Online Appendix - The Optimal Cyclical Design for a Target Benefit Pension Plan

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Appendix A. Stochastic Control Problem

Appendix A.1. Problem Formulation

Before we formulate our stochastic control problem, here we first define the admissible set of the benefit payout strategies.

Definition Appendix A.1. *(Admissible strategies) The strategy* $b = \{b(t)\}_{t\in(0,T]}$ *is called an admissible strategy, if it satisfies the following conditions:*

(1) b *is* $\mathbb{F}$*-progressively measurable;*

(2) $\mathbb{E}_{t,x,l,\varepsilon}\left[\int_t^T (b(s))^2 \mathrm{d}s\right] < \infty$ *for any* $(x, l, \varepsilon) \in \mathbb{R}\times\mathbb{R}^+\times\mathcal{M}$*, where* $\mathbb{E}_{t,x,l,\varepsilon}[\cdot] = \mathbb{E}[\cdot|X(t) = x, L(t) = l, \varepsilon(t) = \varepsilon]$*;*

(3) The stochastic differential Eq. (3) has a unique strong solution $X(t)$ *for any* $(t, x, l, \varepsilon) \in (0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathcal{M}$*.*

We denote the set of all admissible strategies by Π*.*

As mentioned in the main text, the income stability is measured by the quadratic function. Therefore, given the asset value x, the inflation index l, and the market condition $\varepsilon \in \mathcal{M}$ at time t, we adopt a similar objective function as Wang *et al.* (2018) such that:

$$J(t, x, l, \varepsilon) = \mathbb{E}_{t,x,l,\varepsilon}\bigg\{ \int_t^T e^{-r_0 s}\Big(b(s)L(s)\mathcal{R}(s) - \bar{b}L(s)\mathcal{R}(s)\Big)^2 ds + \lambda_{\varepsilon(T)} e^{-r_0 T}\Big(X(T) - X^*_{\varepsilon(T)} \times L(T)\Big)^2\bigg\}.$$

Our problem is to control the actual benefit level b to minimize the income stability:

$$\inf_{b\in\Pi} J(t, x, l, \varepsilon). \tag{A.1}$$

The value function is defined as

$$V(t, x, l, \varepsilon) = \inf_{b\in\Pi} J(t, x, l, \varepsilon). \tag{A.2}$$

Appendix A.2. Solution

By dynamic programming, we derive the following Hamilton-Jacobi-Bellman equation (HJB) equation from the optimal control problem (A.1)

$$\inf_{b}\left\{\mathcal{L}^{b}V(t,x,l,\varepsilon)+e^{-r_0 t}\left(bl\mathcal{R}(t)-\bar{b}l\mathcal{R}(t)\right)^2+\sum_{j=1}^{M}q_{\varepsilon j}V(t,x,l,j)\right\}=0, \tag{A.3}$$

$$V(T,x,l,\varepsilon)=\lambda_\varepsilon e^{-r_0 T}(x-X_\varepsilon^* l)^2, \tag{A.4}$$

where

$$\begin{aligned}\mathcal{L}^{b}V(t,x,l,\varepsilon)=&-r_0 V+V_t+V_x[x(r_\varepsilon+\pi_\varepsilon(\mu_\varepsilon-r_\varepsilon))+c_\varepsilon l\mathcal{A}(t)-bl\mathcal{R}(t)-l\mathcal{D}(t)]\\&+V_l\alpha_\varepsilon l+\frac{1}{2}V_{xx}[x\pi_\varepsilon\sigma_\varepsilon]^2+\frac{1}{2}V_{ll}\eta_\varepsilon^2 l^2+V_{xl}[x\pi_\varepsilon\sigma_\varepsilon]l\eta_\varepsilon\rho_\varepsilon\end{aligned} \tag{A.5}$$

and subscript denotes partial derivatives of V with respect to state variables x and l.

Theorem Appendix A.1. *For the optimization problem (A.1), the optimal benefit strategy is given by*

$$b^*(t)=\bar{b}+\frac{A_{\varepsilon1}(t)}{e^{-r_0 t}}\left(\frac{x-\frac{-A_{\varepsilon3}(t)}{2A_{\varepsilon1}(t)}l}{l\mathcal{R}(t)}\right), \tag{A.6}$$

and the value function is

$$V(t,x,l,\varepsilon)=A_{\varepsilon1}(t)x^2+A_{\varepsilon2}(t)l^2+A_{\varepsilon3}(t)xl,$$

where $A_{\varepsilon i}(t)$, $\varepsilon=1,\cdots,M$ satisfy

$$\begin{cases}A'_{\varepsilon1}(t)+A_{\varepsilon1}(t)\cdot[2(r_\varepsilon+\pi_\varepsilon(\mu_\varepsilon-r_\varepsilon))-r_0+\pi_\varepsilon^2\sigma_\varepsilon^2+q_{\varepsilon\varepsilon}]-\dfrac{A_{\varepsilon1}^2(t)}{e^{-r_0 t}}+\sum\limits_{j\neq\varepsilon}^{M}q_{\varepsilon j}A_{j1}(t)=0,\\ A'_{\varepsilon2}(t)+A_{\varepsilon2}(t)\cdot[2\alpha_\varepsilon-r_0+\eta_\varepsilon^2+q_{\varepsilon\varepsilon}]+\sum\limits_{j\neq\varepsilon}^{M}q_{\varepsilon j}A_{j2}(t)\\ \qquad+A_{\varepsilon3}(t)\cdot[c_\varepsilon\mathcal{A}(t)-\mathcal{D}(t)-\bar{b}\mathcal{R}(t)]-\dfrac{A_{\varepsilon3}^2(t)}{4e^{-r_0 t}}=0,\\ A'_{\varepsilon3}(t)+A_{\varepsilon3}(t)\cdot[(r_\varepsilon+\pi_\varepsilon(\mu_\varepsilon-r_\varepsilon))-r_0+\alpha_\varepsilon+\pi_\varepsilon\sigma_\varepsilon\eta_\varepsilon\rho_\varepsilon+q_{\varepsilon\varepsilon}]-\dfrac{A_{\varepsilon1}(t)A_{\varepsilon3}(t)}{e^{-r_0 t}}\\ \qquad+\sum\limits_{j\neq\varepsilon}^{M}q_{\varepsilon j}A_{j3}(t)+A_{\varepsilon1}(t)\cdot[2c_\varepsilon\mathcal{A}(t)-2\mathcal{D}(t)-2\bar{b}\mathcal{R}(t)]=0\end{cases} \tag{A.7}$$

with boundary conditions $A_{\varepsilon1}(T)=\lambda_\varepsilon e^{-r_0 T}$, $A_{\varepsilon2}(T)=(X_\varepsilon^*)^2\lambda_\varepsilon e^{-r_0 T}$, and $A_{\varepsilon3}(T)=-2X_\varepsilon^*\lambda_\varepsilon e^{-r_0 T}$.

Proof. Consider the HJB equation:

$$\inf_b \Big\{ -r_0 V(t,x,l,\varepsilon) + V_t(t,x,l,\varepsilon) + V_x(t,x,l,\varepsilon)[x(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) + c_\varepsilon l\mathcal{A}(t) - bl\mathcal{R}(t) - l\mathcal{D}(t)]$$
$$+ V_l(t,x,l,\varepsilon)\alpha_\varepsilon l + \frac{1}{2}V_{xx}(t,x,l,\varepsilon)x^2\pi_\varepsilon^2\sigma_\varepsilon^2 + \frac{1}{2}V_{ll}(t,x,l,\varepsilon)\eta^2 l^2 + V_{xl}(t,x,l,\varepsilon)x\pi_\varepsilon\sigma_\varepsilon l\eta_\varepsilon\rho_\varepsilon$$
$$+ e^{-r_0 t}\left(bl\mathcal{R}(t) - \bar{b}l\mathcal{R}(t)\right)^2 + \sum_{j=1}^{M} q_{\varepsilon j}V(t,x,l,j)\Big\} = 0.$$

Suppose that

$$V(t,x,l,\varepsilon) = A_{\varepsilon 1}(t)x^2 + A_{\varepsilon 2}(t)l^2 + A_{\varepsilon 3}(t)xl$$

with boundary conditions $A_{\varepsilon 1}(T) = \lambda_\varepsilon e^{-r_0 T}$, $A_{\varepsilon 2}(T) = (X_\varepsilon^*)^2\lambda_\varepsilon e^{-r_0 T}$, and $A_{\varepsilon 3}(T) = -2X_\varepsilon^*\lambda_\varepsilon e^{-r_0 T}$. Substitute into the HJB equation and use the first order condition, leading to

$$b_0(t) = \frac{A_{\varepsilon 1}(t)}{e^{-r_0 t}\mathcal{R}(t)} \cdot \frac{x}{l} + \frac{1}{\mathcal{R}(t)}\left(\frac{A_{\varepsilon 3}(t)}{2e^{-r_0 t}} + \bar{b}\mathcal{R}(t)\right) = \bar{b} + \frac{A_{\varepsilon 1}(t)}{e^{-r_0 t}}\left(\frac{x - \frac{-A_{\varepsilon 3}(t)}{2A_{\varepsilon 1}(t)}l}{l\mathcal{R}(t)}\right).$$

By separating variables, we obtain

$$\begin{cases} A_{\varepsilon 1}'(t) + A_{\varepsilon 1}(t)\cdot[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 + q_{\varepsilon\varepsilon}] - \dfrac{A_{\varepsilon 1}^2(t)}{e^{-r_0 t}} + \sum\limits_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j1}(t) = 0, \\ A_{\varepsilon 2}'(t) + A_{\varepsilon 2}(t)\cdot[2\alpha_\varepsilon - r_0 + \eta_\varepsilon^2 + q_{\varepsilon\varepsilon}] + \sum\limits_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j2}(t) \\ \qquad + A_{\varepsilon 3}(t)\cdot[c_\varepsilon\mathcal{A}(t) - \mathcal{D}(t) - \bar{b}\mathcal{R}(t)] - \dfrac{A_{\varepsilon 3}^2(t)}{4e^{-r_0 t}} = 0, \\ A_{\varepsilon 3}'(t) + A_{\varepsilon 3}(t)\cdot[(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \alpha_\varepsilon + \pi_\varepsilon\sigma_\varepsilon\eta_\varepsilon\rho_\varepsilon + q_{\varepsilon\varepsilon}] - \dfrac{A_{\varepsilon 1}(t)A_{\varepsilon 3}(t)}{e^{-r_0 t}} \\ \qquad + \sum\limits_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j3}(t) + A_{\varepsilon 1}(t)\cdot[2c_\varepsilon\mathcal{A}(t) - 2\mathcal{D}(t) - 2\bar{b}\mathcal{R}(t)] = 0, \end{cases} \tag{A.8}$$

with the boundary conditions $A_{\varepsilon 1}(T) = \lambda_\varepsilon e^{-r_0 T}$, $A_{\varepsilon 2}(T) = (X_\varepsilon^*)^2\lambda_\varepsilon e^{-r_0 T}$, and $A_{\varepsilon 3}(T) = -2X_\varepsilon^*\lambda_\varepsilon e^{-r_0 T}$.

□

Theorem Appendix A.2. *There exists a unique solution* $(\mathbf{A_1}(t), \mathbf{A_2}(t), \mathbf{A_3}(t))$ *satisfying ODEs* (A.7) *for each* $t \in [0,T]$.

Proof. We only prove the uniqueness and existence of the solution for the first ODEs in (A.8) which is non-linear from the quadratic terms $A_{\varepsilon 1}^2(t)$. The second and third ODEs in (A.8) are both linear with respect to the variables $A_{\varepsilon 2}(t)$ and $A_{\varepsilon 3}(t)$, and the existence of the solutions can be directly obtained from Waltman (2004). We first define recursively a sequence of ODEs as

$$\dot{X}_\varepsilon^{(k)}(t) + \left[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 - \frac{2X_\varepsilon^{(k-1)}(t)}{e^{-r_0 t}}\right]\cdot X_\varepsilon^{(k)}(t)$$
$$+ \sum_{j=1}^{M} q_{\varepsilon j}X_j^{(k)}(t) + e^{-r_0 t}\cdot\left(\frac{X_\varepsilon^{(k-1)}(t)}{e^{-r_0 t}}\right)^2 = 0, \tag{A.9}$$

where $X_\varepsilon^{(0)}(t) \equiv \Psi_\varepsilon$ and Ψ_ε are positive constants for $\varepsilon = 1, 2, \ldots, M$. Obviously, for each k, the linear ODEs (A.9) has uniquely continuous solutions $X_\varepsilon^{(k)}(t)$ as

$$\begin{aligned}
X_\varepsilon^{(k)}(t) &= \exp\left\{\int_t^T 2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 + q_{\varepsilon\varepsilon} - \frac{2X_\varepsilon^{(k-1)}(s)}{e^{-r_0 s}}ds\right\} \cdot \lambda_\varepsilon e^{-r_0 T} \\
&\quad + \int_t^T \exp\left\{\int_t^s 2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 + q_{\varepsilon\varepsilon} - \frac{2X_\varepsilon^{(k-1)}(\upsilon)}{e^{-r_0\upsilon}}d\upsilon\right\} \\
&\qquad \times\Big[\sum_{j\neq\varepsilon} q_{\varepsilon j}X_j^{(k)}(s) + \frac{(X_\varepsilon^{(k-1)}(s))^2}{e^{-r_0 s}}\Big]ds \\
&= \exp\left\{[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2]\cdot[T-t]\right\} \cdot \lambda_\varepsilon e^{-r_0 T} \\
&\quad + \int_t^T \exp\left\{[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2]\cdot[s-t]\right\} \\
&\quad \times\Big[\sum_{j=1}^M q_{\varepsilon j}X_j^{(k)}(s) - 2X_\varepsilon^{(k)}(s)\cdot\frac{X_\varepsilon^{(k-1)}(s)}{e^{-r_0 s}} + \frac{(X_\varepsilon^{(k-1)}(s))^2}{e^{-r_0 s}}\Big]ds, \qquad \text{(A.10)}
\end{aligned}$$

which from the first equality gives the non-negativity of $X_\varepsilon^{(k)}(t)$ for $\varepsilon = 1, 2, \ldots, M$. We can also find from (A.9) that

$$\begin{aligned}
0 &= \dot{X}_\varepsilon^{(k)}(t) + \Big[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 - \frac{2X_\varepsilon^{(k-1)}(t)}{e^{-r_0 t}}\Big]\cdot X_\varepsilon^{(k)}(t) \\
&\qquad + \sum_{j=1}^M q_{\varepsilon j}X_j^{(k)}(t) + e^{-r_0 t}\cdot\left(\frac{X_\varepsilon^{(k-1)}(t)}{e^{-r_0 t}}\right)^2 \\
&= \dot{X}_\varepsilon^{(k)}(t) + \Big[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 - \frac{2X_\varepsilon^{(k)}(t)}{e^{-r_0 t}}\Big]\cdot X_\varepsilon^{(k)}(t) + \sum_{j=1}^M q_{\varepsilon j}X_j^{(k)}(t) \\
&\qquad + e^{-r_0 t}\cdot\left(\frac{X_\varepsilon^{(k)}(t)}{e^{-r_0 t}}\right)^2 + e^{r_0 t}\cdot(X_\varepsilon^{(k-1)}(t) - X_\varepsilon^{(k)}(t))^2.
\end{aligned}$$

Then by defining $Q_\varepsilon(t) = X_\varepsilon^{(k)}(t) - X_\varepsilon^{(k+1)}(t)$ for $\varepsilon = 1, 2, \ldots, M$,

$$\begin{aligned}
0 &= \dot{X}_\varepsilon^{(k+1)}(t) + \Big[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 - \frac{2X_\varepsilon^{(k)}(t)}{e^{-r_0 t}}\Big]\cdot X_\varepsilon^{(k+1)}(t) \\
&\qquad + \sum_{j=1}^M q_{\varepsilon j}X_j^{(k+1)}(t) + e^{-r_0 t}\cdot\left(\frac{X_\varepsilon^{(k)}(t)}{e^{-r_0 t}}\right)^2 \\
&\quad - \Big\{\dot{X}_\varepsilon^{(k)}(t) + \Big[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 - \frac{2X_\varepsilon^{(k)}(t)}{e^{-r_0 t}}\Big]\cdot X_\varepsilon^{(k)}(t) + \sum_{j=1}^M q_{\varepsilon j}X_j^{(k)}(t) \\
&\qquad + e^{-r_0 t}\cdot\left(\frac{X_\varepsilon^{(k)}(t)}{e^{-r_0 t}}\right)^2 + e^{r_0 t}\cdot(X_\varepsilon^{(k-1)}(t) - X_\varepsilon^{(k)}(t))^2\Big\} \\
&= -\dot{Q}_\varepsilon(t) - \Big[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 - \frac{2X_\varepsilon^{(k)}(t)}{e^{-r_0 t}}\Big]\cdot Q_\varepsilon(t) \\
&\qquad - \sum_{j=1}^M q_{\varepsilon j}Q_j^{(k)}(t) - e^{r_0 t}\cdot(X_\varepsilon^{(k-1)}(t) - X_\varepsilon^{(k)}(t))^2,
\end{aligned}$$

that is

$$\dot{Q}_\varepsilon(t) + \Big[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 - \frac{2X_\varepsilon^{(k)}(t)}{e^{-r_0 t}}\Big] \cdot Q_\varepsilon(t)$$
$$+ \sum_{j=1}^{M} q_{\varepsilon j} Q_j^{(k)}(t) + e^{r_0 t} \cdot (X_\varepsilon^{(k-1)}(t) - X_\varepsilon^{(k)}(t))^2 = 0, \quad Q_\varepsilon(T) = 0.$$

The above ODEs have the following solution

$$\begin{aligned} Q_\varepsilon(t) &= \int_t^T \exp\Big\{\int_t^s 2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 + q_{\varepsilon\varepsilon} - e^{r_0 \upsilon} X_\varepsilon^{(k)}(\upsilon) d\upsilon\Big\} \\ &\quad \times \Big[\sum_{j\neq\varepsilon}^{M} q_{\varepsilon j} Q_j(s) + e^{r_0 s}(X_\varepsilon^{(k)}(s) - X_\varepsilon^{(k-1)}(s))^2\Big] ds \geq 0, \end{aligned}$$

which means $X_\varepsilon^{(k)}(t) \geq X_\varepsilon^{(k+1)}(t) \geq 0$. By the monotone convergence theorem, there exists $\mathbf{A}(t) = (A_1(t), \cdots, A_M(t))$ such that $\lim_{k\to\infty} \mathbf{X}^{(\mathbf{k})}(t) = \mathbf{A}(t)$. From the bounded convergence theorem, taking the limit as $k \to \infty$ in the second equality of (A.10), it follows that

$$\begin{aligned} A_\varepsilon(t) &= \exp\Big\{[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2] \cdot [T - t]\Big\} \cdot \lambda_\varepsilon e^{-r_0 T} \\ &\quad + \int_t^T \exp\Big\{[2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2] \cdot [s - t]\Big\}\Big[\sum_{j=1}^{M} q_{\varepsilon j} A_j(s) - \frac{(A_\varepsilon(s))^2}{e^{-r_0 s}}\Big] ds, \end{aligned} \tag{A.11}$$

which is obviously a continuous solution of the first ODEs in (A.8). Next we show the uniqueness of the solution. Suppose there is another solution $\mathbf{Y}(t) = (Y_1(t), \cdots, Y_M(t))$, that is

$$\dot{Y}_\varepsilon(t) + [2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2] Y_\varepsilon(t) + \sum_{j=1}^{M} q_{\varepsilon j} Y_j(t) - e^{r_0 t} Y_\varepsilon^2(t) = 0, \quad Y_\varepsilon(T) = \lambda_\varepsilon e^{-r_0 T},$$

which is equivalent to

$$\dot{Y}_\varepsilon(t) + [2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 - 2e^{r_0 t} A_\varepsilon(t)] Y_\varepsilon(t) + e^{r_0 t} A_\varepsilon^2(t)$$
$$+ \sum_{j=1}^{M} q_{\varepsilon j} Y_j(t) - e^{r_0 t}(A_\varepsilon(t) - Y_\varepsilon(t))^2 = 0, \ Y_\varepsilon(T) = \lambda_\varepsilon e^{-r_0 T}.$$

The ODEs for $A_\varepsilon(t)$ can be written as

$$A'_\varepsilon(t) + A_\varepsilon(t) \cdot [2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 - 2e^{r_0 t} A_\varepsilon(t)]$$
$$+ e^{r_0 t} A_\varepsilon^2(t) + \sum_{j=1}^{M} q_{\varepsilon j} A_j(t) = 0, \ A_\varepsilon(T) = \lambda_\varepsilon e^{-r_0 T}.$$

Then by defining $B_\varepsilon(t) = A_\varepsilon(t) - Y_\varepsilon(t)$, it is easy to find that $B_\varepsilon(t)$ satisfies

$$\dot{B}_\varepsilon(t) + [2(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) - r_0 + \pi_\varepsilon^2\sigma_\varepsilon^2 - 2e^{r_0 t} A_\varepsilon(t)] \cdot B_\varepsilon(t) + \sum_{j=1}^{M} q_{\varepsilon j} B_j(t)$$
$$+ e^{-r_0 t}(A_\varepsilon(s) - Y_\varepsilon(s))^2 = 0, \quad B_\varepsilon(T) = 0.$$

and has the following representation

$$B_\varepsilon(t) = \int_t^T \exp\left\{\int_t^s 2(r_\varepsilon+\pi_\varepsilon(\mu_\varepsilon-r_\varepsilon))-r_0+\pi_\varepsilon^2\sigma_\varepsilon^2+q_{\varepsilon\varepsilon}-e^{r_0\upsilon}A_\varepsilon(\upsilon)d\upsilon\right\} \times\Big[\sum_{j\neq\varepsilon}^{M} q_{\varepsilon j}B_j(s)+e^{r_0 s}(A_\varepsilon(t)-Y_\varepsilon(t))^2\Big]ds \geq 0,$$

which gives $A_\varepsilon(t)\geq Y_\varepsilon(t)$. Similarly, we can also obtain $Y_\varepsilon(t)\geq A_\varepsilon(t)$ which shows the uniqueness. □

Remark 1. *Given the penalty weight λ_ε, if the equity share π_ε satisfies the following conditions:*

$$\left[\pi_\varepsilon^2\sigma_\varepsilon^2+\pi_\varepsilon\times 2(\mu_\varepsilon-r_\varepsilon)+2(r_\varepsilon-r_0)+q_{\varepsilon\varepsilon}\right]\lambda_\varepsilon-\lambda_\varepsilon^2+\sum_{i\neq\varepsilon} q_{\varepsilon i}\lambda_i=0, \qquad \forall\varepsilon=1,\cdots,M, \tag{A.12}$$

then we have $\beta_\varepsilon(t)=\beta_\varepsilon=\lambda_\varepsilon$. In addition, if the population is stationary such that $\mathcal{A}(t)=\mathcal{A}$, $\mathcal{R}(t)=\mathcal{R}$ and $\mathcal{D}(t)=\mathcal{D}$, and X_ε^* satisfies

$$\zeta_\varepsilon X_\varepsilon^*-\left[c_\varepsilon\mathcal{A}-\mathcal{D}-\bar{b}\mathcal{R}\right]+\sum_{i\neq\varepsilon}\frac{q_{\varepsilon i}\lambda_i}{\lambda_\varepsilon}X_i^*=0, \qquad \forall\varepsilon=1,\cdots,M, \tag{A.13}$$

where $\zeta_\varepsilon=(r_\varepsilon+\pi_\varepsilon(\mu_\varepsilon-r_\varepsilon))-2r_0+\alpha_\varepsilon+\pi_\varepsilon\sigma_\varepsilon\eta_\varepsilon\rho_\varepsilon+q_{\varepsilon\varepsilon}-\lambda_\varepsilon$, then $\psi_{\varepsilon(t)}\times Liability_{\varepsilon(t)}(t)=X_\varepsilon^*\times L(t)$.

Appendix B. Equity Share as Control Variable

Similar to the value function (A.2), we define the objective function as:

$$J(t,x,l,\varepsilon)=\mathbb{E}_{t,x,l,\varepsilon}\left\{\int_t^T e^{-r_0 s}\left(b(s)L(s)\mathcal{R}(s)-\bar{b}L(s)\mathcal{R}(s)\right)^2 ds+\lambda_\varepsilon e^{-r_0 T}(X(T)-X_\varepsilon^*\times L(T))^2\right\},$$
$$J(T,x,l,\varepsilon)=\lambda_\varepsilon e^{-r_0 T}(x-X_\varepsilon^*\times l)^2$$

and the value function as:

$$V(t,x,l,\varepsilon)=\inf_{b,\pi} J(t,x,l,\varepsilon). \tag{B.1}$$

The solution of the problem (B.1) can be derived using the standard HJB technique. We get the following HJB equation satisfied by the value function $V(t,x,l,\varepsilon)$:

$$\inf_{b,\pi}\left\{\mathcal{L}^{b,\pi}V(t,x,l,\varepsilon)+e^{-r_0 t}\left(bl\mathcal{R}(t)-\bar{b}l\mathcal{R}(t)\right)^2+\sum_{j=1}^{M}q_{\varepsilon j}V(t,x,l,j)\right\}=0, \tag{B.2}$$

$$V(T,x,l,\varepsilon)=\lambda_\varepsilon e^{-r_0 T}(x-X_\varepsilon^* l)^2, \tag{B.3}$$

where

$$\mathcal{L}^{b,\pi}V(t,x,l,\varepsilon) = -r_0V + V_t + V_x[x(r_\varepsilon + \pi_\varepsilon(\mu_\varepsilon - r_\varepsilon)) + c_\varepsilon l\mathcal{A}(t) - bl\mathcal{R}(t) - l\mathcal{D}(t)] + V_l\alpha_\varepsilon l + \frac{1}{2}V_{xx}[x\pi_\varepsilon\sigma_\varepsilon]^2 + \frac{1}{2}V_{ll}\eta_\varepsilon^2 l^2 + V_{xl}[x\pi_\varepsilon\sigma_\varepsilon]l\eta_\varepsilon\rho_\varepsilon, \quad \text{(B.4)}$$

and subscript denotes the partial derivatives. The solution of the optimization problem is given as:

Theorem Appendix B.1. *For the optimization problem (B.1), the optimal benefit and investment strategies are given by*

$$b^*(t) = \bar{b} + \frac{A_{\varepsilon 1}(t)}{e^{-r_0 t}}\left(\frac{x - \frac{-A_{\varepsilon 3}(t)}{2A_{\varepsilon 1}(t)}l}{l\mathcal{R}(t)}\right),$$

$$\pi^*(t) = -\frac{2(\mu_\varepsilon - r_\varepsilon)A_{\varepsilon 1}(t)x + (\mu_\varepsilon - r_\varepsilon + \eta_\varepsilon\rho_\varepsilon\sigma_\varepsilon)lA_{\varepsilon 3}(t)}{2\sigma_\varepsilon^2 A_{\varepsilon 1}(t)x},$$

and the value function is

$$V(t,x,l,\varepsilon) = A_{\varepsilon 1}(t)x^2 + A_{\varepsilon 2}(t)l^2 + A_{\varepsilon 3}(t)xl,$$

where $A_{\varepsilon i}(t)$, $\varepsilon = 1, 2, \cdots, M$ satisfy

$$\begin{cases} A'_{\varepsilon 1}(t) + A_{\varepsilon 1}(t)\cdot[2r_\varepsilon - r_0 - \frac{(\mu_\varepsilon - r_\varepsilon)^2}{\sigma_\varepsilon^2} + q_{\varepsilon\varepsilon}] - \frac{A_{\varepsilon 1}^2(t)}{e^{-r_0 t}} + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j1}(t) = 0, \\ A'_{\varepsilon 2}(t) + A_{\varepsilon 2}(t)\cdot[2\alpha_\varepsilon - r_0 + \eta_\varepsilon^2 + q_{\varepsilon\varepsilon}] + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j2}(t) \\ \quad +A_{\varepsilon 3}(t)\cdot[c_\varepsilon\mathcal{A}(t) - \mathcal{D}(t) - \bar{b}\mathcal{R}(t)] - \frac{A_{\varepsilon 3}^2(t)}{4e^{-r_0 t}} - \frac{(\mu_\varepsilon - r_\varepsilon + \eta_\varepsilon\rho_\varepsilon\sigma_\varepsilon)^2 A_{\varepsilon 3}^2(t)}{4\sigma_\varepsilon^2 A_{\varepsilon 1}(t)} = 0, \\ A'_{\varepsilon 3}(t) + A_{\varepsilon 3}(t)\cdot[r_\varepsilon - r_0 + \alpha_\varepsilon - \frac{(\mu_\varepsilon - r_\varepsilon)(\mu_\varepsilon - r_\varepsilon + \eta_\varepsilon\rho_\varepsilon\sigma_\varepsilon)}{\sigma_\varepsilon^2} + q_{\varepsilon\varepsilon}] + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j3}(t) \\ \quad -\frac{A_{\varepsilon 1}(t)A_{\varepsilon 3}(t)}{e^{-r_0 t}} + A_{\varepsilon 1}(t)\cdot[2c_\varepsilon\mathcal{A}(t) - 2\mathcal{D}(t) - 2\bar{b}\mathcal{R}(t)] = 0, \end{cases} \quad \text{(B.5)}$$

and $A_{\varepsilon 1}(T) = \lambda_\varepsilon e^{-r_0 T}$, $A_{\varepsilon 2}(T) = (X_\varepsilon^*)^2\lambda_\varepsilon e^{-r_0 T}$, $A_{\varepsilon 3}(T) = -2X_\varepsilon^*\lambda_\varepsilon e^{-r_0 T}$.

The HJB equation becomes

$$\inf_{b,\pi}\Big\{ -r_0V(t,x,l,\varepsilon) + V_t(t,x,l,\varepsilon) + V_x(t,x,l,\varepsilon)[x(r_\varepsilon + \pi(\mu_\varepsilon - r_\varepsilon)) + c_\varepsilon l\mathcal{A}(t) - bl\mathcal{R}(t) - l\mathcal{D}(t)] + V_l(t,x,l,\varepsilon)\alpha_\varepsilon l + \frac{1}{2}V_{xx}(t,x,l,\varepsilon)x^2\pi^2\sigma_\varepsilon^2 + \frac{1}{2}V_{ll}(t,x,l,\varepsilon)\eta_\varepsilon^2 l^2 + V_{xl}(t,x,l,\varepsilon)x\pi\sigma_\varepsilon l\eta_\varepsilon\rho_\varepsilon + e^{-r_0 t}\left(bl\mathcal{R}(t) - \bar{b}l\mathcal{R}(t)\right)^2 + \sum_{j=1}^{M} q_{\varepsilon j}V(t,x,l,j)\Big\} = 0$$

with the boundary condition $V(T,x,l,i) = \lambda_\varepsilon e^{-r_0 T}(x - X_\varepsilon^* l)^2$.

The first-order conditions of b and π are

$$b_0(t) = \frac{V_x(t,x,l,\varepsilon)}{2e^{-r_0 t} l \mathcal{R}(t)} + \bar{b}, \qquad \pi^*(t) = -\frac{(\mu_\varepsilon - r_\varepsilon)V_x(t,x,l,\varepsilon) + l\eta_\varepsilon\rho_\varepsilon\sigma_\varepsilon V_{xl}(t,x,l,\varepsilon)}{\sigma_\varepsilon^2 x V_{xx}(t,x,l,\varepsilon)}.$$

Suppose that

$$V(t,x,l,\varepsilon) = A_{\varepsilon1}(t)x^2 + A_{\varepsilon2}(t)l^2 + A_{\varepsilon3}(t)xl + A_{\varepsilon4}(t)x + A_{\varepsilon5}(t)l + A_{\varepsilon6}(t)$$

with the boundary conditions $A_{\varepsilon1}(T) = \lambda_\varepsilon e^{-r_0 T}$, $A_{\varepsilon2}(T) = (X_\varepsilon^*)^2 \lambda_\varepsilon e^{-r_0 T}$, and $A_{\varepsilon3}(T) = -2X_\varepsilon^*\lambda_\varepsilon e^{-r_0 T}$, $A_{\varepsilon4}(T) = A_{\varepsilon5}(T) = A_{\varepsilon6}(T) = 0$.

Substituting b_0, π^* and $V(t,x,l,\varepsilon)$ into the HJB equation and separating the variables, we have

$$\begin{cases}
A'_{\varepsilon1}(t) + A_{\varepsilon1}(t)\cdot\left[2r_\varepsilon - r_0 - \frac{(\mu_\varepsilon - r_\varepsilon)^2}{\sigma_\varepsilon^2} + q_{\varepsilon\varepsilon}\right] - \frac{A_{\varepsilon1}^2(t)}{e^{-r_0 t}} + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j1}(t) = 0, \\
A'_{\varepsilon2}(t) + A_{\varepsilon2}(t)\cdot\left[2\alpha_\varepsilon - r_0 + \eta_\varepsilon^2 + q_{\varepsilon\varepsilon}\right] + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j2}(t) \\
\qquad + A_{\varepsilon3}(t)\cdot\left[c_\varepsilon\mathcal{A}(t) - \mathcal{D}(t) - \bar{b}\mathcal{R}(t)\right] - \frac{A_{\varepsilon3}^2(t)}{4e^{-r_0 t}} - \frac{(\mu_\varepsilon - r_\varepsilon + \eta_\varepsilon\rho_\varepsilon\sigma_\varepsilon)^2 A_{\varepsilon3}^2(t)}{4\sigma_\varepsilon^2 A_{\varepsilon1}(t)} = 0, \\
A'_{\varepsilon3}(t) + A_{\varepsilon3}(t)\cdot\left[r_\varepsilon - r_0 + \alpha_\varepsilon - \frac{(\mu_\varepsilon - r_\varepsilon)(\mu_\varepsilon - r_\varepsilon + \eta_\varepsilon\rho_\varepsilon\sigma_\varepsilon)}{\sigma_\varepsilon^2} + q_{\varepsilon\varepsilon}\right] + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j3}(t) \\
\qquad - \frac{A_{\varepsilon1}(t)A_{\varepsilon3}(t)}{e^{-r_0 t}} + A_{\varepsilon1}(t)\cdot\left[2c_\varepsilon\mathcal{A}(t) - 2\mathcal{D}(t) - 2\bar{b}\mathcal{R}(t)\right] = 0, \\
A'_{\varepsilon4}(t) + A_{\varepsilon4}(t)\cdot\left[r_\varepsilon - r_0 - \frac{(\mu_\varepsilon - r_\varepsilon)^2}{\sigma_\varepsilon^2} + q_{\varepsilon\varepsilon}\right] + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j4}(t) - \frac{A_{\varepsilon1}(t)A_{\varepsilon4}(t)}{e^{-r_0 t}} = 0, \\
A'_{\varepsilon5}(t) + A_{\varepsilon5}(t)\cdot\left[\alpha_\varepsilon - r_0 + q_{\varepsilon\varepsilon}\right] + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j5}(t) - \frac{(\mu_\varepsilon - r_\varepsilon)(\mu_\varepsilon - r_\varepsilon + \eta_\varepsilon\rho_\varepsilon\sigma_\varepsilon)A_{\varepsilon3}(t)A_{\varepsilon4}(t)}{2\sigma_\varepsilon^2 A_{\varepsilon1}(t)} \\
\qquad - \frac{A_{\varepsilon3}(t)A_{\varepsilon4}(t)}{e^{-r_0 t}} + A_{\varepsilon4}(t)\cdot\left[c_\varepsilon\mathcal{A}(t) - \mathcal{D}(t) - \bar{b}\mathcal{R}(t)\right] = 0, \\
A'_{\varepsilon6}(t) + A_{\varepsilon6}(t)\cdot\left[-r_0 + q_{\varepsilon\varepsilon}\right] + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j}A_{j6}(t) - \frac{(\mu_\varepsilon - r_\varepsilon)^2 A_{\varepsilon4}^2(t)}{4\sigma_\varepsilon^2 A_{\varepsilon1}(t)} = 0.
\end{cases} \tag{B.6}$$

Considering the boundary conditions, we obtain $A_{\varepsilon4}(t) = A_{\varepsilon5}(t) = A_{\varepsilon6}(t) = 0$ and

$$b_0(t) = \frac{2A_{\varepsilon1}(t)x + A_{\varepsilon3}(t)l}{2e^{-r_0 t} l\mathcal{R}(t)} + \bar{b} = \bar{b} + \frac{A_{\varepsilon1}(t)}{e^{-r_0 t}}\left(\frac{x - \frac{-A_{\varepsilon3}(t)}{2A_{\varepsilon1}(t)}l}{l\mathcal{R}(t)}\right),$$

$$\pi^*(t) = -\frac{2(\mu_\varepsilon - r_\varepsilon)A_{\varepsilon1}(t)x + (\mu_\varepsilon - r_\varepsilon + \eta_\varepsilon\rho_\varepsilon\sigma_\varepsilon)lA_{\varepsilon3}(t)}{2\sigma_\varepsilon^2 A_{\varepsilon1}(t)x},$$

where

$$
\begin{cases}
A'_{\varepsilon 1}(t) + A_{\varepsilon 1}(t) \cdot [2r_\varepsilon - r_0 - \frac{(\mu_\varepsilon - r_\varepsilon)^2}{\sigma_\varepsilon^2} + q_{\varepsilon\varepsilon}] - \frac{A_{\varepsilon 1}^2(t)}{e^{-r_0 t}} + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j} A_{j1}(t) = 0, \\
A'_{\varepsilon 2}(t) + A_{\varepsilon 2}(t) \cdot [2\alpha_\varepsilon - r_0 + \eta_\varepsilon^2 + q_{\varepsilon\varepsilon}] + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j} A_{j2}(t) \\
\quad +A_{\varepsilon 3}(t) \cdot [c_\varepsilon \mathcal{A}(t) - \mathcal{D}(t) - \bar{b}\mathcal{R}(t)] - \frac{A_{\varepsilon 3}^2(t)}{4e^{-r_0 t}} - \frac{(\mu_\varepsilon - r_\varepsilon + \eta_\varepsilon \rho_\varepsilon \sigma_\varepsilon)^2 A_{\varepsilon 3}^2(t)}{4\sigma_\varepsilon^2 A_{\varepsilon 1}(t)} = 0, \\
A'_{\varepsilon 3}(t) + A_{\varepsilon 3}(t) \cdot [r_\varepsilon - r_0 + \alpha_\varepsilon - \frac{(\mu_\varepsilon - r_\varepsilon)(\mu_\varepsilon - r_\varepsilon + \eta_\varepsilon \rho_\varepsilon \sigma_\varepsilon)}{\sigma_\varepsilon^2} + q_{\varepsilon\varepsilon}] + \sum_{j\neq\varepsilon}^{M} q_{\varepsilon j} A_{j3}(t) \\
\quad -\frac{A_{\varepsilon 1}(t) A_{\varepsilon 3}(t)}{e^{-r_0 t}} + A_{\varepsilon 1}(t) \cdot [2c_\varepsilon \mathcal{A}(t) - 2\mathcal{D}(t) - 2\bar{b}\mathcal{R}(t)] = 0
\end{cases}
\tag{B.7}
$$

and $A_{\varepsilon 1}(T) = \lambda_\varepsilon e^{-r_0 T}$, $A_{\varepsilon 2}(T) = (X_\varepsilon^*)^2 \lambda_\varepsilon e^{-r_0 T}$, $A_{\varepsilon 3}(T) = -2X_\varepsilon^* \lambda_\varepsilon e^{-r_0 T}$.

References and Notes

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