Online Appendix

Increasing Life Expectancy and NDC Pension Systems

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D Stable and unstable NDC systems

In this appendix I make general assumptions concerning the notional interest rate $\rho(a,t)$, the adjustment rate $\vartheta(a,t)$ and the annuity conversion factor $\Gamma(a,t)$ in order to see which combination of the parameters will lead to a balanced deficit ratio of $d(t) = 1$. This will allow to see why the conventional wisdom mostly fails (as discussed in section 5 and proposition 2) and how to construct stable NDC systems (as discussed in section 6).

These general factors are:

$$\rho(a,t) = g(t) + \mu(a,t-a) + \theta_1 \frac{\gamma}{\omega^c(t)}, \tag{27}$$

$$\vartheta(a,t) = g(t) + \theta_2 \frac{\gamma}{\omega^c(t)} + \theta_3 \frac{\gamma}{\omega^c(t) - R^c(t)}, \tag{28}$$

$$\Gamma(a,t) = \frac{\omega^c(t) - a}{1 + \lambda + \chi\gamma}, \tag{29}$$

where $\chi = 0$ if the annuity conversion factor is based on cohort life expectancy (as in 22a) while $\chi = 1$ if the factor is based on period life expectancy (as in 22b). To see this note that for assumption (9) $e^c(a,t) = \frac{\omega^c(t)-a}{1+\lambda}$ (see (10a))and $e^p(a,t+a) = \frac{\omega^c(t+a)-(1+\gamma)a}{1+\lambda+\gamma} = \frac{\omega^c(t)-a}{1+\lambda+\gamma}$ (see (10b)). Therefore $\chi = 0$ captures the use of cohort life expectancy (22a) and $\chi = 1$ the use of period life expectancy (22b). The parameters in the expressions of $\rho(a,t)$ and $\vartheta(a,t)$ follow the discussion in the text and some experimentation. Note that the growth rate of life expectancy $g^{\omega^c}(t)$ is given by $g^{\omega^c}(t) = \frac{\gamma}{\omega^c(t)}$ which is just the term that is captured by the parameters θ_1 and θ_2.

D.1 Increasing retirement with $R^c(t) = \psi\omega^c(t)$ (and $\lambda = 0$)

In this part it is assumed that retirement age is proportional to longevity, i.e. $R^c(t) = \psi\omega^c(t)$. Note that I assume here rectangular survivorship and thus the term $\mu(a, t-a)$ drops from equation (27). I will solve for the level of total expenditures as given in (21) step-by-step in order to derive closed form expressions for $E(t)$ and for $d(t)$. For convenience I repeat here the expression for $E(t)$ for the assumption of $\lambda = 0$.

$$E(t) = \tau W(t) N \int_{R^p(t)}^{\omega^p(t)} \frac{R^c(t-a)}{\Gamma(R^c(t-a), t-a)} \left(\frac{1}{R^c(t-a)} \int_0^{R^c(t-a)} e^{\int_x^{R^c(t-a)} (\rho(s,t-a+s) - g(t-a+s))\,\mathrm{d}s}\,\mathrm{d}x \right) \left(e^{\int_{R^c(t-a)}^{a} (\vartheta(s,t-a+s) - g(t-a+s))\,\mathrm{d}s} \right) \mathrm{d}a.$$

The first term in the integral is a constant and given by:

$$\frac{R^c(t-a)}{\Gamma(R^c(t-a), t-a)} = \frac{\psi(1+\chi\gamma)}{1-\psi}.$$

Furthermore it holds that:

$$e^{\int_x^{R^c(t-a)} (\rho(s,t-a+s) - g(t-a+s))\,\mathrm{d}s} = e^{\int_x^{R^c(t-a)} \theta_1 \frac{\gamma}{\omega^c(t-a+s)}\,\mathrm{d}s} = (1+\gamma\psi) \left(\frac{\omega^c(t-a)}{\omega^c(t-a+x)} \right)^{\theta_1}.$$

This can be used to calculate that:

$$\begin{aligned}
&\frac{1}{R^c(t-a)} \int_0^{R^c(t-a)} e^{\int_x^{R^c(t-a)} (\rho(s,t-a+s) - g(t-a+s))\,\mathrm{d}s}\,\mathrm{d}x = \\
&\frac{1}{R^c(t-a)} \int_0^{R^c(t-a)} (1+\gamma\psi) \left(\frac{\omega^c(t-a)}{\omega^c(t-a+x)} \right)^{\theta_1} \mathrm{d}x = \\
&\frac{1+\gamma\psi - (1+\gamma\psi)^{\theta_1}}{\gamma\psi(1-\theta_1)}.
\end{aligned}$$

For the term involving the adjustment rates one can calculate that:

$$\begin{aligned}
&e^{\int_{R^c(t-a)}^{a} (\vartheta(s,t-a+s) - g(t-a+s))\,\mathrm{d}s} = \\
&e^{\int_{R^c(t-a)}^{a} (\theta_2 \frac{\gamma}{\omega^c(t-a+s)} + \theta_3 \frac{\gamma}{\omega^c(t-a+s) - R^c(t-a+s)})\,\mathrm{d}s} = \\
&(1+\gamma\psi)^{-(\theta_2 + \frac{\theta_3}{1-\psi})} \left(\frac{\omega^c(t)}{\omega^c(t-a)} \right)^{\theta_2 + \frac{\theta_3}{1-\psi}}.
\end{aligned}$$

Table 6: The assumptions underlying table 1

		Cohort Life Expectancy ($\chi = 0$)		Period Life Expectancy ($\chi = 1$)	
		Notional Interest Rate			
		Average Wages	Wage Bill	Average Wages	Wage Bill
Adjustment Rate	Average Wages	$\theta_1 = 0, \theta_2 = 0$ (0.80)	$\theta_1 = 1, \theta_2 = 0$ (0.87)	$\theta_1 = 0, \theta_2 = 0$ (1.00)	$\theta_1 = 1, \theta_2 = 0$ (1.09)
	Wage Bill	$\theta_1 = 0, \theta_2 = 1$ (0.82)	$\theta_1 = 1, \theta_2 = 1$ (0.89)	$\theta_1 = 0, \theta_2 = 1$ (1.03)	$\theta_1 = 1, \theta_2 = 1$ (1.12)

Note: This table shows the parametrizations that lead to the corresponding entries in table 1. All entries have $\theta_3 = 0$. In brackets I also show numerical values for the (non-approximated) deficit ratio $d(t)$ for the parameters $\gamma = 0.25$ and $\psi = \frac{3}{4}$.

Finally, one can put all terms together to derive an expression for $E(t)$. Dividing this expression by the level of $I(t) = \tau W N R^p(t) = \tau W N \frac{\psi}{1+\gamma\psi} \omega^c(t)$ gives a closed form solution for the deficit ratio $d(t) = \frac{E(t)}{I(t)}$:

$$d(t) = \frac{(1-\chi\gamma)\left(1+\gamma-(1+\gamma\psi)\left(\frac{1+\gamma}{1+\gamma\psi}\right)^{\theta_2+\frac{\theta_3}{1-\psi}}\right)\left(1+\gamma\psi-(1+\gamma\psi)^{\theta_1}\right)}{\gamma^2(1+\gamma)\psi(1-\theta_1)((1-\psi)(1-\theta_2)-\theta_3)}. \tag{30}$$

This can be approximated around $\gamma = 0$ to derive that:

$$d(t) \approx 1 + \frac{\gamma}{2}\left(2\chi - 2 + \psi(\theta_1 - \theta_2) + \theta_2 + \theta_3\right). \tag{31}$$

The eight entries in table 1 all follow by inserting the appropriate parameters for θ_1 to θ_3 and χ into (30) and then eventually performing a first-order approximation around $\gamma = 0$. The corresponding parameter choices are stated in table 6. In the following I report the deficit ratio for the most interesting cases.

Indexation with the growth rate of the wage bill: For the case where the two indexation rates are based on the growth rate of the wage bill (i.e. $\theta_1 = \theta_2 = 1$) one gets for cohort life expectancy ($\chi = 0$):

$$d(t) = \frac{(1+\psi\gamma)\ln\left(\frac{1+\gamma}{1+\psi\gamma}\right)\ln(1+\psi\gamma)}{\gamma^2\psi(1-\psi)} \approx 1 - \frac{\gamma}{2} + \frac{\gamma^2}{12}\left(4 + \psi(1-\psi)\right),$$

where I use a second-order approximation around $\gamma = 0$. For $\psi = 3/4$ and $\gamma = 0.25$ the exact value for $d(t)$ is 0.893, while the second-order and first-order approximations give 0.897 and 0.875, respectively.

For the use of period life expectancy ($\chi = 1$) the deficit ratio comes out as:

$$d(t) = \frac{(1+\gamma)(1+\psi\gamma)\ln\left(\frac{1+\gamma}{1+\psi\gamma}\right)\ln(1+\psi\gamma)}{\gamma^2\psi(1-\psi)} \approx 1 + \frac{\gamma}{2} - \frac{\gamma^2}{12}\left(2 - \psi(1-\psi)\right).$$

In the case of cohort life expectancy one faces a permanent surplus while for the case of period life expectancy one has to deal with a permanent deficit.

Indexation with the average growth rate: For the case with average wage growth indexation (i.e. $\theta_1 = \theta_2 = 0$) one does not need approximations to get concise results. In fact equation (30) implies $d(t) = \frac{1}{1+\gamma}$ (for cohort life expectancy with $\chi = 0$) and $d(t) = 1$ (for period life expectancy with $\chi = 1$).

Additional indexations: In section 6 I look at additional stable NDC systems.

For the use of cohort life expectancy ($\chi = 0$) and $\theta_1 = \theta_2 = 2$ (and $\theta_3 = 0$) equation (30) implies an exact value of $d(t) = 1$.

For the use of cohort life expectancy ($\chi = 0$) and $\theta_1 = \theta_2 = 0$ and $\theta_3 = 2$ one gets that:

$$d(t) = \frac{\left((1+\gamma\psi)\left(\frac{1+\gamma}{1+\gamma\psi}\right)^{\frac{2}{1-\psi}}\right) - (1+\gamma)}{\gamma(1+\gamma)(1+\psi)} \approx 1 - \frac{\gamma^2\psi}{3},$$

which is close to $d(t) = 1$.

D.2 Constant retirement with $R^c(t) = \overline{R}$ (and $\lambda = 0$)

I follow the same steps as in appendix D.1 with the exception that retirement age is now assumed to be constant, i.e. $R^c(t) = \overline{R}$. I use the general parameters as laid done in (27), (28) and (29) for $\lambda = 0$. The results are again related to the failures of the conventional wisdom (as discussed in section 5.1 and proposition 2) and to the construction of stable NDC systems (as discussed in section 6).

The first term in the expressions for $E(t)$ is no longer constant but given by:

$$\frac{R^c(t-a)}{\Gamma(R^c(t-a),t-a)} = \frac{\overline{R}(1+\chi\gamma)}{\omega^c(t-a)-\overline{R}}.$$

Furthermore it holds that:

$$e^{\int_x^{R^C(t-a)}(\rho(s,t-a+s)-g(t-a+s))\,\mathrm{d}s} = e^{\int_x^{R^C(t-a)}\theta_1\frac{\gamma}{\omega^c(t-a+s)}\,\mathrm{d}s} = \left(\frac{\omega^c(t-a+\overline{R})}{\omega^c(t-a+x)}\right)^{\theta_1}.$$

This can be used to calculate that:

$$\frac{1}{R^c(t-a)}\int_0^{R^c(t-a)} e^{\int_x^{R^C(t-a)}(\rho(s,t-a+s)-g(t-a+s))\,\mathrm{d}s}\,\mathrm{d}x =$$
$$\frac{1}{R^c(t-a)}\int_0^{R^c(t-a)} \left(\frac{\omega^c(t-a+\overline{R})}{\omega^c(t-a+x)}\right)^{\theta_1}\mathrm{d}x =$$
$$\frac{\omega^c(t-a+\overline{R})^{\theta_1}\left(\omega^c(t-a+\overline{R})^{1-\theta_1}-\omega^c(t-a)^{1-\theta_1}\right)}{\overline{R}\gamma(1-\theta_1)}.$$

For the term involving the adjustment rates one can calculate that:

$$e^{\int_{R^C(t-a)}^{a}(\vartheta(s,t-a+s)-g(t-a+s))\,\mathrm{d}s} =$$
$$e^{\int_{R^C(t-a)}^{a}(\theta_2\frac{\gamma}{\omega^c(t-a+s)}+\theta_3\frac{\gamma}{\omega^c(t-a+s)-R^C(t-a+s)})\,\mathrm{d}s} =$$
$$\left(\frac{\omega^c(t)}{\omega^c(t-a+\overline{R})}\right)^{\theta_2}\left(\frac{\omega^c(t)-\overline{R}}{\omega^c(t-a+\overline{R})-\overline{R}}\right)^{\theta_3}.$$

Different to the case with $R^c(t)=\psi\omega^c(t)$ it is not possible to derive a closed-form solution for $E(t)$ for general values of θ_1, θ_2 and θ_3. I want to present the solutions for specific cases that are also discussed in the text.

Indexation with the growth rate of the wage bill or average wage growth: For the case with a constant retirement age these two indexation scheme coincide since $g^L(t)=0$. For $\theta_1=\theta_2=0$ (and $\theta_3=0$) one has that:

$$\frac{1}{R^c(t-a)}\int_0^{R^c(t-a)} e^{\int_x^{R^C(t-a)}(\rho(s,t-a+s)-g(t-a+s))\,\mathrm{d}s}\,\mathrm{d}x = 1.$$

and

$$d(t) = \frac{(1+\chi\gamma)\ln(1+\gamma)}{\gamma}.$$

This implies for cohort life expectancy ($\chi = 0$) a deficit ratio of $d(t) = \frac{\ln(1+\gamma)}{\gamma} \approx 1 - \frac{\gamma}{2}(1 - \frac{\gamma}{3})$ and for period life expectancy ($\chi = 1$) a value of $d(t) = \frac{(1+\gamma)\ln(1+\gamma)}{\gamma} \approx 1 + \frac{\gamma}{2}(1 - \frac{2\gamma}{3})$.

Indexation with the corrected growth rate of the wage bill: This uses equations (23c) and (24c) that correct the wage-bill growth for increases in the labor force that are necessary to hold the dependency ratio constant. As argued in section 6.1 this means that $\theta_1 = \theta_2 = -1$ (and $\theta_3 = 0$) and one can calculate for period life expectancy:

$$d(t) = \frac{\overline{R}(1+\gamma)}{\omega^c(t)} \left(\frac{(2+\gamma)\ln(1+\gamma)}{2\gamma} - 1 \right) + 1, \tag{32}$$

which, using a second-order Taylor expansion around $\gamma = 0$, is approximately $d(t) = 1 + \gamma^2 \frac{\overline{R}}{12\omega^c(t)}$. Since the latter term is only of second order importance and rather small one can conclude that this combination will implement an approximately stable NDC system. For $\gamma = 0.25$, $\overline{R} = 45$ and $\omega^c(t) = 60$ the exact value is $d(t) = 1.0039$, the second-order approximation $d(t) = 1.0039$ and the first-order approximation $d(t) = 1$.

For cohort life expectancy one gets that:

$$d(t) = \frac{\overline{R}}{\omega^c(t)} \left(\frac{(2+\gamma)\ln(1+\gamma)}{2\gamma} - 1 \right) + \frac{1}{1+\gamma} \approx 1 - \gamma + \gamma^2 \left(1 + \frac{\overline{R}}{12\omega^c(t)} \right). \tag{33}$$

Additional indexations: In section 6 I look at additional stable NDC systems.

For the use of cohort life expectancy and $\theta_1 = \theta_2 = 1$ (and $\theta_3 = 0$) the resulting expression for $d(t)$ is rather complicated. It can be approximated as:

$$d(t) = 1 + \chi\gamma - \gamma^2 \frac{\overline{R}(3\omega^c(t) - 2\overline{R})}{12\omega^c(t)}.$$

For cohort life expectancy with $\chi = 0$ one thus gets that the deficit ratio is approximately 1. In fact, the approximation is again quite accurate. For the same parameters as above the exact value is $d(t) = 0.996$ and the second-order approximation $d(t) = 0.994$.

For the use of $\theta_1 = \theta_2 = 0$ and $\theta_3 = -1$ one gets that:

$$d(t) \approx 1 - \gamma(1-\chi) + \gamma^2 \left(\frac{1}{2} - \chi + \frac{\omega^c(t)}{2(\omega^c(t) - \overline{R})} \right).$$

This means that for the use of period life expectancy ($\chi = 1$) this is almost balanced.

In a similar fashion, for the use of $\theta_1 = \theta_2 = 0$ and $\theta_3 = 1$ one can approximate the deficit ratio as:

$$d(t) \approx 1 + \gamma\chi - \gamma^2 \frac{\overline{R}}{2(\omega^c(t) - \overline{R})}.$$

This means that in this case the use of cohort life expectancy ($\chi = 0$) leads to an almost balanced budget.

D.3 Increasing retirement with $R^c(t) = \psi\omega^c(t)$ (and $\lambda = 1$)

In this part it is again assumed that the retirement age is proportional to longevity, i.e. $R^c(t) = \psi\omega^c(t)$. I now assume, however, linear survivorship curves ($\lambda = 1$). In order to simplify the discussion I will assume that $\vartheta(a,t)$ corresponds to $\rho(a,t)$ and thus $\theta_2 = \theta_1$ and $\theta_3 = 0$ in equations (27) and (28).

The expression for $E(t)$ (see (21)) can be written as:

$$E(t) = \frac{\tau W(t) N}{\eta} \int_{R^p(t)}^{\omega^p(t)} \frac{\int_0^{R^c(t-a)} S(x, t-a) e^{\int_x^a \frac{\theta_1 \gamma}{\omega^c(t-a+s)} \mathrm{d}s} \mathrm{d}x}{\int_{R^c(t-a)}^{\omega^c(t-a)} S(x, t-a) \mathrm{d}x} S(a, t-a) \mathrm{d}a,$$

where $\eta = \frac{1+\lambda}{1+\lambda+\chi\gamma}$. If the annuity conversion factor is based on cohort life expectancy ($\chi = 0$) then $\eta = 1$ while $\eta = \frac{1+\lambda}{1+\lambda+\gamma}$ if it is based on period life expectancy ($\chi = 1$).

This can be solved in closed form, although the resulting expressions for $E(t)$ and $d(t)$ are rather lengthy. For $\theta_1 = 2$ and $\eta = 1$, however, it can be shown that $d(t) = 1$ holds exactly. For $\theta_1 = \lambda = 1$ and $\eta = \frac{1+\lambda}{1+\gamma+\lambda}$, on the other hand, one can derive that:

$$E(t) = \tau W(t) N \frac{2\omega^c(t)(\gamma\psi - (1+\gamma)\ln(1+\gamma\psi))(\gamma(1-\psi) - (1+\gamma)(\ln(1+\gamma\psi) - \ln(1+\gamma)))}{\gamma^4 (1-\psi)^2}.$$

This can be used to derive the deficit ratio which can then be approximated to yield: $d(t) \approx 1 + \frac{\gamma(2-3\psi)}{6(2-\psi)}$ which equals $d(t) = 1$ for $\psi = \frac{2}{3}$.

E Turnover duration (section 6.2)

The starting point is the definition of the pension liability as the present value of future benefits to all persons to whom the pension system has a liability at the time of assessment minus the present value of future contributions by the same individuals.

One can distinguish between individuals that work and individuals that are already retired. I focus on the case of rectangular survivorship ($\lambda = 0$) and a retirement pattern following $R^c(t) = \psi\omega^c(t)$.

Workers: For a worker of age $a \in [0, R^p(t)]$ the contributions until retirement at age $R^c(t-a)$ are given by:

$$\int_a^{R^c(t-a)} \tau W(t)\,\mathrm{d}x = \tau W(t)\left(R^c(t-a) - a\right),$$

where I use the growth rate of the wage level as discount factor. The future pension benefits, on the other hand, amount to:

$$\int_{R^c(t-a)}^{\omega^c(t-a)} \frac{\tau W(t) R^c(t-a)}{\omega^c(t-a) - R^c(t-a)}\,\mathrm{d}x = \tau W(t)\psi\omega^c(t-a),$$

where the annuity conversion factor is based on cohort life expectancy.

Total net pension liabilities for workers are thus given by:

$$\tau W(t) \int_0^{R^p(t)} \left(\psi\omega^c(t-a) - \left(R^c(t-a) - a\right)\right)\mathrm{d}a = \tau W(t)\int_0^{R^p(t)} a\,\mathrm{d}a = \tau W(t) R^p(t) A_W^p(t), \tag{34}$$

where for the last step I use the definition of the average age of contributors (see (7)) given by $A_W^p(t) = \frac{\int_0^{R^p(t)} a\,\mathrm{d}a}{R^p(t)}$.

Retirees: The pension liabilities of a retiree of age $a \in [R^p(t), \omega^p(t)]$ are given by:

$$\int_a^{\omega^c(t-a)} \frac{\tau W(t) R^c(t-a)}{\omega^c(t-a) - R^c(t-a)}\,\mathrm{d}x = \tau W(t)\frac{\psi}{1-\psi}\left(\omega^c(t-a) - a\right).$$

The total liabilities of all retired cohorts comes out as:

$$\begin{gathered}
\textstyle\int_{R^p(t)}^{\omega^p(t)} \left(\tau W(t)\frac{\psi}{1-\psi}\left(\omega^c(t-a) - a\right)\right)\mathrm{d}a = \\
\tau W(t)\frac{\psi}{1-\psi}\left(\omega^c(t)\left(\omega^p(t) - R^p(t)\right) - (1+\gamma)\textstyle\int_{R^p(t)}^{\omega^p(t)} a\,\mathrm{d}a\right) = \\
\tau W(t)\frac{\psi}{1-\psi}\left(\omega^p(t) - R^p(t)\right)\left(\omega^c(t) - (1+\gamma)A_R^p(t)\right) = \\
\tau W(t) R^p(t)\left(\omega^p(t) - A_R^p(t)\right),
\end{gathered}$$

where I use $A_R^p(t) = \frac{\int_{R^p(t)}^{\omega^p(t)} a\,\mathrm{d}a}{\omega^p(t) - R^p(t)}$ and $\left(\omega^p(t) - R^p(t)\right) = R^p(t)\frac{1-\psi}{\psi(1+\gamma)}$.

Total pension liabilities: Adding this expression and (34) gives the total net pension liabilities. They come out as:

$$\tau W(t) \left(R^p(t) A_W^p(t) + R^p(t) \left(\omega^p(t) - A_R^p(t)\right)\right) =$$
$$\tau W(t) R^p(t) \left(A_R^p(t) - A_W^p(t)\right) =$$
$$I(t) TD(t),$$

where I use $A_W^p(t) + \omega^P(t) - A_R^p(t) = A_R^p(t) - A_W^p(t)$, $I(t) = \tau W(t) R^p(t)$ and $TD(t) = (A_R^p(t) - A_W^p(t))$.

F Required retirement age (section 7.1)

In this section I generalize method A, i.e. the period-based approach that is capable of implementing a stable NDC system for the case of non-rectangular survivorship. The case with rectangular survivorship has been treated in section 6.1 and is shown in tables 3 and 4. This approach is based on a "required retirement age" $R^{*p}(t)$ that determines how the retirement age has to evolve such that the basic NDC system with $\rho(a,t) = g^W(t) + \mu(a,t)$ and $\vartheta(a,t) = g^W(t)$ stays balanced. In such a system the first pension of cohort t is given by:

$$P^F(t) = \frac{\tau W(t) R^{*c}(t)}{e^p(R^{*c}(t), t + R^{*c}(t))} \frac{1}{R^{*c}(t)} \frac{\int_0^{R^{*c}(t)} S(x,t)\,\mathrm{d}x}{S(R^{*c}(t),t)},$$

where $\frac{1}{R^{*c}(t)} \frac{\int_0^{R^{*c}(t)} S(x,t)\,\mathrm{d}x}{S(R^{*c}(t),t)}$ give the inheritance gains that accrue to the surviving members of cohort t at the moment of retirement at age $R^{*c}(t)$. Since the adjustment factor is simply given by the growth rate of average wages the average pension in *period* t is thus given by:

$$\overline{P}(t) = \frac{1}{M(t)} \int_{R^{*p}(t)}^{\omega^p(t)} \frac{\tau W(t) R^{*c}(t-a)}{e^p(R^{*c}(t-a), t-a+R^{*c}(t-a))} \frac{1}{R^{*c}(t-a)} \frac{\int_0^{R^{*c}(t-a)} S(x,t-a)\,\mathrm{d}x}{S(R^{*c}(t-a),t-a)} S(a,t-a)\,\mathrm{d}a.$$

Note that:

$$e^p(R^{*c}(t-a), t-a+R^{*c}(t-a)) = \frac{\omega^c(t-a) - R^{*c}(t-a)}{1+\gamma+\lambda}.$$

For a stable system it has to hold that total pension payments $M(t)\overline{P}(t)$ are equal to total revenues $\tau W(t)L(t)$ or:

$$\begin{aligned} M(t)\overline{P}(t) = \int_{R^{*p}(t)}^{\omega^p(t)} & \frac{R^{*c}(t-a)(1+\gamma+\lambda)}{\omega^c(t-a) - R^{*c}(t-a)} \\ & \frac{1}{R^{*c}(t-a)} \frac{\int_0^{R^{*c}(t-a)} S(x,t-a)\,\mathrm{d}x}{S(R^{*c}(t-a),t-a)} S(a,t-a)\,\mathrm{d}a \\ = \; & L(t) = \int_0^{R^{*p}(t)} S(a,t-a)\,\mathrm{d}a. \end{aligned} \tag{35}$$

My conjecture for a stable system is that the period retirement age has to be set according to:

$$R^{*p}(t) = \phi\left(\omega^c(t)\right)^{1-\lambda}. \tag{36}$$

For $\lambda = 0$ this reduces to $R^{*p}(t) = \phi\omega^c(t)$. If one sets $\phi = \frac{\psi}{1+\gamma\psi}$ then this implies that $R^{*c}(t) = \psi\omega^c(t)$ which corresponds to the case of assumption (13a). From section 6.1 it is already known that this corresponds to the retirement age that leads to a situation where the aggregate support ratio $\frac{L(t)}{M(t)}$ is stabilized. For $\lambda > 0$ this can only be verified by numerical simulations. It comes out that the use of $R^{*p}(t)$ according to (36) leads to a situation where (35) holds approximately. For $\lambda = 1$ expression (35) can be solved in closed form. The resulting (lengthy) solution indicates that the use of (36) (which means a constant $R^{*p}(t) = \phi$ for $\lambda = 1$) leads to an approximately balanced system for $\psi = \frac{2}{3}$.