \in \mathcal{L}_{\Sigma,\mathsf{N}}$ that can occur in K_{c}. For example (the other cases are similar):

- Let $\beta = A(a) \in \mathrm{K}_{\mathsf{c}}$, then, by rule (prl-instd), $S \models \mathtt{instd}(a, A, \mathsf{c}, \mathsf{main})$. This directly implies that $a^{\mathcal{I}(c)} \in A^{\mathcal{I}(c)}$.
- Let $\beta = A \sqsubseteq B \in \mathrm{K}_{\mathsf{c}}$, then $S \models \mathtt{subClass}(A, B, \mathsf{c})$. If $d \in A^{\mathcal{I}(\mathsf{c})}$, then by definition $S \models \mathtt{instd}(d, A, \mathsf{c}, \mathsf{main})$. By rule (prl-subc) we obtain that $S \models \mathtt{instd}(d, B, \mathsf{c}, \mathsf{main})$ and thus $d \in B^{\mathcal{I}(\mathsf{c})}$.

Condition (ii) can be proved similarly, considering rules of Table B 8. In particular, assuming that

$\mathrm{D}_i(\beta) \in \mathrm{K}_{\mathsf{c}'}$ with $\mathsf{c} \preceq_{-i} \mathsf{c}'$ we can proceed by cases on the possible forms of β and consider the (strict) propagation of defeasible axioms to c along the "parallel" relations. For example:

- Let $\beta = A(a)$. Then, by definition of the translation, we have $S \models \mathtt{def\_insta}(a, A, \mathsf{c}', rel1)$. Moreover, since $\mathsf{c} \preceq_{rel2} \mathsf{c}'$, we have $S \models \mathtt{preceq}(\mathsf{c}, \mathsf{c}', rel2)$ with $rel1 \neq rel2$. By the corresponding instantiation of rule (propp-inst), it holds that $S \models \mathtt{instd}(a, A, \mathsf{c}, \mathsf{main})$. By definition, this means that $a^{\mathcal{I}(\mathsf{c})} \in A^{\mathcal{I}(\mathsf{c})}$.
- Let $\beta = A \sqsubseteq B$. Then, by definition of the translation, $S \models \mathtt{def\_subclass}(A, B, \mathsf{c}', rel1)$. Since $\mathsf{c} \preceq_{rel2} \mathsf{c}'$, we have $S \models \mathtt{preceq}(\mathsf{c}, \mathsf{c}', rel2)$ with $rel1 \neq rel2$. If $a^{\mathcal{I}(\mathsf{c})} \in A^{\mathcal{I}(\mathsf{c})}$, then by definition $S \models \mathtt{instd}(a, A, \mathsf{c}, \mathsf{main})$: by rule (propp-subc), we obtain that $S \models \mathtt{instd}(a, B, \mathsf{c}, \mathsf{main})$ and thus $a^{\mathcal{I}(\mathsf{c})} \in B^{\mathcal{I}(\mathsf{c})}$.

To prove condition (iii), let us assume that $\mathrm{D}_i(\beta) \in \mathrm{K}_{\mathsf{c}'}$ with $\mathsf{c} \prec_i \mathsf{c}'' \preceq_{-i} \mathsf{c}'$. We proceed again by cases on the possible forms of β as in the original proof in (Bozzato et al. 2018a), by considering the defeasible propagation to c along the relation i. For example:

- Let $\beta = A(a)$. Then, by definition of the translation, we have that $S \models \mathtt{def\_insta}(a, A, \mathsf{c}', rel1)$. Suppose that $\langle A(x), a \rangle \notin \chi^S_{rel1}(\mathsf{c})$. Then by definition, $\mathtt{ovr}(\mathtt{insta}, a, A, \mathsf{c}', \mathsf{c}, rel1) \notin OVR(\hat{\mathfrak{I}}(\overline{\chi}))$. By construction, we have $S \models \mathtt{prec}(\mathsf{c}, \mathsf{c}'', rel1)$ and $S \models \mathtt{preceq}(\mathsf{c}'', \mathsf{c}', rel2)$. By the definition of the reduction, the corresponding instantiation of rule (propd-inst) has not been removed from $G_S(PK(\mathfrak{K}))$: this implies that $S \models \mathtt{instd}(a, A, \mathsf{c}, \mathsf{main})$. By definition, this means that $a^{\mathcal{I}(\mathsf{c})} \in A^{\mathcal{I}(\mathsf{c})}$.
- Let $\beta = A \sqsubseteq B$. Then, by definition of the translation, $S \models \mathtt{def\_subclass}(A, B, \mathsf{c}', rel1)$. As above, we also have $S \models \mathtt{prec}(\mathsf{c}, \mathsf{c}'', rel1)$ and $S \models \mathtt{preceq}(\mathsf{c}'', \mathsf{c}', rel2)$. Let us suppose that $b^{\mathcal{I}(\mathsf{c})} \in A^{\mathcal{I}(\mathsf{c})}$: then $S \models \mathtt{instd}(b, A, \mathsf{c}, \mathsf{main})$. Suppose that $\langle A \sqsubseteq B, b \rangle \notin \chi_S(\mathsf{c})$: by definition, $\mathtt{ovr}(\mathtt{subClass}, b, A, B, \mathsf{c}', \mathsf{c}, rel1) \notin OVR(\hat{\mathfrak{I}}(\overline{\chi}))$. By the definition of the reduction, the corresponding instantiation of rule (propd-subc) has not been removed from $G_S(PK(\mathfrak{K}))$: this implies that $S \models \mathtt{instd}(b, B, \mathsf{c}, \mathsf{main})$. Thus, by definition, this means that $b^{\mathcal{I}(\mathsf{c})} \in B^{\mathcal{I}(\mathsf{c})}$.

We have shown that $\mathfrak{I}_S$ is a CAS-model of $\mathfrak{K}$: using the same reasoning in the original proof in (Bozzato et al. 2018a) we can also prove the $\mathfrak{I}_S$ corresponds to the least model and that $\overline{\chi}^S$ is justified, thus proving the result. □

Lemma 2
Let $\mathfrak{K}$ be a multi-relational sCKR in $\mathcal{SROIQ}$-RLD normal form. Then, $\hat{\mathfrak{I}}$ is a CKR model of $\mathfrak{K}$ iff there exists a (named) justified clashing assumption $\overline{\chi}$ s.t. $I(\hat{\mathfrak{I}}(\overline{\chi}))$ is a preferred answer set of $PK(\mathfrak{K}) \cup P_{pref}$.

For the proof we need the following result:

Theorem 5
Let $\mathfrak{K}$ be an *eval*-disconnected sCKR and $\mathfrak{I}_{CAS} = \langle \mathfrak{I}, \chi_1, \ldots, \chi_m \rangle$ a justified model of $\mathfrak{K}$. Then $\mathfrak{I}_{CAS}$ is preferred with respect to $P_{1,i}$ defined by

$P_{1,i}(\langle \mathfrak{I}^1, \chi^1_1, \ldots, \chi^1_m \rangle, \langle \mathfrak{I}^2, \chi^2_1, \ldots, \chi^2_m \rangle)$ iff there exists some $\mathsf{c} \in \mathsf{N}$ s.t. $\chi^1_i(\mathsf{c}) > \chi^2_i(\mathsf{c})$ and not $\chi^2_i(\mathsf{c}) > \chi^1_i(\mathsf{c})$, and for no context $\mathsf{c}' \neq \mathsf{c} \in \mathsf{N}$ it holds that $\chi^1_i(\mathsf{c}') < \chi^2_i(\mathsf{c}')$ and not $\chi^2_i(\mathsf{c}') < \chi^1_i(\mathsf{c}')$.

iff it is preferred with respect to $P_{2,i}$ defined by

$P_{2,i}(\langle \mathfrak{I}^1, \chi^1_1, \ldots, \chi^1_m \rangle, \langle \mathfrak{I}^2, \chi^2_1, \ldots, \chi^2_m \rangle)$ iff there exists some $\mathsf{c} \in \mathsf{N}$ s.t. $\chi^1_i(\mathsf{c}) > \chi^2_i(\mathsf{c})$ and not $\chi^2_i(\mathsf{c}) > \chi^1_i(\mathsf{c})$, and for all contexts $\mathsf{c}' \in \mathsf{N}$ it holds that $\chi^1_i(\mathsf{c}') > \chi^2_i(\mathsf{c}')$ or $\chi^1_i(\mathsf{c}') = \chi^2_i(\mathsf{c}')$.

Proof (sketch) of Theorem 5
$P_{2,i}(\langle\mathfrak{I}^1,\chi_1\rangle,\langle\mathfrak{I}^2,\chi_2\rangle)$ implies $P_{1,i}(\langle\mathfrak{I}^1,\chi_1\rangle,\langle\mathfrak{I}^2,\chi_2\rangle)$. So we consider the other direction.

Let $\mathfrak{I}_{CAS}$ be preferred with respect to $P_{2,i}$. Assume that there exists a justified model $\mathfrak{I}'_{CAS}$ of $\mathfrak{K}$ such that $P_{1,i}(\mathfrak{I}'_{CAS},\mathfrak{I}_{CAS})$ holds.

Let $\mathfrak{I}_{CAS}=\langle\{\mathcal{I}(\mathsf{c})\}_{\mathsf{c}\in\mathsf{N}},\chi\rangle$ and $\mathfrak{I}'_{CAS}=\langle\{\mathcal{I}'(\mathsf{c})\}_{\mathsf{c}\in\mathsf{N}},\chi'\rangle$. We know there exists some $\mathsf{c}^*\in\mathsf{N}$ such that $\chi'(\mathsf{c}^*)>\chi(\mathsf{c}^*)$. This implies that some $\mathrm{D}(\alpha)\in\mathcal{K}_{\mathsf{c}'}$ and $\mathbf{e}$ exist such that $\langle\alpha,\mathbf{e}\rangle\in\chi(\mathsf{c}^*)\setminus\chi'(\mathsf{c}^*)$. Let C be the component of $DEP(\mathfrak{K})$ that contains X_{c^*}, where X is any concept or role appearing in α. Note that C is independent of the choice of X, since any two possible choices X,X' satisfy that X_{c^*} and X'_{c^*} are reachable from one another.

We take $\mathfrak{I}''_{CAS}=\langle\{\mathcal{I}''(\mathsf{c})\}_{\mathsf{c}\in\mathsf{N}},\chi''\rangle$ such that $X^{\mathcal{I}''(\mathsf{c})}=X^{\mathcal{I}(\mathsf{c})}$ for $X_{\mathsf{c}}\notin C$ and $X^{\mathcal{I}''(\mathsf{c})}=X^{\mathcal{I}'(\mathsf{c})}$ otherwise, and we let $\chi''(\mathsf{c})=\chi(\mathsf{c})$ for $\mathsf{c}\neq\mathsf{c}^*$ and $\chi''(\mathsf{c})=\chi'(\mathsf{c})$ otherwise. That is, we take the original justified model $\mathfrak{I}_{CAS}$ and swap the interpretations of all the concepts and roles that were changed in order to satisfy $\alpha(\mathbf{e})$ at context c^* by their changed interpretation in $\mathfrak{I}'_{CAS}$. The result, $\mathfrak{I}''_{CAS}$, is still a model of $\mathfrak{K}$, as we exchanged the interpretation for the whole component and therefore any relevant axioms stay satisfied, since they were satisfied in $\mathfrak{I}'_{CAS}$. Furthermore, since $\mathfrak{K}$ is *eval*-disconnected, χ'' is justified because the default $\mathrm{D}(\alpha)$ does not use any concept/role X such that X_{c^*} is connected to X'_{c} such that $\mathsf{c}\neq\mathsf{c}^*$ and X is used in another default $\mathrm{D}(\beta)$. This implies that only the clashing assumptions for c^* were changed.

Now, we however know that $P_{2,i}(\mathfrak{I}''_{CAS},\mathfrak{I}_{CAS})$. This is a contradiction to our original assumption. Therefore, there cannot exist some $\mathfrak{I}'_{CAS}$ such that $P_{1,i}(\mathfrak{I}'_{CAS},\mathfrak{I}_{CAS})$ and $\mathfrak{I}_{CAS}$ is preferred with respect to $P_{1,i}$. $\square$

Proof (sketch) of Lemma 2
Our definition of the preferences in P_{pref} mirrors the definition of preference: both go from local preference on the clashing assumptions per context, i.e. $\chi_i(\mathsf{c})$, to per relation preference and finally to the global preference. We show that the definitions correspond for each step.

We start with the local preference. So let X,Y be two interpretations of $PK(\mathfrak{K})$, c a context and i a relation. Then it holds that $X>_{\mathtt{LocPref(c,}i\mathtt{)}}Y$ iff:

- X and Y do not have the same clashing assumptions at c w.r.t. relation i;
- for each $\neg\mathtt{ovr}(\alpha_1,e,\mathsf{c}_1,\mathsf{c},i)$ s.t. $X\not\models\neg\mathtt{ovr}(\alpha_1,e,\mathsf{c}_1,\mathsf{c},i)$ and $Y\models\neg\mathtt{ovr}(\alpha_1,e,\mathsf{c}_1,\mathsf{c},i)$ there exists $\neg\mathtt{ovr}(\alpha_2,f,\mathsf{c}_2,\mathsf{c},i)>\neg\mathtt{ovr}(\alpha_1,e,\mathsf{c}_1,\mathsf{c},i)$ s.t. $X\models\neg\mathtt{ovr}(\alpha_2,f,\mathsf{c}_2,\mathsf{c},i)$ and $Y\not\models\neg\mathtt{ovr}(\alpha_2,f,\mathsf{c}_2,\mathsf{c},i)$.

or equivalently:

- X and Y do not have the same clashing assumptions at c w.r.t. relation i;
- for each α_1,e, where α_1 is from context $\mathsf{c}_1\succeq_{-i}\mathsf{c}_{1b}\succ_i\mathsf{c}$, s.t. $X\models\mathtt{ovr}(\alpha_1,e,\mathsf{c}_1,\mathsf{c},i)$ and $Y\not\models\mathtt{ovr}(\alpha_1,e,\mathsf{c}_1,\mathsf{c},i)$ there exists α_2,f, where α_2 is from context $\mathsf{c}_2\succeq_{-i}\mathsf{c}_{2b}\succ_i\mathsf{c}$, s.t. $\mathsf{c}_{1b}\succ_i\mathsf{c}_{2b}$ and $X\not\models\mathtt{ovr}(\alpha_2,f,\mathsf{c}_2,\mathsf{c},i)$ and $Y\models\mathtt{ovr}(\alpha_2,f,\mathsf{c}_2,\mathsf{c},i)$.

The second item is equivalent to

> for every $\eta=\langle\alpha_1,\mathbf{e}\rangle\in\chi_i^1(\mathsf{c})\setminus\chi_i^2(\mathsf{c})$ with $\mathrm{D}_i(\alpha_1)$ at a context $\mathsf{c}_1\succeq_{-i}\mathsf{c}_{1b}\succ_i\mathsf{c}$, there exists an $\eta'=\langle\alpha_2,\mathbf{f}\rangle\in\chi_i^2(\mathsf{c})\setminus\chi_i^1(\mathsf{c})$ with $\mathrm{D}_i(\alpha_2)$ at context $\mathsf{c}_2\succeq_{-i}\mathsf{c}_{2b}\succ_i\mathsf{c}$ such that $\mathsf{c}_{1b}\succ_i\mathsf{c}_{2b}$.

So, we see that the only difference between $>_{\mathtt{LocPref(c,}i\mathtt{)}}$ and the order on the context c is the first condition, i.e. that the clashing assumptions on c must be different. However, this does not affect us, since the definition of preference for justified interpretations always uses $E=$"$\chi_i^1(\mathsf{c})<\chi_i^2(\mathsf{c})$ and not $\chi_i^2(\mathsf{c})<\chi_i^1(\mathsf{c})$". This is equivalent to "$X>_{\mathtt{LocPref(c,}i\mathtt{)}}Y$ and not $Y>_{\mathtt{LocPref(c,}i\mathtt{)}}X$", since E can only hold when the clashing assumption sets at c w.r.t. relation i are different.

Next, we consider the preference per relation. As we have shown in Theorem 5 the preferred models with respect to the original preference relation P_1 are the same as the preferred models with respect to the preference relation $P_{2,i}$. However, as can be easily seen from the definition, $P_{2,i}$ is the order that has the models that are pareto optimal with respect to the local preference orders `LocPref(c,`i`)` per context as its optimal models. We see that `RelPref(`i`)` correctly captures this, as it is the pareto combination of the orders `LocPref(c,`i`)` for each context `c`.

Last but not least, we consider the global preference. In our definition, we say that we prioritize the preference on the clashing assumptions with respect to the relations with a lower index. This corresponds to the lexicographical combination of the orders `LocPref(`i`)` for each relation i, when assigning the weight i to relation i, when it is the preference relation with index i. □

Appendix D Proofs for Overall Weight Queries

Before we define the semiring, we ensure that the preference relation `LocPref(`*rel*`)` is transitive.

Lemma 3
The preference relation `LocPref(`*rel*`)` defined in Section 5 is transitive.

We use the transitivity of the local preference:

Lemma 4
Let $\chi_i^1(\mathtt{c}) > \chi_i^2(\mathtt{c})$ and $\chi_i^2(\mathtt{c}) > \chi_i^3(\mathtt{c})$. Then $\chi_i^1(\mathtt{c}) > \chi_i^3(\mathtt{c})$.

Proof
Assume $\chi_i^1(\mathtt{c}) > \chi_i^2(\mathtt{c})$, $\chi_i^2(\mathtt{c}) > \chi_i^3(\mathtt{c})$ and $\langle\alpha_1,\mathbf{e}\rangle \in \chi_i^1(\mathtt{c}) \setminus \chi_i^3(\mathtt{c})$ with $\mathrm{D}_i(\alpha_1)$ at a context $\mathtt{c}_1 \succeq_{-i} \mathtt{c}_{1b} \succ_i \mathtt{c}$.

Case 1: If $\langle\alpha_1,\mathbf{e}\rangle \notin \chi_i^2(\mathtt{c})$ then since $\chi_i^1(\mathtt{c}) > \chi_i^2(\mathtt{c})$ there exists $\langle\alpha_2,\mathbf{f}\rangle \in \chi_i^2(\mathtt{c}) \setminus \chi_i^1(\mathtt{c})$ with $\mathrm{D}_i(\alpha_2)$ at context $\mathtt{c}_2 \succeq_{-i} \mathtt{c}_{2b} \succ_i \mathtt{c}$ such that $\mathtt{c}_{1b} \succ_i \mathtt{c}_{2b}$.

Case 1.1: If $\langle\alpha_2,\mathbf{f}\rangle \in \chi_i^3(\mathtt{c})$ we are done.

Case 1.2: Else, $\langle\alpha_2,\mathbf{f}\rangle \in \chi_i^2(\mathtt{c}) \setminus \chi_i^3(\mathtt{c})$. Then since $\chi_i^2(\mathtt{c}) > \chi_i^3(\mathtt{c})$ there exists $\langle\alpha_3,\mathbf{g}\rangle \in \chi_i^3(\mathtt{c}) \setminus \chi_i^2(\mathtt{c})$ with $\mathrm{D}_i(\alpha_3)$ at context $\mathtt{c}_3 \succeq_{-i} \mathtt{c}_{3b} \succ_i \mathtt{c}$ such that $\mathtt{c}_{2b} \succ_i \mathtt{c}_{3b}$.

Case 1.2.1: If $\langle\alpha_3,\mathbf{g}\rangle \notin \chi_i^1(\mathtt{c})$ we are done.

Case 1.2.2: Otherwise, $\langle\alpha_3,\mathbf{g}\rangle \in \chi_i^1(\mathtt{c}) \setminus \chi_i^2(\mathtt{c})$. Note that this is the same situation as in case 1 except that $\mathrm{D}_i(\alpha_3)$ is at context $\mathtt{c}_{3b} \succ_i \mathtt{c}$ such that $\mathtt{c}_{1b} \succ_i \mathtt{c}_{2b} \succ_i \mathtt{c}_{3b}$. Since $\succ_i$ is a strict (partial) order and we only have finitely many contexts this can only occur finitely often. Since in all other cases below case 1 we have that $\chi_i^1(\mathtt{c}) > \chi_i^3(\mathtt{c})$ we are done with case 1.

Case 2: If $\langle\alpha_1,\mathbf{e}\rangle \in \chi_i^2(\mathtt{c})$ we are in a similar situation as in case 1.2 the statement follows by analogous reasoning. □

Proof
As we have seen, `LocPref(c,`*rel*`)` is transitive for each context `c` and relation *rel*. Thus their pareto combination is also transitive. □

As the domain of the semiring we choose $R = \{(S,\chi) \mid S \in \mathbb{N}^B, \chi \text{ clashing assumption multiset-map}\}$. Here, we need S to be a multiset and χ to map to multisets of clashing assumptions for technical reasons (namely so that our semiring satisfies the distributive law). We generalize the definition of the local preference to multisets by using

$\chi_i^1(\mathsf{c}) > \chi_i^2(\mathsf{c})$, if for every $\eta = \langle \alpha_1, \mathbf{e} \rangle$ s.t. the multiplicity of η in $\chi_i^1(\mathsf{c})$ is greater than its multiplicity in $\chi_i^2(\mathsf{c})$ with $\mathrm{D}_i(\alpha_1)$ at a context $\mathsf{c}_1 \succeq_{-i} \mathsf{c}_{1b} \succ_i \mathsf{c}$, there exists an $\eta' = \langle \alpha_2, \mathbf{f} \rangle$ s.t. the multiplicity of η' in $\chi_i^2(\mathsf{c})$ is greater than its multiplicity in $\chi_i^1(\mathsf{c})$ with $\mathrm{D}_i(\alpha_2)$ at context $\mathsf{c}_2 \succeq_{-i} \mathsf{c}_{2b} \succ_i \mathsf{c}$ such that $\mathsf{c}_{1b} \succ_i \mathsf{c}_{2b}$.

With this in mind, we can define the semiring $\mathcal{R}_{\text{one}}(\mathfrak{K}) = (R \cup \{\mathbf{0}, \mathbf{1}\}, \oplus, \otimes, \mathbf{0}, \mathbf{1})$ by letting

$$a \oplus b = \begin{cases} a & \text{if } a \succ_{\texttt{LocPref}(rel)} b \\ b & \text{if } b \succ_{\texttt{LocPref}(rel)} a \\ \text{lex}-\min(a,b) & \text{otherwise.} \end{cases} \qquad \begin{aligned} \mathbf{0} \oplus a &:= a =: a \oplus \mathbf{0}, \\ \mathbf{1} \oplus a &:= \mathbf{1} =: a \oplus \mathbf{1}, \\ \mathbf{0} \otimes a &:= \mathbf{0} =: a \otimes \mathbf{0}, \\ \mathbf{1} \otimes a &:= a =: a \otimes \mathbf{1}. \end{aligned}$$

$$(S_1, \chi^{(1)}) \otimes (S_2, \chi^{(2)}) = (S_1 + S_2, \chi^{(1)} + \chi^{(2)})$$

Here, $\text{lex}-\min(a,b)$ takes the lexicographical minimum of a, b and the addition refers to pointwise multiset union, i.e., $(\chi^{(1)} + \chi^{(2)})(\mathsf{c}) = \chi^{(1)}(\mathsf{c}) + \chi^{(2)}(\mathsf{c})$.

Now we can define the following weighted formula:

$$\begin{aligned} \alpha_{\text{one}} &= \alpha_1 * \alpha_2 \\ \alpha_1 &= \Pi_{a \in B}(a * (\{a\}, \emptyset) + \neg a \\ \alpha_2 &= \Pi_{c \in C} \Pi_{\langle \alpha, e, i \rangle \in pclash(c)} \texttt{ovr}(\alpha, e, c, i) * (\emptyset, \{c \mapsto \{\{\langle \alpha, e \rangle\}\}) + \neg \texttt{ovr}(\alpha, e, c, i), \end{aligned}$$

where B is the Herbrand base and $pclash(c) = \{\langle \alpha, e, i \rangle \mid \langle \alpha, e \rangle$ is a possible clashing assumption for c and $i\}$. Intuitively, α_1 collects the atoms that are true in the given interpretation and α_2 builds the clashing assumption map, which is used to decide whether one interpretation is preferred over the other.

Theorem 6

$\mathcal{R}_{\text{one}}(\mathfrak{K})$ is a semiring and the overall weight of $\mu = \langle PK(\mathfrak{K}), \alpha_{\text{one}}, \mathcal{R}_{\text{one}}(\mathfrak{K}) \rangle$ is (I, χ), where I is the minimum lexicographical preferred model of $\mathfrak{K}$ and χ its clashing assumption map or $\mathbf{0}$ if there is no preferred model.

Proof

Associativity of $\oplus$ follows from transitivity of $\texttt{LocPref}(\texttt{c}, rel)$ and the lexicographical order. Commutativity of $\oplus$ is clear. $\mathbf{0}$ and $\mathbf{1}$ are identities and annihilators of $\otimes, \oplus$ by definition. Associativity of $\otimes$ is clear.

It remains to prove that multiplication distributes over addition. So let $A_i = (I_i, \chi_i)$ for $i = 1, 2, 3$. Then, in the expression

$$A_1 \otimes (A_2 \oplus A_3)$$

Assume w.l.o.g. that $(A_2 \oplus A_3)$ evaluates to A_2. If $A_2 >_{\texttt{LocPref}(rel)} A_3$ then there exists a context c such that $\chi_2(\mathsf{c}) >_{\texttt{LocPref}(\texttt{c},rel)} \chi_3(\mathsf{c})$. Then it also holds that $(\chi_1 + \chi_2)(\mathsf{c}) >_{\texttt{LocPref}(\texttt{c},rel)} (\chi_1 + \chi_3)(\mathsf{c})$ and thus

$$A_1 \otimes (A_2 \oplus A_3) = A_1 \otimes A_2 = A_1 \otimes A_2 \oplus A_1 \otimes A_3.$$

If $A_2 \not>_{\texttt{LocPref}(rel)} A_3$ this implies that A_2 is either equal to A_3 (in this case we are done) or that A_2 is smaller lexicographically. In the latter case the sum $A_1 \otimes A_2$ is however also lexicographically smaller than $A_1 \otimes A_3$ since we add A_1 both times.

Thus we have established that $\mathcal{R}_{\text{one}}(\mathfrak{K})$ is a semiring. For each answer set I of $PK(\mathfrak{K})$, we know that I corresponds to a (least) CAS model. Thus,

$$[\![\alpha_{\text{one}}]\!]_{\mathcal{R}_{\text{one}}}(I) = (\mathfrak{I}, \chi),$$

where $\mathfrak{I} \in \{0,1\}^B$ and χ only maps to multisets that can be interpreted as sets (i.e. each of their elements has at most multiplicity one). The lexicographical minimum CKR model $\langle \mathfrak{I}^*, \chi^* \rangle$ satisfies that $(\mathfrak{I}, \chi) \oplus (\mathfrak{I}^*, \chi^*) = (\mathfrak{I}^*, \chi^*)$ for all $(\mathfrak{I}, \chi)$ that are the semantics of α_{one} w.r.t. some answer set of $PK(\mathfrak{K})$. Therefore, if there exists a CKR model, the overall weight is $(\mathfrak{I}^*, \chi^*)$. Otherwise, it is $\mathbf{0}$. □

We continue with the $\mathcal{R}_{\text{all}}$ semiring. Again, we need some additional lemma

Lemma 5
Let $\mathfrak{K}$ be a single relational sCKR without *eval* expressions. Then a CAS model $(\{\mathcal{I}(\mathsf{c})\}_{\mathsf{c} \in \mathsf{N}}, \chi)$ is a CKR model iff no CAS model $(\{\mathcal{I}'_{\mathsf{c}}\}_{\mathsf{c} \in \mathsf{N}}, \chi')$ and $\mathsf{c} \in \mathsf{N}$ exist such that $\chi'(\mathsf{c}) > \chi(\mathsf{c})$.

Therefore, we can take the locally optimal models $\mathcal{I}(\mathsf{c})$ for each context c and obtain the global optimal models as arbitrary combinations of locally preferred models.

In the following, we let $\mathcal{D}$ be the Herbrand base.

Using this notation, we define the semiring $\mathcal{R}_{\mathsf{c}} = (R_{\mathsf{c}}, \oplus_{\mathsf{c}}, \otimes_{\mathsf{c}}, e_{\oplus}, e_{\otimes})$ that collects all locally optimal models $\mathcal{I}(\mathsf{c})$. Here,

$$\begin{aligned}
R_{\mathsf{c}} &= \{opt(B) \mid A \subseteq 2^{\mathcal{D}}, B = \{(S, \chi) \mid S \in A, \chi \text{ multiset of clashing assumptions}\}\} \\
A \oplus_{\mathsf{c}} B &= opt_{\mathsf{c}}(A \cup B) \\
A \otimes_{\mathsf{c}} B &= opt_{\mathsf{c}}(\{(S_1 \cup S_2, \chi_1 + \chi_2) \mid (S_1, \chi_1) \in A, (S_2, \chi_2) \in B\} \\
e_{\oplus} &= \emptyset \\
e_{\otimes} &= \{(\emptyset, \{\!\{\}\!\})\} \\
opt_{\mathsf{c}}(A) &= \{(S, \chi) \in A \mid \forall (S', \chi') \in A : \neg(\chi'(\mathsf{c}) > \chi(\mathsf{c}))\}
\end{aligned}$$

We again have to use multisets for χ instead of sets. This is necessary because otherwise multiplication and addition do not satisfy the distributive law.

Then, we can define the measure $\mu_{\mathsf{c}} = \langle PK(\mathfrak{K}), \alpha_{\text{all}}, \mathcal{R}_{\mathsf{c}} \rangle$, where

$$\begin{aligned}
\alpha_{\text{all}} &= \alpha_1 * \alpha_2 \\
\alpha_1 &= \Pi_{d \in \mathcal{D}} d * \{(\{d\}, \{\!\{\}\!\})\} + \neg d \\
\alpha_2 &= \Pi_{\langle \alpha, e, i \rangle \in pclash(c)} \mathtt{ovr}(\alpha, e, c) * \{(\emptyset, \{c \mapsto \{\!\{\langle \alpha, e \rangle\}\!\}\})\} + \neg \mathtt{ovr}(\alpha, e, c).
\end{aligned}$$

where $pclash(c)$ is the set of all possible clashing assumptions $\langle \alpha, e \rangle$ for c. We obtain

Theorem 7
$\mathcal{R}_{\mathsf{c}}$ is a semiring and the overall weight $\mu_{\mathsf{c}}(PK(\mathfrak{K}))$ is equal to the set containing for each locally optimal interpretation $\mathcal{I}(\mathsf{c})$ of $\mathfrak{K}$ the pair $(\mathcal{I}(\mathsf{c}), \chi_{\mathcal{I}(\mathsf{c})})$, where $\chi_{\mathcal{I}(\mathsf{c})}$ is the unique multiset containing each justified clashing assumption of $\mathcal{I}(\mathsf{c})$ once.

We take $\mathcal{R}_{\text{all}}$ to be the crossproduct semiring $(\mathcal{R}_{\mathsf{c}})_{\mathsf{c} \in \mathsf{N}}$ defined by

$$(\mathcal{R}_{\mathsf{c}})_{\mathsf{c} \in \mathsf{N}} = ((R_{\mathsf{c}})_{\mathsf{c} \in \mathsf{N}}, \oplus, \otimes, (\emptyset)_{\mathsf{c} \in \mathsf{N}}, (\{(\emptyset, \{\!\{\}\!\})\})_{\mathsf{c} \in \mathsf{N}}), \text{ where}$$
$$(A_{\mathsf{c}})_{\mathsf{c} \in \mathsf{N}} \odot (B_{\mathsf{c}})_{\mathsf{c} \in \mathsf{N}} = (A_{\mathsf{c}} \odot_{\mathsf{c}} B_{\mathsf{c}})_{\mathsf{c} \in \mathsf{N}}, \text{ for } \odot \in \{\oplus, \otimes\}$$

Using it, we can obtain the locally optimal interpretations for each context as the crossproduct of measures $\mu^* = (\mu_{\mathsf{c}})_{\mathsf{c}\in\mathsf{N}}$ which is a measure over the crossproduct semiring $(\mathcal{R}_{\mathsf{c}})_{\mathsf{c}\in\mathsf{N}}$. As we have shown in Lemma 5, this gives us all the preferred models. Namely, let $\mu^*(PK(\mathfrak{K})) = (A_{\mathsf{c}})_{\mathsf{c}\in\mathsf{N}}$, then $\{(\mathcal{I}(\mathsf{c}))_{\mathsf{c}\in\mathsf{N}} \mid (\mathcal{I}(\mathsf{c}),\chi(\mathsf{c})) \in A_{\mathsf{c}}\}$ is the set of preferred models.

Example 6
The sCKR $\mathfrak{K}$ defined in Example 5 has five contexts c_{world}, $\mathsf{c}_{branch1}$, $\mathsf{c}_{branch2}$, c_{local1}, and c_{local2}. Therefore, the measure μ^* is a crossproduct of the five measures $\mu_{\mathsf{c}_{world}}, \mu_{\mathsf{c}_{branch1}}, \mu_{\mathsf{c}_{branch2}}, \mu_{\mathsf{c}_{local1}}$ and $\mu_{\mathsf{c}_{local2}}$. Their overall weight is given by

$$\mathsf{c}_{world}(PK(\mathfrak{K})) = \mu_{\mathsf{c}_{branch1}}(PK(\mathfrak{K})) = \mu_{\mathsf{c}_{branch2}}(PK(\mathfrak{K})) = \mu_{\mathsf{c}_{local2}}(PK(\mathfrak{K})) = \{(\emptyset, \{\{\}\}\}$$
$$\mu_{\mathsf{c}_{local1}}(PK(\mathfrak{K})) = \{(\{S(i), M(i)\}, \{\{\mathtt{ovr}(S \sqsubseteq R, i, \mathsf{c}_{branch2}), \mathtt{ovr}(S \sqsubseteq E, i, \mathsf{c}_{world})\}\})\}$$

Accordingly, there is exactly one preferred model $(\mathcal{I}(\mathsf{c}))_{\mathsf{c}\in\mathsf{N}}$, where

$$\mathcal{I}_{\mathsf{c}_{world}} = \mathcal{I}_{\mathsf{c}_{branch1}} = \mathcal{I}_{\mathsf{c}_{branch2}} = \mathcal{I}_{\mathsf{c}_{local2}} = \emptyset \qquad \mathcal{I}_{\mathsf{c}_{local1}} = \{S(i), M(i)\}$$

Theorem 8
Let $\mathfrak{K}$ be a single-relational, *eval*-free sCKR, then $\mathcal{R}_{\text{all}}$ is a semiring and the overall weight of $\mu_{\text{all}} = \langle PK(\mathfrak{K}), \alpha_{\text{all}}, \mathcal{R}_{\text{all}}(\mathfrak{K})\rangle$ is $(A_{\mathsf{c}})_{\mathsf{c}\in\mathsf{N}}$ and the set of preferred models corresponds to $\{(\mathcal{I}(\mathsf{c}))_{\mathsf{c}\in\mathsf{N}} \mid$ for each $c \in \mathsf{N} : (\mathcal{I}(\mathsf{c}),\chi(\mathsf{c})) \in A_{\mathsf{c}}\}$.

Proof
The reasoning that $\mathcal{R}_{\text{all}}$ is a semiring is along the same lines as that for $\mathcal{R}_{\text{one}}$. The fact that the result is as desired can be clearly seen during the construction of the semiring. □

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