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To the dynamics of the two-body problem with variable masses in the presence of reactive forces

Abstract. The dynamics of celestial bodies with variable masses, especially the non-stationary stage of the gravitating system is little studied. We have considered the gravitational system consisting of two spherical celestial bodies with variable masses in the relative coordinate system. We studied the general case where the masses of bodies change non-isotropically at different rates, in the presence of reactive forces. The problem was investigated by methods of perturbation theory based on aperiodic motion along a quasi-conic section. We used perturbed motion swith variables $a, e, i, \pi, \Omega, \lambda$, which are analogs of Keplerian elements. The equations of perturbed motion of the osculating variables are obtained. Averaging over the mean longitude we obtained the evolution equations of the two-body problem with variable masses in the presence of reactive forces. The evolution equations have an exact analytic integral $a^3e^4 = const$. The derived evolution equations of the two-body problem with variable masses in the presence of reactive forces will be used to study binary systems with variable masses.

Since there is no general solution for arbitrary mass change laws of the Gulden-Meshchersky

1. Introduction

Real celestial bodies are nonstationary, their masses, sizes, shapes and structures changes in the evolution process. The mass variability of celestial bodies, especially in the nonstationary stage of gravitational systems, significantly affects the further dynamical evolution of this system generally [1, 2]. For the first time this problem appeared in astronomy due to the hypothesis of C. Dufour [3] about the possible role of meteor accretion in the parameters evolution of the Earth-Moon system. Later I.V. Meshchersky found an exact solution to this problem in special cases of mass change laws [4]. So the twobody problem with isotropically varying masses, the Gulden-Meshchersky problem, a general solution of this problem, for arbitrary changes of masses, is unknown.

The Gulden-Meshchersky problem in astronomy is especially actual when the body masses change according to the Eddington-Jeans law

$$\frac{dm}{dt} = \alpha m^n, \qquad \alpha = const, \qquad n = const. \tag{1.1}$$

B.E. Gelfgat found an integrable case of the Gulden-Meshchersky problem for the case n=1 and n = 3/2 [5]. An overview of integrable cases for concrete mass change laws and interesting results obtained by the autonomization method for the Gulden-Meshchersky problem are given in the paper by L.M. Berkovich [6].

The method of autonomization for the Binet equation describing the Gulden-Meshchersky problem was applied by Bekov to determine the exact special solutions. In his work [7] new exact special solutions of the Gulden-Meshchersky problem in parametric form were found.

Many aspects of the Gulden-Meshchersky problem with an extensive bibliography are given in A. Deprit [8]. A classification of the two-body problem with variable masses is given in E.P. Razbitnaya [9]. Some aspects of the two-body problem with variable masses are described in E.P. Polyakhova [10]. Note also the review work by J. Hadjidimetriou [11] where, in particular, a new interpretation of the Gulden-Meshchersky problem was obtained simultaneously and independently by T.B. Omarov [12] and J. Hadjidimetriou [13].

T.B. Omarov's work [12] is interesting because here for the first time the aperiodic motion along a conic section is proposed as an initial unperturbed motion for studying the dynamics of gravitating systems with variable masses and the corresponding perturbation theory in the Newton equation form is constructed [14, 15]. Relatively recently this perturbation theory has been used in D. Veras, J.D. Hadjidemetriou, S.A. Tout [16].

problem, it seems promising to study the dynamics of gravitating systems with variable masses by methods of perturbation theories. The development of this idea was carried out in papers [17-19] in which the perturbation theory based on aperiodic motion along the quasiconical section in different systems of oscillating elements and in different forms, including the canonical perturbation theory were constructed. The perturbation theory based on aperiodic motion along a conic section developed by T.B. Omarov [12] is contained as a special case of the perturbation theory based on aperiodic motion along a quasi-conic section [18]. In [20-24], various forms of the equation of perturbed motion were used to study the dynamics of two, three, and many bodies with variable masses in various cases. In [19], new forms of the equation of perturbed motion based on aperiodic motion along a quasi-conic section, in the form of Newton's equation were obtained, which were used in this work. In the current paper, we investigated the problem of two spherical bodies with non-isotropically

varying masses in the presence of reactive forces, using the methods of perturbation theory [19]. The perturbation equations of secular perturbations were obtained. One exact first integral of the system of differential equations of secular perturbations was found.

2. Model description. Differential equations of motion.

2.1. Physical statement of the problem.

Consider a gravitating system consisting of two celestial bodies with variable masses. We assume that the bodies are spherical with spherical mass distributions. Let, B_1 - the more massive body (hereinafter the central body), B_2 - the less massive body (hereinafter the satellite). Accordingly, we denote the masses, which are functions of time

$$m_1 = m_1(t)$$
, $m_2 = m_2(t)$, $m_1 > m_2$. (2.1.1)
Let's assume that the body mass decreases due to separating particles and increases due to joining (sticking) particles. In this case, in general, the relative velocity of separating particles from the body differs from the relative velocity of joining (sticking) particles to the body. Let us consider the general case when the masses of bodies do not change isotropically at different rates

$$\frac{\dot{m}_1}{m_1} \neq \frac{\dot{m}_2}{m_2} \ . \tag{2.1.2}$$

A mathematical description of the dynamic evolution of two mutually gravitating spherical bodies with non-isotropically varying variable masses, in the presence of reactive forces under the above assumptions, is needed

 $m_1 = m_1(t) = m_1(t_0) - (|m_{1-}|) + (m_{1+}) = m_1(t_1) - \int (|\dot{m}_{1-}|) dt + \int (\dot{m}_{1+}) dt$ (2.1.3)

$$m_2 = m_2(t) = m_2(t_0) - (|m_{2-}|) + (m_{2+}) = m_2(t_0) - \int_0^t (|\dot{m}_{2-}|) dt + \int_0^t (\dot{m}_{2+}) dt,$$
 (2.1.4)

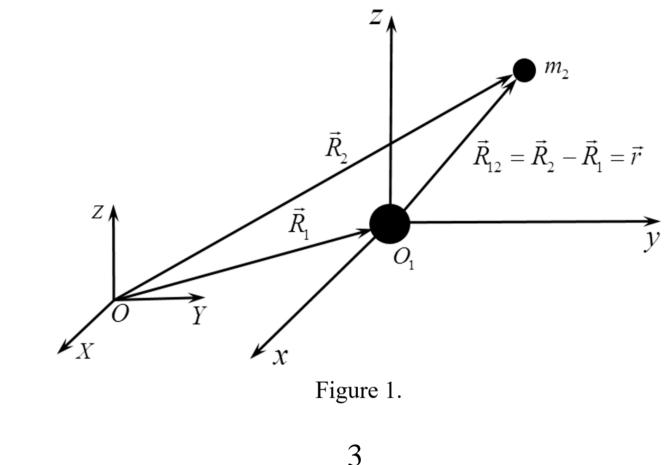
$$-(|m_{1-}|) = -\int_{1}^{t} (|\dot{m}_{1-}|) dt < 0, \qquad -(|m_{2-}|) = -\int_{1}^{t} (|\dot{m}_{2-}|) dt < 0$$
(2.

- the masses of particles separating from the body B_1 and B_2 during t_1

$$(m_{1+}) = + \int_{t}^{t} (\dot{m}_{1+}) dt > 0,$$
 $(m_{2+}) = + \int_{t}^{t} (\dot{m}_{2+}) dt > 0$ (2.1.6)

- the masses of the particles joining the body B_1 and B_2 during t,

 $\vec{V}_{1-} = \vec{u}_{1-} - \dot{\vec{R}}_{1}, \qquad \vec{V}_{2-} = \vec{u}_{2-} - \dot{\vec{R}}_{2}, \qquad \qquad \vec{V}_{1+} = \vec{u}_{1+} - \dot{\vec{R}}_{1}, \qquad \vec{V}_{2+} = \vec{u}_{2+} - \dot{\vec{R}}_{2}$ (2.1.7)relative velocities of separating and attaching (sticking) particles.



2.2 Motion equations of bodies in a relative coordinate system

Let us introduce a relative coordinate system O_1xyz with the origin in the center of the central body B_1 , the axes of which are parallel to the corresponding axes of the absolute coordinate system. Let us denote (Fig. 1)

$$\vec{r} = \vec{R}_2 - \vec{R}_1 = \vec{R}_{12} \,. \tag{2.2.1}$$
 Then, in relative coordinates, the equations of motion of the two-body problem have the form

$$\ddot{\vec{r}} + fm\frac{\vec{r}}{r^3} = \vec{F} , \qquad r = \sqrt{x^2 + y^2 + z^2} , \qquad (2.2.3)$$

where the total mass of the bodies and the equilibrium of the reactive forces (per unit mass) are denoted as follows

$$m = m(t) = m_1(t) + m_2(t),$$

$$\vec{F} = \vec{F}(t) = \vec{F}(F_x, F_y, F_z) = \vec{F}_+ + \vec{F}_-,$$
(2.2.4)

$$\vec{F}_{+} = \frac{\dot{m}_{2+}}{m_2} \vec{V}_{2+} - \frac{\dot{m}_{1+}}{m_1} \vec{V}_{1+}, \qquad \qquad \vec{F}_{-} = \frac{\left| \dot{m}_{1-} \right|}{m_1} \vec{V}_{1-} - \frac{\left| \dot{m}_{2-} \right|}{m_2} \vec{V}_{2-}. \qquad (2.2.5)$$

In the orbital coordinate system the reactive force (2.2.4) can be written in the form

$$\vec{F} = \vec{F}(t) = \vec{F}_r + \vec{F}_\tau + \vec{F}_n = \vec{F}(F_r, F_\tau, F_n), \qquad (2.2.6)$$

where the \vec{F}_r radial, \vec{F}_τ transversal, and \vec{F}_n normal components of the reactive forces are denoted by radial, transversal, and normal components. Usually the observation determines the relative velocities of separating and attaching (sticking) particles (2.1.7) with respect to each particular body. Therefore, it is convenient to express the reactive forces in an orbital coordinate system. Let us assume that quantities (2.2.3)-(2.2.6) are known and given functions of time.

Relations (2.2.2), (2.2.3)-(2.2.6) describe the two-body problem with non-isotropically varying masses in the presence of reactive forces in the relative coordinate system. These equations (2.2.2)-(2.2.6) are the basic initial equations in this paper whose solutions will be sought by perturbation theory methods.

3. Equations of perturbed motion in osculating elements.

Using earlier obtained equations of aperiodic motion along a quasi-conical section [18], in the presence of a perturbing force [19], let us consider the two-body problem with variable masses in the presence of reactive forces (2.2.2) as equations of perturbed motion along a quasi-conical section in osculating elements [19]. Let us rewrite equation (2.2.2) as

$$\ddot{\vec{r}} + fm\frac{\vec{r}}{r^3} - mA\vec{r} = \vec{W}, \qquad \qquad \vec{W} = -mA\vec{r} + \vec{F} = \vec{W}(W_r, W_\tau, W_\eta). \tag{3.1}$$

$$W_r^* = W_r \cdot \frac{\sqrt{p}}{\sqrt{\mu_0}} \frac{m_0}{m}, \qquad W_\tau^* = W_\tau \cdot \frac{\sqrt{p}}{\sqrt{\mu_0}} \frac{m_0}{m}, \qquad W_n^* = W_n \cdot \frac{\sqrt{p}}{\sqrt{\mu_0}} \frac{m_0}{m}, \qquad p = a(1 - e^2)$$
 (3.2)

$$A = d^{2}(1/m)/dt^{2}, W_{r} = F_{r}(t) - mAr, W_{\tau} = F_{\tau}(t), W_{n} = F_{n}(t), (3.3)$$

In the case when the perturbing force is zero
$$W_r^* = 0 , \qquad W_\tau^* = 0 , \qquad W_n^* = 0 \qquad (3.4)$$

$$a, e, i, \pi, \Omega,$$
 (3.5)

are constants. Here a is analog of the semi-major axis, p is analog of the orbital parameter, e is analog of eccentricity, i is analog of orbital inclination to the plane, $\lambda = M + \pi$ is analog of the mean longitude in the orbit, M is analog of the mean anomaly, π analog of pericenter longitude, and Ω analog of the longitude of the ascending node. The average longitude in the orbit is an increasing function of time and is defined by the following

$$\dot{\lambda} = n \left(\frac{m(t)}{m(t_0)} \right)^2 > 0, \qquad n = \frac{\sqrt{\mu_0}}{a^{3/2}} = const, \qquad (3.6)$$

where n is the analog of the mean orbital motion.

then

formula

Accordingly, the equations of perturbed motion in the Newton equations form [19] have the form

$$\dot{a} = Q \left\{ \frac{2a^2 e \sin \theta}{p} \left(F_r(t) - mAr \right) + \frac{2a^2}{\rho} F_\tau(t) \right\}, \tag{3.4}$$

$$\dot{e} = Q \left\{ \sin \theta \left(F_r(t) - mAr \right) + \left(\cos \theta + \frac{e + \cos \theta}{1 + e \cos \theta} \right) F_\tau(t) \right\}, \tag{3.5}$$

$$\frac{di}{dt} = Q \frac{\cos u}{1 + e \cos \theta} F_n(t), \tag{3}$$

$$\dot{\pi} = Q \left\{ -\frac{\cos \theta}{e} \left(F_r(t) - mAr \right) + \frac{\sin \theta}{e} \left(1 + \frac{mr}{m_0 p} \right) F_\tau(t) + \frac{mr}{m_0 p} \sin u \cdot \operatorname{tg} \frac{i}{2} F_n(t) \right\}, \quad (3)$$

$$\dot{\Omega} = Q \frac{\sin u}{\left(1 + e \cos \theta \right) \sin i} F_n(t), \quad (3)$$

$$\dot{\lambda} = n \left(\frac{m}{m_0}\right)^2 - Q \left\{ 2 \frac{mr}{m_0 p} \sqrt{1 - e^2} \left(F_r(t) - mAr\right) - \frac{mr}{m_0 p} \sin u \cdot \operatorname{tg} \frac{i}{2} F_n(t) \right\} +$$
(3.

$$+\frac{e}{1+\sqrt{1-e^2}}\left[\left(1+\frac{mr}{m_0p}\right)\sin\theta F_{\tau}(t)-\cos\theta\left(F_{\tau}(t)-mAr\right)\right]Q,$$

where $Q = \frac{\sqrt{p}}{\sqrt{\mu_0}} \frac{m_0}{m}$.

4. System of differential equations of secular perturbations in oscillating elements. To obtain the equations of motion of the secular perturbation, the equation system (3.4)-(3.9) is averaged over the mean longitude along the orbit. Let us consider equation (3.4), analogous to the semi-major axis, one of the elements which determines the size of the orbit, and calculate the secular perturbations

$$\dot{a}^{\text{sec}} = \frac{1}{2\pi} \int_{0}^{2\pi} \left[\left\{ \left[\sin \theta \right] \frac{2a^{2}e}{p} F_{r}(t) - \left[\frac{\sin \theta}{1 + e \cos \theta} \right] 2ea^{2}m_{0}A + \left[\frac{1}{\rho} \right] 2a^{2}F_{\tau}(t) \right\} \frac{\sqrt{p}}{\sqrt{\mu_{0}}} \frac{m_{0}}{m} \right] d\lambda$$

Let's use the well-known formulas of transition from θ to λ , also θ to E [18], [25]

$$d\lambda = \frac{\left(1 - e^2\right)^{3/2} d\theta}{\left(1 + e\cos\theta\right)^2}, \qquad d\theta = \frac{\sqrt{1 - e^2}}{1 - e\cos E} dE,$$

$$\sin\theta = \frac{\sqrt{1 - e^2}\sin E}{1 - e\cos E}, \qquad \cos\theta = \frac{\cos E - e}{1 - e\cos E}, \qquad \frac{1}{1 + e\cos\theta} = \frac{1 - e\cos E}{1 - e^2}.$$
The result is
$$\dot{a}^{\text{sec}} = \frac{2a^{3/2}m_0}{m\sqrt{\mu_0}} \sqrt{1 - e^2} F_{\tau}(t). \tag{4.1}$$

Similarly, performing all calculations, we obtain the secular perturbations for the other osculating elements. As a result, we have the following system of differential equations describing the secular perturbations

$$\dot{e}^{\text{sec}} = -\frac{3}{2} \frac{e\sqrt{a(1-e^2)}}{\sqrt{\mu_0}} \frac{m_0}{m} F_{\tau}(t), \qquad (4.2)$$

$$\frac{di^{\text{sec}}}{dt} = -\frac{3}{2} \frac{e\sqrt{a}\cos\left(\pi - \Omega\right)}{\sqrt{\mu_0} \sqrt{1 - e^2}} \frac{m_0}{m} F_n(t), \tag{4.3}$$

$$\dot{\pi}^{\text{sec}} = \frac{m_0}{m} \frac{\sqrt{(t-t)^2}}{\sqrt{\mu_0}} \left\{ F_r(t) - \frac{3}{2} m_0 a A - \frac{3c}{2(1-e^2)} \sin(\pi - \Omega) \cdot \text{tg} \frac{t}{2} F_n(t) \right\}, \tag{4.4}$$

$$\dot{\Omega}^{\text{sec}} = -\frac{3}{2} \frac{e\sqrt{a}}{\sqrt{1-e^2}} \frac{m_0}{\sqrt{\mu_0}} \frac{\sin(\pi - \Omega)}{m} F_n(t), \tag{4.5}$$

$$\dot{\lambda}^{\text{sec}} = n \left(\frac{m}{m_0}\right)^2 - \frac{m_0}{m} \frac{\sqrt{a}}{\sqrt{\mu_0}} \left\{ F_r(t) \left(2 + e^2\right) - \left(2 + 3e^2\right) m_0 A a \right\} - \frac{3}{2} \frac{m_0}{m} \frac{e\sqrt{a} \sin\left(\pi - \Omega\right)}{\sqrt{\mu_0} \left(1 - e^2\right)} \operatorname{tg} \frac{i}{2} F_n(t) + \frac{m_0}{m} \frac{\sqrt{a(1 - e^2)}}{1 + \sqrt{1 - e^2}} \frac{e^2}{\sqrt{\mu_0}} \left(F_r(t) - \frac{3}{2} m_0 A a \right).$$
(4.6)

The obtained secular equations (4.1)-(4.6) can be used in the study of the dynamic evolution of the two-body problem with variable masses in the presence of reactive forces. They are convenient for describing the dynamics of wide double systems with variable masses.

5. The first integral of the differential equation system of secular perturbations Consider together the equation of the analog of the semi-major axis (4.2) and the analog of the eccentricity (4.3), we obtain the following expression

$$\frac{a}{\dot{e}^{\text{sec}}} = -\frac{4}{3}\frac{a}{e}, \qquad (5.1)$$

$$a^{3}e^{4} = const = a^{3}e^{4}, \qquad e_{0} = e(t_{0}), \qquad a_{0} = a(t_{0}). \qquad (5.2)$$

$$a^3 e^4 = const = a_0^3 e_0^4, \qquad e_0 = e(t_0), \qquad a_0 = a(t_0),$$
 (5.2)

$$\left(\frac{a}{a_0}\right)^3 = \left(\frac{e_0}{e}\right)^4. \tag{5.3}$$

The obtained integral (5.2) has a simple analytical form and can be used in the analysis of dynamic evolution of binary systems with variable masses. This integral (5.2) can be used to simplify the system of secular differential equations (4.1)-(4.6).

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6. Conclusion

We have considered a gravitational system consisting of two spherical celestial bodies with variable masses in a relative coordinate system. We have studied the general case where the masses of the bodies change non-isotropically at different rates in the presence of reactive forces.

The problem was investigated by perturbation theory methods based on aperiodic motion along a quasi-conic section. We used perturbed motion equations in the Newton equations form with variables a, e, i, π , Ω , λ , which are analogs of Keplerian elements. The equations of perturbed motion of the osculating variables are obtained.

Averaging over the mean longitude, we obtained the evolution equations of the two-body problem with variable masses in the presence of reactive forces. The evolution equations have the exact analytic integral a^3e^4 =const.

The derived evolution equations can be successfully used in the study of the dynamic evolution of

the two-body problem with variable masses in the presence of reactive forces. They are convenient for describing the dynamics of wide binary systems with variable masses. In the case of close binary systems, it is also necessary to consider tidal forces. These and other

aspects of the two-body problem with variable masses in the presence of reactive forces will be studied in the following papers.

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