Online appendix for the paper

Strong Equivalence of Logic Programs with Ordered Disjunction: a Logical Perspective

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A Proofs of Theorem 2 and Theorem 3

This appendix contains the proofs of Theorems 2 and 3 from Section 3.

Theorem 2
Two LPODs P_1, P_2 are strongly equivalent under all the answer sets if and only if they are logically equivalent in four-valued logic.

Proof
($\Leftarrow$) Assume that P_1 and P_2 are logically equivalent in four-valued logic. Then, every four-valued model that satisfies one of them, also satisfies the other. This means that for all programs P, $P_1 \cup P$ has the same models as $P_2 \cup P$. But then, $P_1 \cup P$ has the same answer sets as $P_2 \cup P$ (because the answers sets of a program are the $\preceq$-minimal models among all the models of the program). Therefore, $P_1 \cup P$ and $P_2 \cup P$ are strongly equivalent under all the answer sets.
($\Rightarrow$) Assume that P_1 and P_2 are strongly equivalent under all the answer sets. Assume, for the sake of contradiction, that P_1 has a model M which is not a model of P_2. We will show that we can construct an interpretation M' and a program P such that M' is a $\preceq$-minimal model of one of $P_1 \cup P$ and $P_2 \cup P$ but not of the other, contradicting our assumption of strong equivalence under all the answer sets. The construction of M' and the proof that M' is a model of P_1, are identical to the corresponding ones in the proof of Theorem 1. We distinguish two cases.

Case 1: M' is not a model of P_2. We define exactly the same program P as in Case 1 of Theorem 1 and we demonstrate, following the same steps, that M' is a $\preceq$-minimal model of $P_1 \cup P$. This contradicts our assumption of strong equivalence because M' is not even a model of $P_2 \cup P$ (since we have assumed that it is not a model of P_2).

Case 2: M' is a model of P_2. We define exactly the same program P as in Case 2 of Theorem 1 and we demonstrate, following the same steps, that M' is a $\preceq$-minimal model of $P_2 \cup P$. We then show, following the same steps as in the proof of Theorem 1, that M' is not a $\preceq$-minimal model of $P_1 \cup P$. This contradicts our assumption of strong equivalence under all answer sets.

In conclusion, P_1 and P_2 are logically equivalent. $\square$

For the proof of Theorem 3 we will make use of the following lemma from the paper by Charalambidis et al. (2021):

Lemma A.1
Let P be a normal logic program. Then, the answer sets of P (see Definition 7) coincide with the standard answer sets of P.

Theorem 3
Let P_1, P_2 be normal logic programs. Then, P_1 and P_2 are strongly equivalent under the standard answer set semantics if and only if they have the same three-valued models.

Proof
($\Leftarrow$) Assume that P_1 and P_2 have the same three-valued models. This means that for all programs P, $P_1 \cup P$ has the same three-valued models as $P_2 \cup P$. Since $P_1 \cup P$ and $P_2 \cup P$ are normal programs, by Lemma A.1 the answer sets coincide with the standard answer sets which are two-valued by definition and therefore the answer sets are the $\preceq$-minimal models among the three-valued models of the program. But then, $P_1 \cup P$ has the same answer sets (and the same standard answer sets) as $P_2 \cup P$. Therefore, P_1 and P_2 are strongly equivalent under the standard answer set semantics.
($\Rightarrow$) Assume that P_1 and P_2 are strongly equivalent under the standard answer set semantics. Suppose that P_1 has a three-valued model M which is not a model of P_2. Without loss of generality, we may assume that $M(A) = F$, for every atom $A \in \Sigma$ that does not occur in $P_1 \cup P_2$. We will show that we can construct an three-valued interpretation M' and a normal logic program P such that M' is a standard answer set of one of $P_1 \cup P$ and $P_2 \cup P$ but not of the other contradicting our assumption of strong equivalence.

Let M' be the two-valued interpretation defined as:

$$M'(A) = \begin{cases} T & M(A) \geq T^* \\ F & \text{otherwise} \end{cases}$$

We claim that M' is a model of P_1. Since P_1 is a normal logic program all rules are of the form $C \leftarrow A_1, \ldots, A_m, not\, B_1, \ldots, not\, B_k$. If $M'(A_1, \ldots, A_m, not\, B_1, \ldots, not\, B_k) = F$ then the rule is trivially satisfied. If $M'(A_1, \ldots, A_m, not\, B_1, \ldots, not\, B_k) = T$ then it follows that $M(A_i) \geq T^*$ and $M(B_j) = F$ for every A_i and B_j in the body of the rule and $M(A_1, \ldots, A_m, not\, B_1, \ldots, not\, B_k) \geq T^*$. Since M is a model of P_1 it satisfies the rule and thus $M(C) \geq T^*$. By the construction of M' it follows that $M'(C) = T$ and consequently the rule is satisfied. Lastly, notice that no other values are possible for the body of the rule and therefore we conclude that M' is a model of P_1.

We proceed by distinguishing two cases that depend on whether M' is a model of P_2 or not.

Case 1: M' is not a model of P_2. We take P to be $\{A \leftarrow | M'(A) = T\}$. It is easy to see that M' is a model of P and thus model of $P_1 \cup P$. We show that M' is also a $\preceq$-minimal model of $P_1 \cup P$ and since $P_1 \cup P$ is a normal logic program M' is also a standard answer set of $P_1 \cup P$. Let N be a model of $P_1 \cup P$ and $N \prec M'$. It must exist atom A such that $N(A) \prec M'(A)$. Since M' assigns only values T and F, it must be $N(A) = F$ and $M(A) = T$. But then, N is not a model of P because there is a rule $A \leftarrow$ in P which leads to contradiction. Therefore, M' is $\preceq$-minimal and a standard answer set of $P_1 \cup P$. By our initial assumption, M' is not a model of P_2 and thus not a model of $P_2 \cup P$ which leads to the contradiction that P_1 and P_2 are strongly equivalent.

<u>*Case 2*</u>: M' is a model of P_2. Let D be an atom in Σ that does not occur in $P_1 \cup P_2$. Such atom always exists, since Σ is countably infinite set and P_1, P_2 are finite; moreover, $M(D) = F$ by our assumption about M. We take P to be

$$\begin{aligned} P = &\{A \leftarrow \mid M(A) = T\} \cup \\ &\{B \leftarrow A \mid A \neq B \text{ and } M(A) = T^* \text{ and } M(B) = T^* \} \cup \\ &\{D \leftarrow \textit{not}\, A \mid M(A) = T^*\} \end{aligned}$$

It is easy to see that M' satisfies every rule in P and therefore is a model of both $P_1 \cup P$ and $P_2 \cup P$. We show that M' is a standard answer set of $P_2 \cup P$ but not of $P_1 \cup P$.

We proceed by showing that M' is a $\preceq$-minimal model of $P_2 \cup P$ and therefore an answer set of $P_2 \cup P$ which by Lemma A.1 is also a standard answer set of $P_2 \cup P$. Assume there exists a model N of $P_2 \cup P$ such that $N \prec M'$.

We first show that there exists an atom A such that $M(A) = T^*$ and $N(A) = T$. Consider an arbitrary atom C. If $M(C) = T$ then it is also $N(C) = T$, because P contains $C \leftarrow$ and N is a model of P. If $M(C) = F$ then, by the construction of M' it is $M'(C) = F$ and since $N \prec M'$ we get $N(C) = F$. Therefore if $M(C) \neq T^*$ then $M(C) = N(C)$. There should be, however, an atom A that occurs in P_2 such that $N(A) \neq M(A)$ because N is a model of P_2 and M is not. Obviously, for that atom it must be $M(A) = T^*$ and $N(A) \neq T^*$. Notice that there exists a rule $D \leftarrow \textit{not}\, A$ in P where $M(D) = F$ and must be satisfied by N since it is also a model of P. Since $M(D) = F$ implies $N(D) = F$, the only possibility is $N(A) = T$.

We next show that there exists an atom B such that $M(B) = N(B) = T^*$. Since $N \prec M'$, there exists B such that $N(B) \prec M'(B)$. The last relation immediately implies that $M'(B) \neq F$ and by the construction of M', it is $M'(B) \neq T^*$. Therefore, the only remaining value is $M'(B) = T$. For that atom, it cannot be $M(B) = T$ because then it is also $N(B) = T$. It follows, by the construction of M' that $M(B) = T^*$. We claim that $N(B) = T^*$, that is, it cannot be $N(B) = F$. Since $M(B) = T^*$ there exists a rule $D \leftarrow \textit{not}\, B$ where $M(D) = F$. Since $M(D) = F$, it is also $N(D) = F$. If we assume that $N(B) = F$ then N does not satisfy this rule which is a contradiction. Therefore, $N(B) = T^*$.

Since $M(A) = M(B) = T^*$ there exists a rule $B \leftarrow A$ in P that is not satisfied by N because we showed that $N(B) = T^*$ and $N(A) = T$. Therefore, N is not a model of $P_2 \cup P$ and M' is $\preceq$-minimal model of $P_2 \cup P$.

In order to conclude the proof, it suffices to show that M' is not a standard answer set of $P_1 \cup P$. By the definition of M', it is $M \preceq M'$. But since M' is a model of P_2 and M is not, it must be $M' \neq M$ and thus $M \prec M'$. M also satisfies the rules of P and therefore it is a model of $P_1 \cup P$. We conclude that M' is not $\preceq$-minimal model of $P_1 \cup P$ and thus not a standard answer set of $P_1 \cup P$. $\square$