Defined Benefit Pension Schemes:

A Welfare Analysis of Risk Sharing

and Labour Market Distortions

**A: Derivation household behaviour**

Section 1 derives intertemporal budget equation (21) in the main text. Section 2 derives leisure demand equation (17). The derivation of consumption equation (22), portfolio equation (20) and stochastic discount factor is the subject of section 3.

**Financial wealth, pension rights and human wealth**

This section derives the intertemporal budget restriction. Assume the stochastic discount factor is exogenous and and hold. These assumptions will be verified later on (see section 3). Multiply equation (14) in the main text with the stochastic discount factor, take expectations and reorder to obtain

|  |  |  |
| --- | --- | --- |
|  |  | (A.1) |

Forward solution leads to

|  |  |  |
| --- | --- | --- |
|  |  | (A.2) |

*i.e.* financial wealth equals the discounted value of the difference between future consumption and income in case households don’t leave bequests. Non-capital income, consists of net wage income in the working ages () and pension income in the retirement period (). The discounted value of pension income can be split up into pension rights related to past work, , and future work

|  |  |  |
| --- | --- | --- |
|  |  | (A.3) |

in which denotes the discounted value of future pension income that can be attributed to the marginal hour of work as defined in equation (19) in the main text. Use the price of leisure definition and equation (A.3) to write the discounted value of income as

|  |  |  |
| --- | --- | --- |
|  |  | (A.4) |

Substitute this result (A.4) into equation (A.2)

|  |  |  |
| --- | --- | --- |
|  |  | (A.5) |

Define human wealth, diminished with labour-induced consumption

|  |  |  |
| --- | --- | --- |
|  |  | (A.6) |

reorder, substitute equation (21) of the main text to obtain the intertemporal budget equation.

**Leisure demand**

This section derives leisure demand equation (17) in the main text. Leisure demand is a static decision due to the assumptions made. Define the value of per-period total expenditures as the sum of good consumption and leisure consumption

|  |  |  |
| --- | --- | --- |
|  |  | (A.7) |

The per-period decision problem is

|  |  |  |
| --- | --- | --- |
|  |  | (A.8) |

given the restriction (A.7) and given leisure less than the total available time, which is normalized to one, *i.e.* . The Lagrangian of this static problem reads as

|  |  |  |
| --- | --- | --- |
|  |  | (A.9) |

with and Lagrangian parameters. First-order conditions are

|  |  |  |
| --- | --- | --- |
|  |  | (A.10) |

|  |  |  |
| --- | --- | --- |
|  |  | (A.11) |

The Kuhn Tucker condition is

|  |  |  |
| --- | --- | --- |
|  |  | (A.12) |

Define the price of leisure, inclusive the shadow price as

|  |  |  |
| --- | --- | --- |
|  |  | (A.13) |

Combining the two first-order conditions (A.10) and (A.11) leads, after substitution of both marginal utilities, to the following leisure demand equation:

|  |  |  |
| --- | --- | --- |
|  |  | (A.14) |

In case then . This is leisure demand equation (17) in the main text.

**The intertemporal consumption problem**

This section derives consumption equation (22), portfolio equation (20) and the stochastic discount factor. Define above labour induced consumption

|  |  |  |
| --- | --- | --- |
|  |  | (A.15) |

in which labour induced consumption is defined in (13). Substitute this definition into equation (A.5) after substitution of equation (A.6) and write it in difference equation format

|  |  |  |
| --- | --- | --- |
|  |  | (A.16) |

The market value of total wealth is determined using the stochastic discount factor approach. Alternatively, a replicating portfolio of equity and bonds could be used to determine this market value. For each asset a replicating portfolio exists. The sum of the equity investments in the replicating portfolios is the total risk exposure, which implies for the development of total wealth over time

|  |  |  |
| --- | --- | --- |
|  |  | (A.17) |

The instantaneous utility function of the dynamic allocation problem can be written as

|  |  |  |
| --- | --- | --- |
|  |  | (A.18) |

Substitution into the intertemporal utility function gives

|  |  |  |
| --- | --- | --- |
|  |  | (A.19) |

The recursive restatement of the maximum problem is

|  |  |  |
| --- | --- | --- |
|  |  | (A.20) |

Subject to the intertemporal budget equation. Assume, the value function can be written as

|  |  |  |
| --- | --- | --- |
|  |  | (A.21) |

with constants for to be defined later on. The assumption will be checked in section 5.

First-order conditions

Substitute the value function assumption (A.21) into the Bellman equation (A.20) just as the expression for total wealth (A.17) to write the maximum problem as

|  |  |  |
| --- | --- | --- |
|  |  | (A.22) |

First order conditions are

|  |  |  |
| --- | --- | --- |
|  |  | (A.23) |

|  |  |  |
| --- | --- | --- |
|  |  | (A.24) |

|  |  |  |
| --- | --- | --- |
|  |  | (A.25) |

Portfolio decision

Substitute (A.17) into the implicit portfolio equation (A.24) gives

|  |  |  |
| --- | --- | --- |
|  |  | (A.26) |

Divide (A.26) by the sum of the certain terms

|  |  |  |
| --- | --- | --- |
|  |  | (A.27) |

with

|  |  |  |
| --- | --- | --- |
|  |  | (A.28) |

A second order approximation of equation (A.27) gives the equity demand equation (20) in the main text (see Draper (2008) for the derivation). The direct investments in equity can be obtained by inverting the budget equation

|  |  |  |
| --- | --- | --- |
|  |  | (A.29) |

with

|  |  |  |
| --- | --- | --- |
|  |  | (A.30) |

The unique stochastic discount factor

Equation (A.24) and (A.25) can be written as and respectively with the relative utility. Equation (A.27) makes identification of the stochastic discount factor possible

|  |  |  |
| --- | --- | --- |
|  |  | (A.31) |

with a proportionality factor. This proportionality factor can be derived from This implies for the stochastic discount factor

|  |  |  |
| --- | --- | --- |
|  |  | (A.32) |

The expected value can be obtained using numerical integration. Because is determined by exogenous variables in equation (A.27), can be considered as given for the households.

Consumption-saving decision

Starting point for the non-labour related consumption decision is equation (A.23) Substitution of the marginal utility and the derivative of the value function gives

|  |  |  |
| --- | --- | --- |
|  |  | (A.33) |

make use of definition

|  |  |  |
| --- | --- | --- |
|  |  | (A.34) |

to write the non-labour-related total consumption equation as

|  |  |  |
| --- | --- | --- |
|  |  | (A.35) |

with the price index of total wealth defined as

|  |  |  |
| --- | --- | --- |
|  |  | (A.36) |

Equation (A.35) is consumption equation (22) in the main text, with the proportionality factor

|  |  |  |
| --- | --- | --- |
|  |  | (A.37) |

Value function

We check now the assumption of the value function. Total wealth develops according to

|  |  |  |
| --- | --- | --- |
|  |  | (A.38) |

Use this relation, substitute the assumption for the value function for and the consumption relation into the Belmann equation to obtain

|  |  |  |
| --- | --- | --- |
|  |  | (A.39) |

Note, are constants (*i.e.* exogenous given) for .

**B: Parametrized expectations**

**Conditional expectations human wealth**

This section follows, Heer and Mausner (2005) and Judd *ea* (2009)[[1]](#footnote-1). Variable is the conditional expectations given the information at age , *i.e.* the state of the economy at age

|  |  |  |
| --- | --- | --- |
|  |  | (B.1) |

Define with , the average and the standard deviation of . The state variables are collected in . With a stable population the funding ratio of pension funds, , is the most obvious relevant statistic for the state of the economy. We use the transformed funding ratio of pension funds, defined as with the average and the standard deviation of . The conditional expectations is a function of , such that for all . We seek a function which approximates . For instance . A constant term is not necessary due to the transformation. Assume both and depend on random variable . The least square parameters are

|  |  |  |
| --- | --- | --- |
|  |  | (B.2) |

We use the Monte Carlo approach to generate pairs of and . Then we regress in our example

|  |  |  |
| --- | --- | --- |
|  |  | (B.3) |

The conditional expectation equals

|  |  |  |
| --- | --- | --- |
|  |  | (B.4) |

which is the same as (B.3) except for the error term. The conditional epectations of human wealth become

|  |  |  |
| --- | --- | --- |
|  |  | (B.5) |

Judd *ea* (2009) advocates to use Hermite or Chebyshev polynomials in case higher order approximations are necessary instead of ordinary polynomials. Next section details on the construction of a complete set of Hermite polynomials

**Hermite polynomial representation**

To approximates we consider Hermite polynomials.

|  |  |  |
| --- | --- | --- |
|  |  | (B.6) |

We construct a complete set of polynomials of degree in variables using the ordinar polynomials

|  |  |  |
| --- | --- | --- |
|  |  | (B.7) |

In fact this set is not the complete set because one is not included. One is not included due to the normalization of the variables. In the same way a complete set of Ordinary polynomials can be defined.With one exogenous variable and ordinary polynomials we get as in the example in previous section. The complete set of polynomials of degree 2 in 2 variables is

|  |  |  |
| --- | --- | --- |
|  |  | (B.8) |

**Kalman filter**

Rational expectations imply that the agents understand the working of the economy, i.e. they have a very good model to predict the future, given the state of the economy. To get a good approximation a Kalman filter approach seems necessary. This section follows Harvey (1986) page 106 up to 110[[2]](#footnote-2). Equation (B.3) can be summarized by

|  |  |  |
| --- | --- | --- |
|  |  | (B.9) |

with in which indicates the variable has a mean and a variance; stands for ’wide sense’. Suppose constant parameters

|  |  |  |
| --- | --- | --- |
|  |  | (B.10) |

We use for the the minimum mean square linear estimator (MMSLE) of at time . The covariance matrix of is . The covariance matrix of the estimation error is

|  |  |  |
| --- | --- | --- |
|  |  | (B.11) |

The error made in prediction at time is

|  |  |  |
| --- | --- | --- |
|  |  | (B.12) |

with variance

|  |  |  |
| --- | --- | --- |
|  |  | (B.13) |

Updating rule for the covariance matrix

|  |  |  |
| --- | --- | --- |
|  |  | (B.14) |

The updating rule for the state vector is

|  |  |  |
| --- | --- | --- |
|  |  | (B.15) |

**C: Sensitivity analysis**

In order to assess whether our results depend on specific model parameters, we perform a sensitivity analysis. Table 2 presents the influence of parameter changes. It presents the expected values and standard deviations in steady state of the funding ratio of the pension scheme, the pension contribution rate, consumption and labour supply of a household at the age of 20 and two aspects of the distribution of equivalent variations of a steady-state generation. It also presents the percentage changes of consumption and labour supply with the economy with individual DC pension schemes and the welfare effects that correspond with the pension reform.

The second column, denoted (0), gives the outcomes for our benchmark scenario.[[3]](#footnote-3) Column (1) presents the effects of a shorter recovery period for funding deficits: here, equals 0.75 instead of 0.5 as in the benchmark simulation. This narrows the distribution of the funding ratio (the standard deviation drops to 0.21) and widens the distribution of the pension contribution rate (the standard deviation increases to 21). The counterpart is less risk sharing with future generations. Indeed, the welfare gain that we calculated for the benchmark case disappears completely. This explains the fall of average consumption and the increase in its variance.

Column (2) presents the effects of a different leisure demand elasticity: takes value 0.4 rather than 0.33 as in the benchmark case. In this simulation, we also adjust the preference parameter so that initial labour supply remains unaffected. The welfare gain from the pension reform is now larger than in the benchmark case. Although the higher price elasticity implies more variation in labour supply and consumption, it also increases average consumption and labour supply as the implicit subsidy of labour through the equity premium has a stronger labour supply effect.

A higher risk aversion of households has huge effects. Column (3) displays the effects of a simulation with equal to 5 rather than 3. Now, the introduction of a DB scheme is no longer welfare-improving, but welfare-reducing. The reason is a mechanism that has been noted before (Krueger *ea* (2006), Gollier (2008) and Bonenkamp *ea* (2014)): a higher degree of risk aversion reduces the demand for equity investment. Hence, the gain in risk reduction that can be achieved by setting up a DB scheme is less large. The effect is so huge that now the gain from risk sharing does not compensate for the losses due to labour supply distortions. Now, average consumption drops with the pension reform; labour supply remains largely unaffected.

The simulation in column (4) reveals that the upper and lower bound of the catching up contribution rate () have minor effects on the simulation outcomes. The same holds true for a smaller DB pension scheme () (column (5)). The effects of simulations with a higher equity premium (column (6)) and a lower volatility of equity returns (column (7)) echo the results of the simulation with a higher risk aversion. Both a higher equity premium and a lower volatility of equity returns increase the demand for equity investment and blow up the welfare gains from better risk sharing (in case of a higher equity premium with a factor of six!).

The simulation presented in column (8) refers to the idea that households may be myopic. It increases the rate of time preference of households () from 0.993 to 1.051. This reduces savings (see the increase in average consumption at the age of 20) with little effect upon labour supply. The effect upon welfare is negligible, however.

Overall, we conclude that our quantitative results are not robust to all parameter changes. In particular, alternative assumptions on the degree of risk aversion, the equity premium and the volatility of equity returns have a big impact upon the outcomes; even the sign of the welfare effect may change. On the other hand, the effect of alternative assumptions about the values of other parameters is much smaller. In general, we conclude that introduction of a DB scheme which combines better intergenerational risk with labour supply distortions will be welfare-improving.

Table 2. Sensitivity simulation results for parameter changes

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | (0)  | (1)  | (2)  | (3)  | (4)  | (5)  | (6)  | (7)  | (8)  |
| Funding ratio  |
| -  | 1.00  | 1.00  | 1.00  | 1.00  | 1.00  | 1.00  | 1.00  | 1.00  | 1.00  |
| -  | 0.24  | 0.21  | 0.24  | 0.24  | 0.24  | 0.24  | 0.24  | 0.21  | 0.24  |
|  |  |  |  |  |  |  |  |  |  |
| Pension premium  |
| -  | 13  | 14  | 13  | 13  | 13  | 10  | 9  | 13  | 13  |
| -  | 17  | 21  | 17  | 17  | 17  | 13  | 17  | 15  | 17  |
|  |  |  |  |  |  |  |  |  |  |
| Consumption at age 20  |
| -  | 4.54  | 4.49  | 4.61  | 4.56  | 4.54  | 4.52  | 4.92  | 4.63  | 5.06  |
| -  | 0.43  | 0.49  | 0.48  | 0.43  | 0.43  | 0.34  | 0.45  | 0.39  | 0.47  |
| - %  | 3.1  | 2.0  | 6.1  | 0.3  | 3.1  | 2.6  | 7.9  | 4.3  | 3.0  |
|  |  |  |  |  |  |  |  |  |  |
| Employment at age 20  |
| -  | 2.5  | 2.5  | 3.3  | 2.5  | 2.5  | 2.5  | 2.6  | 2.5  | 2.5  |
| -  | 0.1  | 0.2  | 0.1  | 0.1  | 0.1  | 0.1  | 0.1  | 0.1  | 0.1  |
| - %  | 1.3  | 0.8  | 1.5 | 1.3  | 1.3  | 1.2  | 2.5  | 1.5  | 1.3  |
|  |  |  |  |  |  |  |  |  |  |
| Ex post distribution equivalent variation of steady-state generation  |
| -  | 0.48  | 0.14  | 1.64  | 0.95  | 0.48  | 0.40  | 1.95  | 0.95  | 0.48  |
| -  | 43  | 51  | 14  | 76  | 43  | 42  | 11  | 28  | 43  |
|  |  |  |  |  |  |  |  |  |  |
| Welfare  | 19  | 1  | 24  | 80  | 19  | 18  | 115  | 52  | 19  |

 is the expected value, the standard deviation, the percentage change implied by the pension reform and the probability of a welfare loss. is the base run, , , and .

1. Heer, B. and A. Maussner, 2005, *Dynamic General Equilibrium Modelling*, Springer-Verlag, Berlin. Judd, K., L. Maliar, and S. Maliar, 2009, Numerically stable stochastic simulation ap-

proaches for solving dynamic economic models, NBER Working Paper 15296, NBER. [↑](#footnote-ref-1)
2. Harvey, A., 1986, *Time Series Models*, Philip Allan. [↑](#footnote-ref-2)
3. The parameter values for the benchmark scenario are summarized in Table 1. [↑](#footnote-ref-3)