Technical appendix

ONLINE APPENDIX, NOT INTENDED FOR PUBLICATION

A.1 Introduction

In this appendix, we present more detailed derivations and information about the stochastic overlapping-generations model presented in the main part of the paper. Section A.2 explains the solution method which is based on a log-linearization technique around the stochastic steady state. Section A.3 provides the derivations of the analytical results and propositions presented in Section 3 of the paper. In Section A.4 we present a baseline simulation of the model.

Throughout this appendix, we use exactly the same notational convention as in the paper. In addition, we use the notation $\hat{r} \equiv \log(1+r)$ for log returns and $\hat{y}_t \equiv \log y_t$ for the log of any other variable y_t. Variables without a time index refer to steady-state values.

A.2 Solving the model

A.2.1 The model

The model presented in the paper is summarized in Table A.1.[24] There are various ways to solve this model. One way is to solve the model numerically using dynamic programming methods or using perturbation methods around the deterministic steady state (see, for instance, Collard and Juillard, 2001 or Schmitt-Grohé and Uribe, 2004). Another possibility is to approximate the model using log-linearization around the steady state. The latter gives a bit more insight into the working of the model, and it is the road we will take. It should be understood that log-linearization is a small-shock approximation or an approximation to shocks with bounded support (Samuelson, 1970). Despite these limitations of log-linear approximations, this method clearly helps to explore the most important economic factors that affect the interaction between retirement behaviour and portfolio choice. As such, it provides a useful starting point for further qualitative explorations with higher-order numerical techniques.[25]

[24] To construct equation (A.1.1) we have substituted equations (15) and (16) in (9). Equation (A.1.2) is the result of inserting the portfolio rate of return (8) and the equilibrium conditions (16) and (17) into equation (10). The remaining equations, equation (A.1.4)-(A.1.7b), just repeat equation (11) (for $j = k$ and $j = b$) and equations (2), (3), (12) and equation (13).

[25] We also checked our results with higher order approximations using Dynare++. Although quantitatively the results give some small differences, the qualitative observations are exactly the same.

Table A.1: The model

$$w_t - c_{y,t} - r_{b,t}b = b + k_{t+1} \tag{A.1.1}$$

$$c_{o,t} = (1 + r_{b,t})b + (1 + r_{k,t})k_t + z_t w_t \tag{A.1.2}$$

$$c_{y,t}^{-\gamma} = \beta \mathrm{E}_t \left[(1 + r_{k,t+1}) c_{o,t+1}^{-\rho} u(c_{o,t+1}, z_{t+1})^{\rho-\varphi} \right] \tag{A.1.3}$$

$$c_{y,t}^{-\gamma} = \beta(1 + r_{b,t+1}) \mathrm{E}_t \left[c_{o,t+1}^{-\rho} u(c_{o,t+1}, z_{t+1})^{\rho-\varphi} \right] \tag{A.1.4}$$

$$w_t = (1 - \alpha) A_t k_t^{\alpha} (1 + z_t)^{-\alpha} \tag{A.1.5}$$

$$r_{k,t} + \delta_t = \alpha A_t k_t^{\alpha-1} (1 + z_t)^{1-\alpha} \tag{A.1.6}$$

$$\left(\frac{c_{o,t+1}}{1 - z_{t+1}} \right)^{\rho} = \frac{w_{t+1}}{\theta} \tag{A.1.7a}$$

$$\begin{aligned} &\mathrm{E}_t \left[w_{t+1} c_{o,t+1}^{-\rho} u(c_{o,t+1}, z_{t+1})^{\rho-\varphi} \right] = \\ &\theta(1 - z_{t+1})^{-\rho} \mathrm{E}_t \left[u(c_{o,t+1}, z_{t+1})^{\rho-\varphi} \right] \end{aligned} \tag{A.1.7b}$$

A.2.2 Steady state

A linearization around a deterministic steady state is sufficient for understanding macroeconomic dynamics, but it is not necessarily sufficient for an economic analysis involving uncertainty, such as questions about precautionary savings and asset-pricing issues. Following Juillard and Kamenik (2005), Bovenberg and Uhlig (2008) and Beetsma and Bovenberg (2009), we therefore use the concept of a *stochastic* steady state. This concept is defined as a situation in which each period shocks are equal to their expectations but agents are not aware of this (i.e., conditional variances are not zero). This point is solved from a non-linear system, and hence the solution does not generally correspond to the expected values of the variables involved.[26]

This non-linear system of steady-state equations is described in Table A.2. Notice that equations (A.2.1), (A.2.2), (A.2.5), (A.2.6) and (A.2.7a) have the same form as the original model equations of Table A.1. The remaining expectational equations, i.e., equations (A.2.3), (A.2.4) and (A.2.7b), are derived using second-order Taylor approximations of the original first-order conditions.

[26]Since the solution is not necessarily equal to expected values of the variables, Beetsma and Bovenberg (2009) label this solution as the *median solution*. We prefer to use the term stochastic steady state to indicate that the steady state is adjusted for risk.

Table A.2: The steady state

$$w - c_y - r_b b = b + k \tag{A.2.1}$$

$$c_o = (1 + r_b)b + (1 + r_k)k + zw \tag{A.2.2}$$

$$c_y^{-\gamma} = \beta(1 + r_k)c_o^{-\varphi}(1 - z)^{\theta(\rho-\varphi)} \exp\left(\frac{1}{2}\sigma^2_{r_k - u}\right) \tag{A.2.3}$$

$$c_y^{-\gamma} = \beta(1 + r_b)c_o^{-\varphi}(1 - z)^{\theta(\rho-\varphi)} \exp\left(\frac{1}{2}\sigma^2_{u}\right) \tag{A.2.4}$$

$$w = (1 - \alpha)Ak^{\alpha}(1 + z)^{-\alpha} \tag{A.2.5}$$

$$r_k + \delta = \alpha Ak^{\alpha-1}(1 + z)^{1-\alpha} \tag{A.2.6}$$

$$\left(\frac{c_o}{1 - z}\right)^{\rho} = \frac{w}{\theta} \tag{A.2.7a}$$

$$\left(\frac{c_o}{1 - z}\right)^{\rho} = \frac{w}{\theta} \exp\left[\frac{1}{2}\left(\sigma^2_{w-c_o} - \sigma^2_{c_o}\right)\right] \tag{A.2.7b}$$

Derivation expectational equations

We can write equation (A.1.3) as,

$$1 = \mathrm{E}_t\Big\{\exp\big[\underbrace{\log\beta + \log\hat{r}_k + \gamma\hat{c}_{y,t} - \rho\hat{c}_{o,t+1} + (\rho - \varphi)\hat{u}_{t+1}}_{x_{t+1}}\big]\Big\} \tag{A.1}$$

Taking a second-order Taylor expansion of $\exp(x_{t+1})$ around $\mathrm{E}_t x_{t+1} \equiv \bar{x}_t$, we obtain,

$$\begin{aligned} 1 &\approx \mathrm{E}_t\left\{\exp(\bar{x}_t)\left[1 + x_{t+1} - \bar{x}_t + \frac{1}{2}(x_{t+1} - \bar{x}_t)^2\right]\right\} \\ &= \exp(\bar{x}_t)\left(1 + \frac{1}{2}\mathrm{Var}_t x_{t+1}\right) \end{aligned} \tag{A.2}$$

Then, a first-order Taylor expansion around zero gives the result,

$$\begin{aligned} 1 &\approx 1 + \bar{x}_t + \frac{1}{2}\mathrm{Var}_t x_{t+1} \\ &= \exp\left(\bar{x}_t + \frac{1}{2}\mathrm{Var}_t x_{t+1}\right) \end{aligned} \tag{A.3}$$

Recall that the per-period utility function is given by:

$$u(c_o, 1-z) = \left[(1-\eta)c_o^{1-\rho} + \eta(1-z)^{1-\rho}\right]^{\frac{1}{(1-\rho)(1-\eta)}} \tag{A.4}$$

where η defines the relative preference for leisure and ρ represents the inverse of the elasticity of substitution between consumption and leisure in the second period. Taking logs of equation (A.4) gives:

$$\hat{u} = \frac{\log\left\{\exp\left[\log(1-\eta) + (1-\rho)\hat{c}_y\right] + \exp\left[\log\eta + (1-\rho)\log(1-z)\right]\right\}}{(1-\rho)(1-\eta)} \tag{A.5}$$

Taking a first-order Taylor expansion around zero then gives:

$$\hat{u} \approx \hat{c}_y + \theta \log(1-z) \tag{A.6}$$

with $\theta \equiv \eta/(1-\eta)$. Combining equations (A.3) and (A.6), we obtain the steady-state Euler equation regarding capital investments, equation (A.2.3):

$$c_y^{-\gamma} = \beta(1+r_k)c_o^{-\varphi}(1-z)^{\theta(\rho-\varphi)} \exp\left(\frac{1}{2}\sigma^2_{r_k-u}\right) \tag{A.7}$$

with $\sigma^2_{r_k-u}$ defined by:

$$\sigma^2_{r_k-u} \equiv \text{Var}_t\left[\log(1+r_{k,t+1}) - \varphi \log c_{o,t+1} + \theta(\rho-\varphi)\log(1-z_{t+1})\right] \tag{A.8}$$

The derivation of the Euler equation regarding government bonds investments, equation (A.2.4), and that of the optimality condition with respect to fixed retirement, equation (A.2.7b), are similar to the one above and will therefore not be discussed. The conditional (co)variances σ^2_u, $\sigma^2_{w-c_o}$ and $\sigma^2_{c_o}$ which appear in these other first-order conditions are defined as:

$$\sigma^2_u \equiv \text{Var}_t\left[-\varphi \log c_{o,t+1} + \theta(\rho-\varphi)\log(1-z_{t+1})\right] \tag{A.9}$$

$$\sigma^2_{w-c_o} \equiv \text{Var}_t\left(\log w_{t+1} - \varphi \log c_{o,t+1}\right) \tag{A.10}$$

$$\sigma^2_{c_o} \equiv \text{Var}_t\left[(\rho-\varphi)\log c_{o,t+1}\right] \tag{A.11}$$

At this point, we implicitly assume that the variances are constant over time. This will be justified in Section A.2.4, when solving for the linear recursive law of motion of the log-linearized system.

Special case: deterministic steady state

In general, the system in Table A.2 cannot be solved analytically. Only for a particular situation we are able to obtain explicit solutions, namely if: *i*) life-time utility is log-linear in consumption and leisure ($\gamma = \rho = 1$); *ii*) there is full depreciation ($\delta = 1$) and *iii*) all conditional covariances are perceived to be zero (deterministic steady state).

In this specific case, ignoring the risk terms or assuming a non-stochastic steady state implies that $r_k = r_b \equiv r$. Then inserting equation (A.2.1) and equation (A.2.2) in the Euler equation (A.2.3) (or equation (A.2.4)) gives:

$$\frac{1+\beta}{\beta} k = w - rb - \frac{1+\beta}{\beta} b - \frac{w}{(1+r)\beta} z \tag{A.12}$$

From the optimality condition with respect to leisure, equation (A.2.7a) (or equation (A.2.7b)), we derive:

$$k = \frac{w}{(1+r)\theta}(1-z) - \frac{w}{1+r} z - b \tag{A.13}$$

Substituting equation (A.13) in (A.12) and solving for z gives:

$$z = \frac{1+\beta-\beta\theta(1+r)\left(1-\frac{rb}{w}\right)}{1+\beta+\beta\theta} \tag{A.14}$$

Inserting equation (A.13) in equation (A.12) and solving for k leads to:

$$k = \frac{\beta(1+\theta)w\left(1-\frac{rb}{w}\right) - \frac{w}{1+r} - (1+\beta+\beta\theta)b}{1+\beta+\beta\theta} \tag{A.15}$$

Using the factor prices, equation (A.2.5) and equation (A.2.6), we can rewrite equation (A.15) into:

$$1+z = \frac{\beta(1+\theta)\left(1-\frac{rb}{w}\right)(1-\alpha)\left(\frac{k}{1+z}\right)^{\alpha-1} - \frac{1-\alpha}{\alpha}}{(1+\beta+\beta\theta)\left(1+\frac{b}{k}\right)} \tag{A.16}$$

In the same way, we can rewrite (A.14) into:

$$1+z = \frac{2(1+\beta)+\beta\theta-\beta\theta\left(1-\frac{rb}{w}\right)\alpha A\left(\frac{k}{1+z}\right)^{\alpha-1}}{1+\beta+\beta\theta} \tag{A.17}$$

Equations (A.16) and (A.17) form a closed system in k and z. Solving these equations gives for the capital-labour ratio,

$$\frac{k}{1+z} = \left[\frac{\left(1-\alpha+\theta+\theta\alpha\frac{b}{k}\right)\alpha\beta\left(1-\frac{rb}{w}\right)}{1-\alpha+\left(1+\frac{b}{k}\right)\alpha(2+2\beta+\beta\theta)}\right]^{\frac{1}{1-\alpha}} \tag{A.18}$$

and for labour supply:

$$z = \frac{1-\alpha-\alpha\theta-\alpha\theta\frac{b}{k}}{1+\theta-\alpha+\alpha\theta\frac{b}{k}} \tag{A.19}$$

Using the definition $\lambda \equiv k/(b+k)$ in equation (A.19), gives the labour supply decision as function of the portfolio choice:

$$z = \frac{\lambda(1-\alpha) - \alpha\theta}{\lambda(1+\theta-\alpha) + (1-\lambda)\alpha\theta} \tag{A.20}$$

Notice that equation (A.18) still depends on w and r, which are functions of the capital-labour ratio. Again using equations (A.2.5) and (A.2.6), we derive:

$$\frac{rb}{w} = \frac{\alpha A \left(\frac{k}{1+z}\right)^{\alpha-1} - 1}{(1-\alpha)\left(\frac{k}{1+z}\right)^{\alpha-1}} \frac{b}{k}(1+z) \tag{A.21}$$

Finally, substituting this expression in equation (A.18) and using equation (A.19), we obtain:

$$\frac{k}{1+z} = \left[\frac{\alpha\beta A\left(1+\theta-\alpha-2\alpha\frac{b}{k}\right)}{1+\alpha+\alpha\beta(2+\theta)+2\alpha\frac{b}{k}}\right]^{\frac{1}{1-\alpha}} \tag{A.22}$$

Using the definition λ in equation (A.22), gives the capital-labour ratio as function of the portfolio choice:

$$\frac{k}{1+z} = \left[\frac{\alpha\beta A(1+\theta+\alpha)\lambda - 2\alpha^2\beta A}{(1-\alpha)\lambda + \alpha\beta(2+\theta)\lambda + 2\alpha}\right]^{\frac{1}{1-\alpha}} \tag{A.23}$$

Notice from these expressions that both labour supply and the capital-labour ratio positively depend on the portfolio share λ invested in firm stocks: if λ decreases, for example because of a higher government debt, this leads to a crowding out of firm stocks which reduces the capital-labour ratio. In general equilibrium, a lower capital-labour ratio reduces the wage rate and, hence, labour supply incentives. Simulations confirm that this property of the model also holds under more general assumptions for which analytical results are not available. Given a solution to equations (A.20) and (A.23), all other steady-state variables can be calculated.

A.2.3 The log-linearized model

We now use a "tilde" on variables to denote the log-linearization of the model variables around the stochastic steady state. That is:

$$\tilde{x}_t \equiv \log x_t - \log x$$

The complete log-linearized model is reported in Table A.3.

Table A.3: The log-linearized model

$$w\tilde{w}_t - c_y\tilde{c}_{y,t} = k\tilde{k}_{t+1} + r_b b\tilde{r}_{b,t} \tag{A.3.1}$$

$$c_o\tilde{c}_{o,t} = r_k k\tilde{r}_{k,t} + (1+r_k)k\tilde{k}_t + r_b b\tilde{r}_{b,t} + zw(\tilde{z}_t + \tilde{w}_t) \tag{A.3.2}$$

$$\varphi \mathrm{E}_t\tilde{c}_{o,t+1} - \gamma\tilde{c}_{y,t} = \frac{r_k}{1+r_k}\mathrm{E}_t\tilde{r}_{k,t+1} - \theta(\rho-\varphi)\frac{z}{1-z}\mathrm{E}_t\tilde{z}_{t+1} \tag{A.3.3}$$

$$\varphi \mathrm{E}_t\tilde{c}_{o,t+1} - \gamma\tilde{c}_{y,t} = \frac{r_b}{1+r_b}\tilde{r}_{b,t+1} - \theta(\rho-\varphi)\frac{z}{1-z}\mathrm{E}_t\tilde{z}_{t+1} \tag{A.3.4}$$

$$\tilde{w}_t = \alpha\tilde{k}_t - \alpha\frac{z}{1+z}\tilde{z}_t + \omega_{A,t} \tag{A.3.5}$$

$$\tilde{r}_{k,t} + \frac{\delta}{r_k}\tilde{\delta}_t = \frac{r_k+\delta}{r_k}\left[(1-\alpha)\frac{z}{1+z}\tilde{z}_t - (1-\alpha)\tilde{k}_t + \omega_{A,t}\right] \tag{A.3.6}$$

$$\tilde{z}_{t+1} = \frac{1-z}{\rho z}\tilde{w}_{t+1} - \frac{1-z}{z}\tilde{c}_{o,t+1} \tag{A.3.7a}$$

$$\tilde{z}_{t+1} = \frac{1-z}{\rho z}\mathrm{E}_t\tilde{w}_{t+1} - \frac{1-z}{z}\mathrm{E}_t\tilde{c}_{o,t+1} \tag{A.3.7b}$$

A.2.4 The recursive law of motion

In the usual situation of no uncertainty, the steady state can be computed separately from the recursive law of motion. With a stochastic steady state, though, this procedure does no longer apply. In this case, deriving the recursive law involves the calculation of a fixed point: note from equations (A.2.3), (A.2.4) and (A.2.7b) that the steady state requires knowledge of the conditional variances, which can be calculated, given the log-linear recursive law of motion. But the latter are solutions to a system of equations of which the coefficients depend on the steady state. Hence, we are forced to simultaneously solve for the steady state and the log-linear model.

Solving for the log-linearized equilibrium law involves a three-step procedure:

- **Step 1: Rewriting the linear system.** The first step is to write the log-linearized endogenous variables as function of the endogenous and exogenous state variables. Our model contains two exogenous state variables, productivity shocks ($\omega_{A,t}$) and depreciation shocks (ω_δ) and one endogenous state variable, which is the capital stock ($\tilde{k}_t$). Recall that the return on government bonds ($\tilde{r}_{b,t}$) and labour supply in case of retirement inflexibility ($\tilde{z}_t$) are predetermined variables at time t. It turns out, however, that both variables are proportional to the capital stock so that they can be eliminated from the state space.

The proportional (and negative) relation between the return on bonds and the capital stock follows from capital market equilibrium: a higher capital stock combined with a constant level of government debt has to result in a more aggressive asset portfolio. To make this happen, the risk-free return on bonds will fall. The proportional relation between labour supply and the capital stock in case of retirement inflexibility can either be positive or negative, depending on the relative strength of income and substitution effects: a higher next-period capital stock leads to higher future wage expectations. Hence, rational agents, who plan to retire before shocks are revealed under retirement inflexibility, will postpone retirement if the substitution effect dominates and will advance retirement if the income effect dominates.

Accordingly, the capital stock is the only endogenous state variable in the model and we thus want to rewrite the log-linearized model of Table A.3 in the following linear system:

$$\tilde{k}_{t+1} = \pi_{k,k}\,\tilde{k}_t + \pi_{k,A}\,\omega_{A,t} + \pi_{k,\delta}\,\omega_{\delta,t} \tag{A.24}$$

and:

$$\begin{bmatrix} \tilde{c}_{y,t} \\ \tilde{c}_{o,t} \\ \tilde{r}_{k,t} \\ \tilde{w}_t \\ \tilde{r}_{b,t+1} \\ \tilde{z}_t \text{ or } \tilde{z}_{t+1} \end{bmatrix} = \begin{bmatrix} \pi_{c_y,k} \\ \pi_{c_o,k} \\ \pi_{r_k,k} \\ \pi_{w,k} \\ \pi_{r_b,k} \\ \pi_{z,k} \end{bmatrix} \tilde{k}_t + \begin{bmatrix} \pi_{c_y,A} & \pi_{c_y,\delta} \\ \pi_{c_o,A} & \pi_{c_o,\delta} \\ \pi_{r_k,A} & \pi_{r_k,\delta} \\ \pi_{w,A} & \pi_{w,\delta} \\ \pi_{r_b,A} & \pi_{r_b,\delta} \\ \pi_{z,A} & \pi_{z,\delta} \end{bmatrix} \begin{bmatrix} \omega_{A,t} \\ \omega_{\delta,t} \end{bmatrix} \tag{A.25}$$

where $\pi_{x,y}$ denotes the partial elasticity of endogenous variable x with respect to state variable y. With retirement flexibility, the recursive law for labour supply is based on $\tilde{z}_t$. With retirement inflexibility, it is based on $\tilde{z}_{t+1}$ because retirement is predetermined at time t.

We wish to solve for the partial derivatives $\pi_{x,y}$ such that the linear recursive law of motion satisfies the log-linearized equations. Preserving the derivations to Section A.2.5, the solution of the linear system ultimately provides expressions for the partial derivatives which will depend on the steady state.

- **Step 2: determining the conditional variances.** As a second step, we use the derived recursive law from the previous step to write the conditional variances in terms of the steady-state values and the exogenous shock terms. When doing this, we obtain the following variance terms of the consumption Euler equations:

$$\sigma^2_{r_k-u} = \sum_{i=A,\delta} \left[\frac{r_k}{1+r_k} \pi_{r_k,i} - \varphi \pi_{c_o,i} - \frac{\theta(\rho-\varphi)z}{1-z} \pi_{z,i} \right]^2 \sigma_i^2 \tag{A.26}$$

$$\sigma^2_u = \sum_{i=A,\delta} \left[-\varphi \pi_{c_o,i} - \frac{\theta(\rho-\varphi)z}{1-z} \pi_{z,i} \right]^2 \sigma_i^2 \tag{A.27}$$

Note that these variances are indeed constant over time, as assumed in the previous subsection. Equations (A.26) and (A.27) apply to the flexible retirement setting as well as to the inflexible retirement setting, although the partial elasticities differ in both cases. With retirement inflexibility, we also have to derive the covariance terms that show up in the optimality condition of labour supply. These covariances are equal to:

$$\sigma^2_{w-c_o} = \sum_{i=A,\delta} \left(\pi_{w,i} - \varphi \pi_{c_o,i} \right)^2 \sigma_i^2 \tag{A.28}$$

$$\sigma^2_{c_o} = \sum_{i=A,\delta} \left[(\rho-\varphi) \pi_{c_o,i} \right]^2 \sigma_i^2 \tag{A.29}$$

- **Step 3: Solving the linear system.** In the final step, we numerically solve for the steady-state variables, given the derived expressions for the conditional variances of the previous step. In case of retirement flexibility, this boils down to solving equations (A.2.1)-(A.2.7a), equation (A.26) and equation (A.27). For retirement inflexibility, the complete system of equations is described by equations (A.2.1)-(A.2.6), (A.2.7b) and (A.26)-(A.29). Once solved for the steady state, this solution can be substituted in the expressions for the partial derivatives (to be derived in the next section) and then the whole solution procedure is finished.

A.2.5 Derivation partial derivatives

In this section, we provide more information on how to solve for the partial elasticities (as discussed in Step 1 of Section A.2.4). We start with the flexible retirement setting and then turn to the fixed retirement setting.

Flexible retirement

Note that equations (A.3.2), (A.3.5), (A.3.6) and (A.3.7a) form an independent system of the endogenous variables $\tilde{c}_{o,t}$, $\tilde{w}_t$, $\tilde{r}_{k,t}$ and $\tilde{z}_t$ in the predetermined variables $\tilde{k}_t$ and $\tilde{r_b}_{,t}$ and the exogenous shocks $\omega_{A,t}$ and $\omega_{\delta,t}$. From this system we can infer the partial elasticities with respect to productivity and depreciation shocks. To save on notation, we define the following two variables:

$$\Gamma \equiv w^{1-\frac{1}{\rho}} \theta^{\frac{1}{\rho}} \tag{A.30}$$

$$\Delta \equiv (1-z)\alpha + (1+z)\rho(1+\Gamma) + \rho\alpha\Gamma \tag{A.31}$$

Then the partial elasticities with respect to productivity shocks are:

$$\pi_{c_o,A} = \frac{(1-z+\rho z+\alpha\rho)y}{c_o\Delta} > 0 \tag{A.32}$$

$$\pi_{r_k,A} = \frac{(r_k+\delta)(\rho+\rho z+\rho\Gamma+1-z)}{r_k\Delta} > 0 \tag{A.33}$$

$$\pi_{w,A} = \frac{\rho(1+z)(1+\Gamma-\alpha)}{(1-\alpha)\Delta} > 0 \tag{A.34}$$

$$\pi_{z,A} = \frac{(1+z)\left[(1-z)(1-\alpha)-\rho\Gamma(\alpha+z)\right]}{z(1-\alpha)\Delta} \tag{A.35}$$

Note that the sign of $\pi_{z,A}$ is ambiguous; it can either be positive or negative, depending on the substitution between consumption and leisure. For the partial elasticities with respect to depreciation shocks we have:

$$\pi_{c_o,\delta} = -\frac{\delta k(\rho+\alpha-\alpha z+\rho z)}{c_o\Delta} < 0 \tag{A.36}$$

$$\pi_{r_k,\delta} = -\frac{\delta\left[\rho(1+z)+(1-z)\alpha+\rho\Gamma(1+z-\alpha z)\right]}{r_k\Delta} < 0 \tag{A.37}$$

$$\pi_{w,\delta} = -\frac{\rho(1-z)\delta k\alpha}{c_o\Delta} < 0 \tag{A.38}$$

$$\pi_{z,\delta} = \frac{(1+z)(1-z)\rho\delta k}{c_o z\Delta} > 0 \tag{A.39}$$

Noting that $\mathrm{E}_t\omega_{A,t+1} = \mathrm{E}_t\omega_{\delta,t+1} = 0$ and using the Euler equations (A.3.3) and (A.3.4), we now can express the bond return $\tilde{r}_{b,t+1}$, the conditional expectations $\mathrm{E}_t\tilde{c}_{o,t+1}$ and $\mathrm{E}_t\tilde{c}_{r_k,t+1}$ together with first-period consumption $\tilde{c}_{y,t}$ as functions of the next-period capital stock $\tilde{k}_{t+1}$. This ultimately gives:

$$\tilde{r}_{b,t+1} = \Psi_{r_b}\tilde{k}_{t+1} \tag{A.40}$$

$$\mathrm{E}_t\tilde{c}_{o,t+1} = \Psi_{c_o}\tilde{k}_{t+1} \tag{A.41}$$

$$\tilde{c}_{y,t} = \Psi_{c_y}\tilde{k}_{t+1} \tag{A.42}$$

$$\mathrm{E}_t\tilde{z}_{t+1} = \Psi_z\tilde{k}_{t+1} \tag{A.43}$$

where the partial elasticities are equal to,

$$
\begin{aligned}
\Psi_{r_b} &\equiv -\frac{(1+r_b)\rho(1+z)y\left[(r_k+\delta)(1+\Gamma-\alpha)+\alpha(1-\delta)\Gamma\right]}{r_b y\Delta(1+r_k)+r_b(1+r_b)\rho(r_k+\delta)\Gamma(1+z)b} \\
\Psi_{c_o} &\equiv \frac{\left[\rho+\alpha+z(\rho-\alpha)\right]\left[(1-\delta)k+r_b b\Psi_{r_b}\right]+\alpha\left[1-z+\rho(z+\alpha)\right]y}{c_o\Delta} \\
\Psi_{c_y} &\equiv \frac{1}{\gamma}\left[\varphi\Psi_{c_o}-\frac{r_b\Psi_{r_b}}{1+r_b}+\frac{\theta(\rho-\varphi)z\Psi_z}{1-z}\right] \\
\Psi_z &\equiv \frac{(1-z)(1+z)\left[\alpha c_o-\alpha\rho(y-w)-\rho(1-\delta)k-\rho r_b b\Psi_{r_b}\right]}{c_o z\Delta}
\end{aligned}
$$

Notice from equation (A.40) that $\tilde{r}_{b,t}$ and $\tilde{k}_t$, the two predetermined variables, move proportionally. Therefore, we can substitute out $\tilde{r}_{b,t}$ from the state space.

To obtain the equilibrium law of the capital stock, (A.24), we substitute equation (A.42) in the budget restriction, equation (A.3.1). This gives the following partial elasticities for the capital stock:

$$\pi_{k,k} = \frac{w\pi_{w,k}-r_b b\Psi_{r_b}}{c_y\Psi_{c_y}+k} \tag{A.44}$$

$$\pi_{k,A} = \frac{w\pi_{w,A}}{c_y\Psi_{c_y}+k} \tag{A.45}$$

$$\pi_{k,\delta} = \frac{w\pi_{w,\delta}}{c_y\Psi_{c_y}+k} \tag{A.46}$$

The system is stable if and only if $\pi_{k,k}<1$.

The equilibrium law for $\tilde{k}_t$ pins down the solutions of the remaining endogenous variables in the model as shown in equation (A.25). Notice that the equilibrium laws of $\tilde{r}_{b,t+1}$ and $\tilde{c}_{y,t}$ follow from equations (A.40) and (A.42), respectively. This implies that $\pi_{r_b,i}=\Psi_{r_b}\pi_{k,i}$ and $\pi_{c_y,i}=\Psi_{c_y}\pi_{k,i}$ with $i=\{k,A,\delta\}$. The solutions for $\tilde{c}_{o,t}$, $\tilde{w}_t$, $\tilde{r}_{k,t}$ and $\tilde{z}_t$ then follow from equations (A.3.2), (A.3.5), (A.3.6) and (A.3.7a). This gives the remaining partial elasticities with respect to the capital stock:

$$\pi_{c_o,k} = \Psi_{c_o} \tag{A.47}$$

$$
\begin{aligned}
\pi_{r_k,k} = \frac{r_k+\delta}{r_k}\Bigg[&\frac{\alpha(\rho+\rho z+\rho\Gamma+1-z)}{\Delta} \\
&-\frac{\Gamma\rho(1+z)(k-\delta k+r_b b\Psi_{r_b})}{y\Delta}-1\Bigg]
\end{aligned} \tag{A.48}
$$

$$\pi_{w,k} = \frac{\alpha\rho(1+z)(1+\Gamma-\alpha)}{(1-\alpha)\Delta}+\frac{\alpha\rho(1-z)(k-\delta k+r_b b\Psi_{r_b})}{c_o\Delta} \tag{A.49}$$

$$\pi_{z,k} = \Psi_z \tag{A.50}$$

Fixed retirement

The derivation of the solution with fixed retirement mainly follows the same steps as that of the flexible retirement setting. When retirement is fixed, equations (A.3.2), (A.3.5) and (A.3.6) form an independent system of the endogenous variables $\tilde{c}_{o,t}$, $\tilde{w}_t$ and $\tilde{r}_{k,t}$ in terms of the predetermined variables $\tilde{k}_t$, $\tilde{r}_{b,t}$ and $\tilde{z}_t$ and the exogenous shocks $\omega_{A,t}$ and $\omega_{\delta,t}$. From this system, we can directly solve for the partial elasticities with respect to the shock terms. For productivity shocks we have:

$$\pi_{c_o,A} = \frac{y-w}{c_o} > 0 \tag{A.51}$$

$$\pi_{r_k,A} = \frac{r_k+\delta}{r_k} > 0 \tag{A.52}$$

$$\pi_{w,A} = 1 \tag{A.53}$$

and for depreciation shocks:

$$\pi_{c_o,\delta} = -\frac{\delta k}{c_o} < 0 \tag{A.54}$$

$$\pi_{r_k,\delta} = -\frac{\delta}{r_k} < 0 \tag{A.55}$$

$$\pi_{w,\delta} = 0 \tag{A.56}$$

With inflexible retirement, equations (A.40)-(A.42) do not change except that the definition of Γ (used in the Ψ-terms) now becomes,

$$\Gamma \equiv w^{1-\frac{1}{\rho}}\,\theta^{\frac{1}{\rho}} \exp\left[\frac{1}{2\rho}\left(\sigma_{c_o}^2 - \sigma_{w-c_o}^2\right)\right] \tag{A.57}$$

Consequently, the dynamic solution of the capital stock is still given by equations (A.44)-(A.46). Therefore, we retain the solution $\pi_{r_b,i} = \Psi_{r_b}\pi_{k,i}$ and $\pi_{c_y,i} = \Psi_{c_y}\pi_{k,i}$ with $i = \{k, A, \delta\}$. In addition, the elasticities of $\tilde{c}_{o,t}$, $\tilde{w}_t$ and $\tilde{r}_{k,t}$ with respect to the capital stock, as given by equations (A.47)-(A.49), are also still satisfied. Note from equation (A.53) that $\pi_{w,\delta} = 0$. Equation (A.46) then implies $\pi_{k,\delta} = 0$ which also means that $\pi_{r_b,\delta} = 0$ and $\pi_{c_y,\delta} = 0$.

Equation (A.43) is no longer satisfied, though, and becomes $\tilde{z}_t = \Psi_z\tilde{k}_t$. For the partial elasticities this means: $\pi_{z,k} = \Psi_z\pi_{k,k}$, $\pi_{z,A} = \Psi_z\pi_{k,A}$ and $\pi_{z,\delta} = 0$.

A.3 Appendix to Section 3

Suppose that we have log-linear life-time utility in consumption and leisure (i.e., $\rho = \gamma = 1$). Assume further that wages are non-stochastic.

A.3.1 Flexible retirement

Portfolio choice

Consumption in the second period is given by:

$$c_{o,t+1} = (1 + r_{t+1})s_t + z_{t+1}w_{t+1} \tag{A.58}$$

People invest a fraction λ of private savings in firm stocks (with stochastic return $r_{k,t+1}$) and a fraction $1-\lambda$ in government bonds (with risk-free return $r_{b,t+1}$). Hence, the return on the asset portfolio equals:

$$r_{t+1} \equiv (1 - \lambda_t)r_{b,t+1} + \lambda_t r_{k,t+1} \tag{A.59}$$

Inserting equation (A.1.7a) in equation (A.58), and using equation (A.59), we obtain:

$$c_{o,t+1} = \frac{1}{1+\theta}(1 + r_{T,t+1})\left(s_t + \frac{w_{t+1}}{1 + r_{b,t+1}}\right) \tag{A.60}$$

with,

$$r_{T,t+1} \equiv (1 - a_t)r_{b,t+1} + a_t r_{k,t+1} \tag{A.61}$$

$$a_t \equiv \frac{\lambda_t s_t}{s_t + \frac{w_{t+1}}{1+r_{b,t+1}}} \tag{A.62}$$

Note that $c_{o,t+1}$ is decomposed in non-stochastic terms (the first and third term) and a stochastic term (the second one). Substituting (A.60) in the two Euler equations, equations (A.1.3) and (A.1.4), and subtracting both, we have:

$$\mathrm{E}_t\left[(1 + r_{T,t+1})^{-1}(r_{k,t+1} - r_{b,t+1})\right] = 0 \tag{A.63}$$

Taking logs of equation (A.63), we obtain:

$$\mathrm{E}_t\hat{r}_{k,t+1} + \frac{1}{2}\mathrm{Var}_t\hat{r}_{k,t+1} - \hat{r}_{b,t+1} = \mathrm{Cov}_t(\hat{r}_{T,t+1}, \hat{r}_{k,t+1}) \tag{A.64}$$

where we have used Jensen's inequality condition for a lognormal variable, $\log \mathrm{E}_t x_{t+1} = \mathrm{E}_t \log x_{t+1} + 1/2\mathrm{Var}_t \log x_{t+1}$. To derive the term on the left-hand side of equation (A.64), we follow Campbell and Viceira (2002) and use a second-order Taylor approximation of the portfolio return, $r_{T,t+1}$. This gives,

$$\hat{r}_{T,t+1} \approx \hat{r}_{b,t+1} + a_t(\hat{r}_{k,t+1} - \hat{r}_{b,t+1}) + \frac{1}{2}a_t(1 - a_t)\mathrm{Var}_t\hat{r}_{k,t+1} \tag{A.65}$$

Hence,

$$\mathrm{Cov}_t(\hat{r}_{T,t+1}, \hat{r}_{k,t+1}) = a_t\mathrm{Var}_t\hat{r}_{k,t+1} \tag{A.66}$$

Substituting equation (A.66) into (A.64) then gives:

$$a_t = \frac{\mathrm{E}_t \hat{r}_{k,t+1} - \hat{r}_{b,t+1} + \frac{1}{2}\mathrm{Var}_t \hat{r}_{k,t+1}}{\mathrm{Var}_t \hat{r}_{k,t+1}} \tag{A.67}$$

Finally, inserting (A.67) in (A.62), we end up with the portfolio allocation in terms of financial wealth:

$$\lambda_t^F = \left[1 + \frac{w_{t+1}}{(1+r_{b,t+1})s_t}\right] \frac{\log \mathrm{E}_t(1+r_{k,t+1}) - \log(1+r_{b,t+1})}{\mathrm{Var}_t \log(1+r_{k,t+1})} \tag{A.68}$$

Consumption and leisure

Substituting equation (A.60) in equation (A.1.4) and rearranging gives:

$$c_{y,t}^{-1} = \beta(1+\theta)(1+r_{b,t+1})\mathrm{E}_t(1+r_{T,t+1})^{-1}\left(w_t - \tau_t - c_{y,t} + \frac{w_{t+1}}{1+r_{b,t+1}}\right)^{-1} \tag{A.69}$$

Notice that:

$$\begin{aligned}(1+r_{b,t+1})\mathrm{E}_t(1+r_{T,t+1})^{-1} &= (1+r_{b,t+1})\mathrm{E}_t(1+r_{T,t+1})^{-1} \\ &\quad + a_t \mathrm{E}_t\left[(1+r_{T,t+1})^{-1}(r_{k,t+1} - r_{b,t+1})\right] \\ &= 1\end{aligned} \tag{A.70}$$

Hence, first-period consumption satisfies:

$$c_{y,t} = \frac{1}{1+\beta(1+\theta)}\left(w_t - \tau_t + \frac{w_{t+1}}{1+r_{b,t+1}}\right) \tag{A.71}$$

Note that the propensity to consume is the same as under certainty. Hence, there is no precautionary saving motive, which is a direct implication of the log-utility specification (see Sandmo, 1970). Combining (A.71) and (A.60), we obtain for second-period consumption:

$$c_{o,t+1} = \frac{\beta(1+r_{T,t+1})}{1+\beta(1+\theta)}\left(w_t - \tau_t + \frac{w_{t+1}}{1+r_{b,t+1}}\right) \tag{A.72}$$

Substituting (A.72) in (A.1.7a), we obtain the expression for labour supply:

$$z_{t+1}^F = 1 - \frac{\theta\beta(1+r_{T,t+1})}{1+\beta(1+\theta)}\left(\frac{w_t - \tau_t}{w_{t+1}} + \frac{1}{1+r_{b,t+1}}\right) \tag{A.73}$$

A.3.2 Fixed retirement

Portfolio choice

Consider now the fixed retirement setting. Then the intertemporal budget constraint becomes:

$$c_{o,t+1} = (1 + r_{T,t+1})\left(s_t + \frac{w_{t+1}z_{t+1}}{1 + r_{b,t+1}}\right) \tag{A.74}$$

with $r_{T,t+1}$ again defined as in (A.61) but where a_t now satisfies:

$$a_t = \frac{\lambda_t s_t}{s_t + \frac{w_{t+1}z_{t+1}}{1+r_{b,t+1}}} \tag{A.75}$$

Inserting (A.74) in the two Euler equations (for $j = r_b$ and $j = r_k$) again gives condition (A.63). Hence, a_t is still given by equation (A.67). Inserting (A.67) into (A.75) we end up with the portfolio share in terms of financial wealth:

$$\lambda_t^I = \left[1 + \frac{w_{t+1}z_{t+1}}{(1 + r_{b,t+1})s_t}\right] \frac{\log \mathrm{E}_t(1 + r_{k,t+1}) - \log(1 + r_{b,t+1})}{\mathrm{Var}_t \log(1 + r_{k,t+1})} \tag{A.76}$$

Consumption and leisure

The fact that wages are non-stochastic implies that the first-order condition with respect to leisure consumption, equation (A.1.7b), becomes:

$$\frac{\theta}{1 - z_{t+1}} = w_{t+1}\mathrm{E}_t c_{o,t+1}^{-1} \tag{A.77}$$

Combining (A.77) and (A.1.4) gives:

$$(1 - z_{t+1})w_{t+1} = \theta\beta(1 + r_{b,t+1})c_{y,t} \tag{A.78}$$

Substituting (A.74) in (A.1.4) and rearranging gives:

$$c_{y,t}^{-1} = \beta\left(w_t - \tau_t - c_{y,t} + \frac{w_{t+1}z_{t+1}}{1 + r_{b,t+1}}\right)^{-1} \tag{A.79}$$

where we (again) used equality (A.70). Substitution of (A.78) in (A.79) gives:

$$c_{y,t}^{-1} = \beta\left[w_t - \tau_t + \frac{w_{t+1}}{1 + r_{b,t+1}} - (1 + \theta\beta)c_{y,t}\right]^{-1} \tag{A.80}$$

Hence,

$$c_{y,t} = \frac{1}{1 + \beta(1 + \theta)}\left(w_t - \tau_t + \frac{w_{t+1}}{1 + \tau_{b,t+1}}\right) \tag{A.81}$$

Note that consumption (and thus savings) under fixed labour supply is exactly equal to consumption under flexible labour supply. Substituting (A.81) in (A.78) and solving for z_{t+1}, we ultimately obtain the optimal retirement decision:

$$z^I_{t+1} = 1 - \frac{\theta\beta(1+r_{b,t+1})}{1+\beta(1+\theta)}\left(\frac{w_t-\tau_t}{w_{t+1}} + \frac{1}{1+r_{b,t+1}}\right) \tag{A.82}$$

A.4 Simulation results

Table A.4 shows the unconditional mean and standard deviation of the most important endogenous variables. These moments are calculated by simulating the derived recursive laws.[27] From this table we draw the same conclusions as from the steady-state results, discussed in the main text. With depreciation risk, retirement flexibility indeed offers a way to insure against adverse investment outcomes as stressed by Bodie et al. (1992). In this situation, the equity premium is lower than in case of inflexible retirement and agents are able to retire earlier on average. With productivity risk, however, we again have the opposite result. Then the equity premium under flexible retirement is higher than under inflexible retirement and agents choose to retire later on average. From a welfare perspective, , flexibility is preferable to inflexibility. Note that expected life-time utility is unambiguously higher in the first case, irrespective of whether depreciation risk or productivity risk is the sole risk factor.

Table A.4: Statistical moments of general equilibrium models

	Depreciation risk				Productivity risk			
	Fixed		Flexible		Fixed		Flexible	
	Mean	Stdev	Mean	Stdev	Mean	Stdev	Mean	Stdev
c_y/y	36.72	0.00	37.82	3.36	37.59	1.27	37.47	1.88
c_o/y	53.71	15.71	50.98	11.37	49.59	3.04	49.70	2.13
r_k	2.94	8.20	2.69	7.66	2.64	8.20	2.65	8.22
r_b	2.09	0.00	2.08	0.58	2.18	6.40	2.14	6.27
z	21.12	0.00	20.65	14.50	16.54	0.67	17.13	2.11
k/y	15.63	0.00	16.80	1.23	19.55	10.40	19.53	10.43
U	−6.60	0.97	−6.46	0.98	−6.68	2.37	−6.67	2.34

Note: the return on capital and the return on government debt are annualized figures. All figures are expressed in percentages.

[27] These simulations are based on the same parameterization as in the paper (see Section 4.1).