Online Appendix - How do empirical estimators of popular risk measures impact pro-cyclicality?

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Note that parts of Appendices B, C and D are taken from [2] and included here for self-containedness. We explicitly point this out throughout this online appendix where applicable.

Appendix B Additional material corresponding to Section 4

B.1 Explicit formulas corresponding to examples in Section 4

Here we present the explicit formulas corresponding to the plots in Section 4. More precisely, we consider the asymptotic correlation between either the sample variance or sample MAD with one of the three risk measure estimators $\widehat{\text{VaR}}_{n,t}(p)$, $\widehat{\text{ES}}_{n,t}(p)$ and $\hat{e}_{n,t}(p)$.

In Table 1, we present the expressions for an underlying Gaussian distribution and then in Table 2 for a Student distribution with ν degrees of freedom.

We introduce the standard notation $\Phi, \phi, \Phi^{-1}(p)$ for the cdf, pdf and quantile of order p of the standard normal distribution. Further let us denote by $Y \sim t(0, \frac{\nu-2}{\nu}, \nu)$ a rv Y which follows the Student-t distribution with ν degrees of freedom, mean 0 and variance 1. In contrast to this, denote by $\tilde{Y} \sim t(0,1,\nu)$ a rv $\tilde{Y}$ which follows the standard Student-t distribution with ν degrees of freedom (i.e. mean 0 but variance $\nu/(\nu-2)$. We will usually give expressions for the Student distribution in terms of $\tilde{Y}$ as this is the standard form for the Student distribution.

Before commenting the two tables, let us briefly say a few words on how to compute these expressions and where to find those computations (as, for conciseness, we do not present the calculations here).

For the correlations including the sample VaR, i.e. with the sample variance or the sample MAD, they correspond simply to the asymptotic correlations of the sample quantile with the sample variance or the sample MAD, respectively and were computed in [1], Appendix B.1; note that the corresponding expressions for the VaR with the sample MAD can be also found in equations (B.5), (B.6) in [2]).

The same remarks hold for the expectile estimator, as it is the sample quantile at level $\kappa^{-1}(p)$ with $\kappa(p)$ defined in (7)(of the main manuscript), which simplifies for location-scale distributions, as follows:

$$\kappa(p) = \frac{p q_Y(p) - \int_{-\infty}^{q_Y(p)} y dF_Y(y)}{-2\int_{-\infty}^{q_Y(p)} y dF_Y(y) - (1-2p) q_Y(p)}.$$

Table 1: Asymptotic correlations between the log-ratios of each, three risk measure estimators, and the two measures of dispersion estimator in the case of a standard Gaussian distribution

Correlation $\lim_{n\to\infty} \mathrm{Cor}(\hat{m}(X,n,r,t),...)$	Sample Variance	Sample MAD
...with $\log\left\lvert\frac{\widehat{\mathrm{VaR}}_{n,t+1y}(p)}{\widehat{\mathrm{VaR}}_{n,t}(p)}\right\rvert$	$\frac{-1}{\sqrt{2}}\frac{\phi(\Phi^{-1}(p))\left\lvert\Phi^{-1}(p)\right\rvert}{\sqrt{2p(1-p)}}$ (1)	$\frac{-1}{\sqrt{2}}\frac{\left\lvert\phi(\Phi^{-1}(p))-(1-p)\sqrt{2/\pi}\right\rvert}{\sqrt{p(1-p)}\sqrt{1-2/\pi}}$ (2)
...with $\log\left\lvert\frac{\widehat{\mathrm{ES}}_{n,t+1y}(p)}{\widehat{\mathrm{ES}}_{n,t}(p)}\right\rvert$	$\frac{-1}{\sqrt{2}}\frac{\left\lvert\int_p^1\Phi^{-1}(u)du\right\rvert}{2\sqrt{\int_p^1\int_v^1\frac{v(1-u)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}dudv}}$ (3)	$\frac{-1}{\sqrt{2}}\frac{\left\lvert 1-p-\int_p^1\frac{1-u}{\phi(\Phi^{-1}(u))\sqrt{2/\pi}}du\right\rvert}{2\sqrt{\left(\frac{1}{2}-\frac{1}{\pi}\right)\int_p^1\int_v^1\frac{v(1-u)}{\phi(\Phi^{-1}(v))\phi(\Phi^{-1}(u))}dudv}}$ (4)
...with $\log\left\lvert\frac{\hat{e}_{n,t+1y}(p)}{\hat{e}_{n,t}(p)}\right\rvert$	$\frac{-1}{\sqrt{2}}\frac{\phi(\Phi^{-1}(\kappa^{-1}(p)))\left\lvert\Phi^{-1}(\kappa^{-1}(p))\right\rvert}{\sqrt{2\kappa^{-1}(p)(1-\kappa^{-1}(p))}}$ (5)	$\frac{-1}{\sqrt{2}}\frac{\left\lvert\phi(\Phi^{-1}(\kappa^{-1}(p)))-(1-\kappa^{-1}(p))\sqrt{2/\pi}\right\rvert}{\sqrt{\kappa^{-1}(p)(1-\kappa^{-1}(p))}\sqrt{1-2/\pi}}$ (6)

Table 2: Asymptotic correlations between the log-ratios of each, three risk measure estimators, and the two measures of dispersion estimator in the case of a Student distribution with ν degrees of freedom

Correlation $\lim_{n\to\infty} \mathrm{Cor}(\hat{m}(X,n,r,t),...)$	Sample Variance	Sample MAD
...with $\log\left\lvert\frac{\widehat{\mathrm{VaR}}_{n,t+1y}(p)}{\widehat{\mathrm{VaR}}_{n,t}(p)}\right\rvert$	$\frac{-1}{\sqrt{2}}\frac{f_{\tilde{Y}}(q_{\tilde{Y}}(p))\left\lvert q_{\tilde{Y}}(p)\right\rvert\left(1+\frac{q^2_{\tilde{Y}}(p)}{\nu}\right)}{\sqrt{\frac{\nu-1}{\nu-4}\,2\,p(1-p)}}$ (7)	$\frac{-1}{\sqrt{2}}\frac{\left\lvert\frac{\sqrt{\nu(\nu-2)}}{\nu-1}f_{\tilde{Y}}(q_{\tilde{Y}}(p))\left(1+\frac{q^2_{\tilde{Y}}(p)}{\nu}\right)-(1-p)\sqrt{\frac{\nu-2}{\pi}}\frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\nu/2)}\right\rvert}{\sqrt{p(1-p)}\sqrt{1-\frac{\nu-2}{\pi}\frac{\Gamma^2(\frac{\nu-1}{2})}{\Gamma^2(\nu/2)}}}$ (8)
... with $\log\left\lvert\frac{\widehat{\mathrm{ES}}_{n,t+1y}(p)}{\widehat{\mathrm{ES}}_{n,t}(p)}\right\rvert$	$\frac{-1}{\sqrt{2}}\frac{\left\lvert\int_p^1 q_{\tilde{Y}}(u)\left(1+\frac{q^2_{\tilde{Y}}(u)}{\nu}\right)du\right\rvert}{2\sqrt{\frac{\nu-1}{\nu-4}\int_p^1\int_v^1\frac{v(1-u)}{f_{\tilde{Y}}(q_{\tilde{Y}}(v))f_{\tilde{Y}}(q_{\tilde{Y}}(u))}dudv}}$ (9)	$\frac{-1}{\sqrt{2}}\frac{\left\lvert\int_p^1\sqrt{\nu-2}\left(\frac{\sqrt{\nu}}{\nu-1}\left(1+\frac{q^2_{\tilde{Y}}(u)}{\nu}\right)-\frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})}\frac{(1-u)}{\sqrt{\pi}\,f_{\tilde{Y}}(q_{\tilde{Y}}(u))}\right)\right\rvert}{\sqrt{2\int_p^1\int_v^1\frac{v(1-u)}{f_{\tilde{Y}}(q_{\tilde{Y}}(v))f_{\tilde{Y}}(q_{\tilde{Y}}(u))}dudv}\sqrt{1-\frac{\nu-2}{\pi}\frac{\Gamma((\nu-1)/2)^2}{\Gamma(\nu/2)^2}}}$ (10)
...with $\log\left\lvert\frac{\hat{e}_{n,t+1y}(p)}{\hat{e}_{n,t}(p)}\right\rvert$	$\frac{-1}{\sqrt{2}}\frac{f_{\tilde{Y}}(q_{\tilde{Y}}(\kappa^{-1}(p)))\left\lvert q_{\tilde{Y}}(\kappa^{-1}(p))\right\rvert\left(1+\frac{q^2_{\tilde{Y}}(\kappa^{-1}(p))}{\nu}\right)}{\sqrt{\frac{\nu-1}{\nu-4}\,2\,\kappa^{-1}(p)(1-\kappa^{-1}(p))}}$ (11)	$\frac{-1}{\sqrt{2}}\frac{\left\lvert\frac{\sqrt{\nu(\nu-2)}}{\nu-1}f_{\tilde{Y}}(q_{\tilde{Y}}(\kappa^{-1}(p)))\left(1+\frac{q^2_{\tilde{Y}}(\kappa^{-1}(p))}{\nu}\right)-(1-\kappa^{-1}(p))\sqrt{\frac{\nu-2}{\pi}}\frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\nu/2)}\right\rvert}{\sqrt{\kappa^{-1}(p)(1-\kappa^{-1}(p))}\sqrt{1-\frac{\nu-2}{\pi}\frac{\Gamma^2(\frac{\nu-1}{2})}{\Gamma^2(\nu/2)}}}$ (12)

It gives, in the case of the Gaussian distribution (recall the first truncated moment, e.g. from (B.36) in [1]),

$$\kappa_{norm}(p) = \frac{p\Phi^{-1}(p) + \phi(\Phi^{-1}(p))}{2\phi(\Phi^{-1}(p)) - (1-2p)\Phi^{-1}(p)}.$$

For the Student distribution (assumed to be with mean 0, and recalling the first truncated moment computed in (B.38) in [1]), we obtain

$$\kappa_{stud}(p) = \frac{pq_{\tilde{Y}}(p) + \frac{\nu}{\nu-1}f_{\tilde{Y}}(q_{\tilde{Y}}(p))(1+q^2_{\tilde{Y}}(p)/\nu)}{2\frac{\nu}{\nu-1}f_{\tilde{Y}}(q_{\tilde{Y}}(p))(1+q^2_{\tilde{Y}}(p)/\nu) - (1-2p)q_{\tilde{Y}}(p)}.$$

For the ES estimator $\widehat{\mathrm{ES}}_{n,t}(p)$, note that it is asymptotically equivalent to $\frac{1}{1-p}\int_p^1 q_n(u)du$. Using the Rieman sum (partitioning the interval $[0,1]$ into n intervals of length $1/n$, and chosing $x_i^\star = i/n, i = 1,...,n$ to always be the right end-point of each interval), we can write:

$$\frac{1}{1-p}\int_p^1 q_n(u)du = \frac{1}{1-p}\int_p^1 X_{(\lceil nu \rceil)}du = \frac{1}{1-p}\int_0^1 X_{(\lceil nu \rceil)}\,\mathbb{1}_{(u\geq p)}du = \lim_{\Delta x\to 0}\frac{1}{1-p}\sum_{i=1}^n X_{(\lceil nx_i^\star \rceil)}\,\mathbb{1}_{(x_i^\star\geq p)}\Delta x$$
$$= \lim_{1/n\to 0}\frac{1}{1-p}\sum_{i=1}^n X_{(\lceil ni/n \rceil)}\,\mathbb{1}_{(i/n\geq p)}1/n = \lim_{n\to\infty}\frac{1}{n-np}\sum_{i=1}^n X_{(i)}\,\mathbb{1}_{(i\geq np)}$$
$$= \lim_{n\to\infty}\frac{1}{n-np}\sum_{i=1}^n X_{(i)}\,\mathbb{1}_{(i\geq \lceil np \rceil)} = \lim_{n\to\infty}\frac{1}{n-np}\sum_{i=1}^n X_{(i)}\,\mathbb{1}_{(X_i\geq X_{(\lceil np \rceil)})}.$$

Because of the more compact integral representation of the asymptotic correlation, we keep in the tables the correlation with $\widehat{\mathrm{ES}}_{n,t}(p)$ as in [1] The explicit solutions of this integral representation are very lengthy and can be found in the Appendix D.1 of [1].

The expressions look more complex in the Student case than in the Gaussian one. Still, we recover the Gaussian expressions for $\nu\to\infty$. For this, recall the definition the Gamma function $\Gamma(.)$,

$$\Gamma(x) := \begin{cases}(x-1)! & \text{for integers } x>0,\\ \sqrt{\pi}\frac{(2x-2)!!}{2^{\frac{2x-1}{2}}} & \text{for half-integers } x\text{, i.e. odd integer-multiples of } \frac{1}{2},\end{cases}$$

where ! and !! denote the factorial and double-factorial function, respectively. Further, one might need to recall the asymptotic property of the Gamma function, namely $\lim_{n\to\infty}\frac{\Gamma(n+\alpha)}{\Gamma(n)n^\alpha} = 1$, which we need to use here with $n = \nu\alpha$ and $\alpha = 1/2$.

B.2 Additional plots I: Comparing Student to Gaussian case

As mentioned, we also want to study the Student correlation with degrees of freedom other than $\nu = 5$, and compare it to the Gaussian case. Thus, we look in Figure 1 at the correlation for each pair of risk and dispersion measure separately, but showing the cases $\nu = 3, 4, 5, 10, 40$ and ∞ (Gaussian case) in the same plot.

First, we look at the case with the VaR (Figure 1, 1st row). We see in both plots, for the sample standard deviation (std; left plot) and the sample MAD (right plot), that heavier distributions have a lower degree of pro-cyclicality, except for values far in the tail. This turning point is further in the tail with the MAD than with the sample std, as already observed in Section 4.2. Also, the difference in the degree of pro-cyclicality between the different heavy-tailed distributions is more distinctive with the std in comparison to the MAD. Similar observations hold true for the case of the expectile in the 3rd row of the figure.

Let us turn to the ES. While the behaviour in the case of the std is in line with the observations made for the VaR and expectile, the same does not hold true when considering the sample MAD as realised volatility estimator. Indeed, in Figure 1, we can observe on the right plot in the 2nd row that the degree of pro-cycliclality increases with the heaviness of the tail.

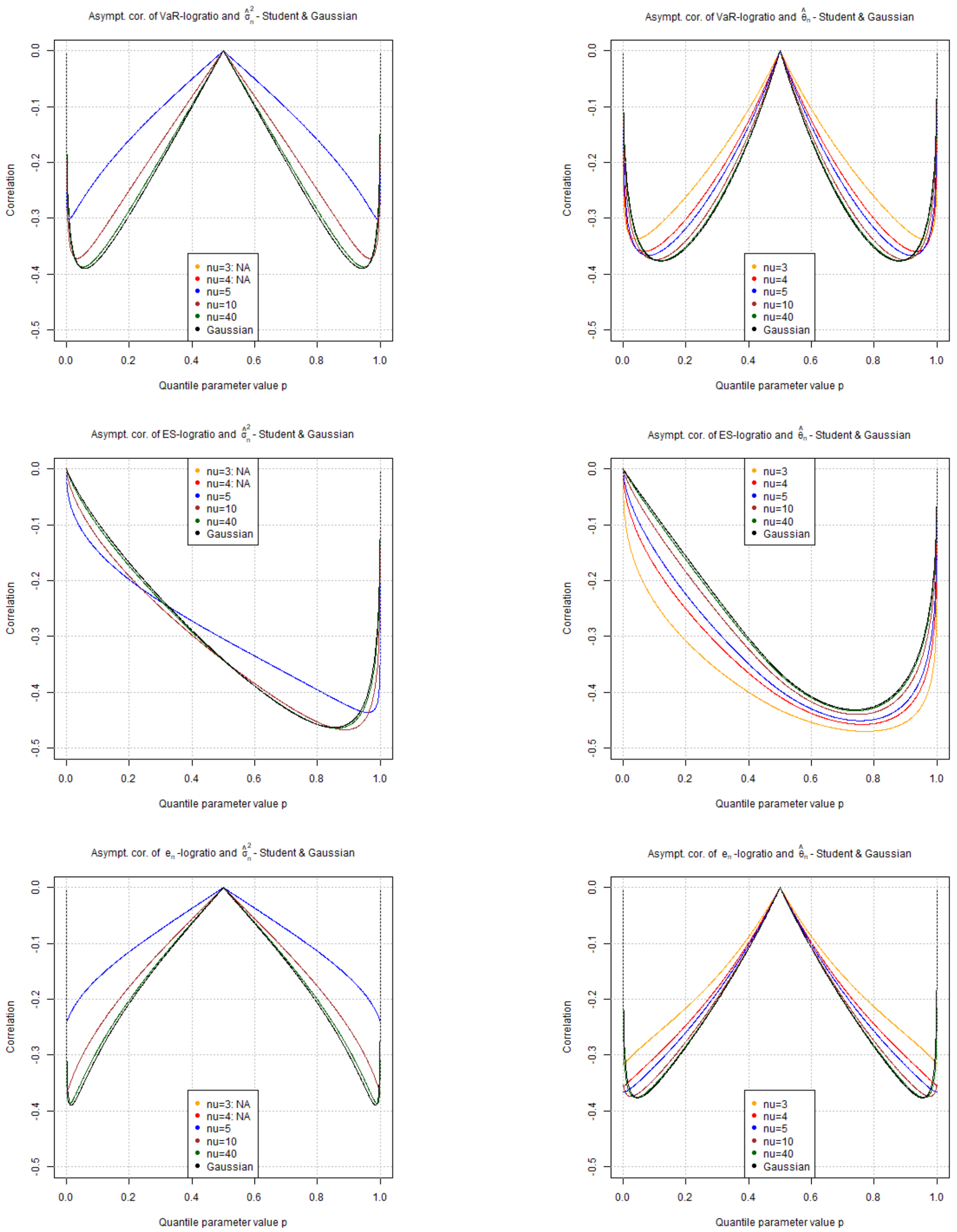

Figure 1: Pro-cyclicality as defined in (13)(of the main manuscript), comparing the case of a Student distribution with $\nu = 3, 4, 5, 10$ and 40 degrees of freedom as well as the Gaussian distribution. From top to bottom: sample quantile, ES, expectile; and from left to right: sample variance, sample MAD.

Appendix C Additional material corresponding to Section 5

The empirical results, formulas and plots presented in this section are only recalled for self-containedness of this paper. They originate from [2] for the data considered, the GARCH parameters and all VaR-related results, and from [1] otherwise.

C.1 Empirical results

Data considered - See [2, Appendix E].

Table 3: For each index we provide the country, country code, the Bloomberg index (the name under which one can find the index on Bloomberg) and a short description of the index.

Country	Country Code	Bloomberg Index	Description
Australia	AUS	AS30	The All Ordinaries Index (AOI) is made up of the 500 largest companies listed on the Australian Securities Exchange (ASX)
Canada	CAN	SPTSX	The Canada S&P/TSX Composite Index comprises around 70% of market capitalization for Toronto Stock Exchange listed companies
France	FRA	SBF250	The CAC All-Tradable Index is a French stock market index representing all sectors of the French economy (traded on Euronext Paris)
Germany	DEU	CDAX	CDAX is a composite index of all stocks traded on the Frankfurt Stock Exchange in the Prime Standard and General Standard market segment
Great Britain	GBR	ASX	The FTSE All-Share Index (ASX) comprises more than 600 (out of 2000+) companies traded on the London Stock Exchange
Italy	ITA	ITSMBCIG	This index is called Comit Globale R it includes all the shares listed on the MTA and is calculated using reference prices
Japan	JPN	TPX	The Tokyo Price Index (TPX) is a capitalization-weighted price index of all First Section stocks (1600+ companies) of the Tokyo Stock Exchange
Netherlands	NLD	AEX	The AEX index is a stock market index composed of a maximum of 25 of the most actively traded shares listed on Euronext Amsterdam
Singapore	SGP	STI	FTSE Straits Times Index (STI) tracks the performance of the top 30 companies listed on the Singapore Exchange
Sweden	SWE	OMXAFGX	Sweden OMX Affärsvärldens General Index is a stock market index designed to measure the performance of the Stockholm Stock Exchange
United States of America	USA	SPX	S&P 500 is an American stock market index based on 500 large companies listed on the NYSE or NASDAQ (around 80% market capitalization)

GARCH parameters - See [2, Appendix D].

Table 4: Fitting results of the GARCH$(1, 1)$ models for the 11 indices considered

	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA
ω $[10^{-6}]$	3.45	1.11	2.54	2.18	2.42	5.29	2.21	2.78	3.29	1.81	1.70
α $[10^{-1}]$	1.58	1.03	0.93	0.87	0.80	1.42	1.07	1.23	1.05	1.06	0.99
β $[10^{-1}]$	8.06	8.85	8.90	8.99	9.06	8.27	8.81	8.60	8.75	8.77	8.88
$\alpha + \beta$	*0.96*	*0.99*	*0.98*	*0.99*	*0.99*	*0.97*	*0.99*	*0.98*	*0.98*	*0.98*	*0.99*

Measured pro-cyclicality for the VaR -

Table 5: Pearson correlation between the logarithm of look-forward ratios (using the VaR) and the volatility, for each index, over the whole historical sample, and for four thresholds (95%, 97.5%, 99% and 99.5%). In the last column, the average over all indices ± the standard deviation over the 11 displayed values.

α	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG (± sd)**
95%	-0.51	-0.46	-0.50	-0.49	-0.52	-0.58	-0.52	-0.50	-0.51	-0.50	-0.50	**-0.51** ± 0.03
95%	-0.55	-0.47	-0.49	-0.44	-0.52	-0.58	-0.53	-0.50	-0.50	-0.53	-0.52	**-0.51** ± 0.04
99%	-0.57	-0.46	-0.51	-0.47	-0.52	-0.52	-0.55	-0.54	-0.48	-0.50	-0.54	**-0.51** ± 0.04
99.5%	-0.52	-0.46	-0.47	-0.45	-0.51	-0.48	-0.54	-0.50	-0.45	-0.49	-0.53	**-0.49** ± 0.03

Measured pro-cyclicality for the ES -

Table 6: Pearson correlation between the logarithm of look-forward ratios (using the ES) and the volatility, for each index, over the whole historical sample, and for four thresholds (95%, 97.5%, 99% and 99.5%). In the last column, the average over all indices ± the standard deviation over the 11 displayed values.

α	AUS	CAN	FRA	DEU	ITA	JPN	NLD	SGP	SWE	GBR	USA	**AVG (± sd)**
95%	-0.58	-0.48	-0.50	-0.48	-0.55	-0.55	-0.55	-0.54	-0.51	-0.53	-0.54	**-0.53** ± 0.03
97.5%	-0.56	-0.47	-0.49	-0.46	-0.55	-0.52	-0.55	-0.53	-0.49	-0.52	-0.54	**-0.52** ± 0.03
99%	-0.53	-0.45	-0.46	-0.41	-0.53	-0.44	-0.53	-0.51	-0.44	-0.51	-0.51	**-0.48** ± 0.04
99.5%	-0.50	-0.44	-0.43	-0.37	-0.50	-0.41	-0.51	-0.48	-0.41	-0.50	-0.49	**-0.46** ± 0.05

C.2 Confidence intervals for sample correlation

Here, we refer directly to [2, Section B.2], and only recall the confidence interval for the sample Pearson linear correlation coefficient between the log-ratio and the realised volatility, for a level α and an underlying iid sample, namely:

$$\left[\rho(\alpha) + \tanh\left(\tanh^{-1}(\rho(\alpha)) - \frac{\Phi^{-1}\left((1+\alpha)/2\right)}{\sqrt{l-3}}\right)\ ;\ \rho(\alpha) + \tanh\left(\tanh^{-1}(\rho(\alpha)) + \frac{\Phi^{-1}\left((1+\alpha)/2\right)}{\sqrt{l-3}}\right)\right]$$

where $\rho(\alpha)$ denotes the asymptotic correlation between the log-ratio and the realised volatility, and l is the sample size on which the correlation is computed.

C.3 Additional plots II: Pro-cyclicality of empirical residuals

Here we show all the 22 plots for the pro-cyclicality of the empirical residuals for each of the 11 indices when either considering the VaR (Figure 2) or the ES (Figure 3). For completeness, we can reiterate here the observations of Section C on the totality of the 11 indices: In the case of the VaR (Figure 2) in 38 out of 44 cases (86%), the sample correlation of the residuals falls in the 95% confidence interval of the sample correlation of iid rv's. But in none of the cases, the sample correlation of the real data falls in these confidence intervals. In the case of the ES (Figure 3) this holds in 37 out of 44 cases (84%) - and in exactly one (out of 44) cases, the sample correlation of the real data falls in these confidence intervals.

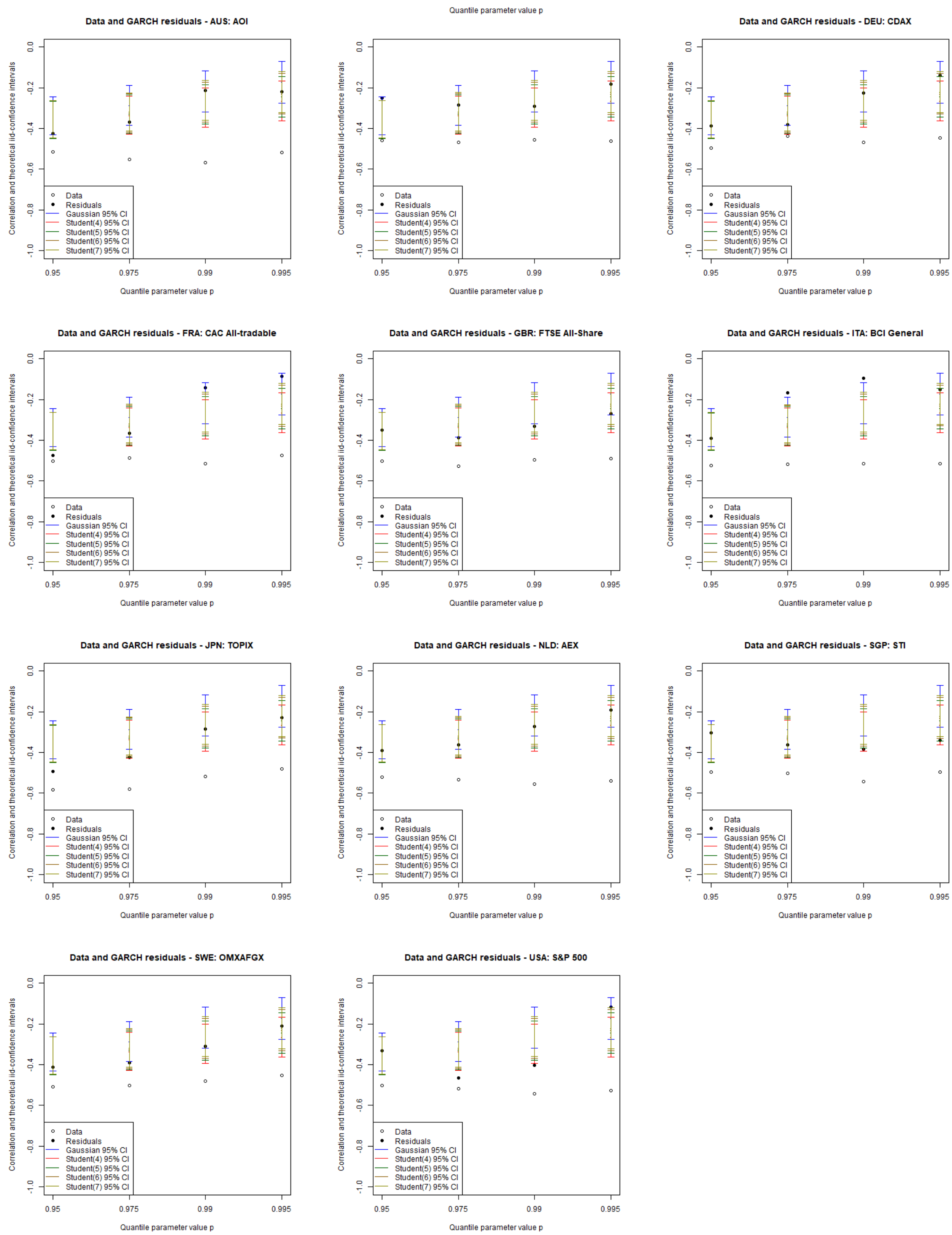

Figure 2: Comparison of pro-cyclicality for the real data (blank circle) with the pro-cyclicality for the GARCH$(1,1)$-residuals (filled circle) for each index separately, when considering the VaR as risk measure (estimator $\widehat{\mathrm{VaR}}_n(p)$). Each plot contains the correlation for the four different quantile values p. For each of them, corresponding theoretical confidence intervals (for the sample correlation) assuming a specific underlying distribution (Gaussian or Student with different degrees of freedom) are plotted.

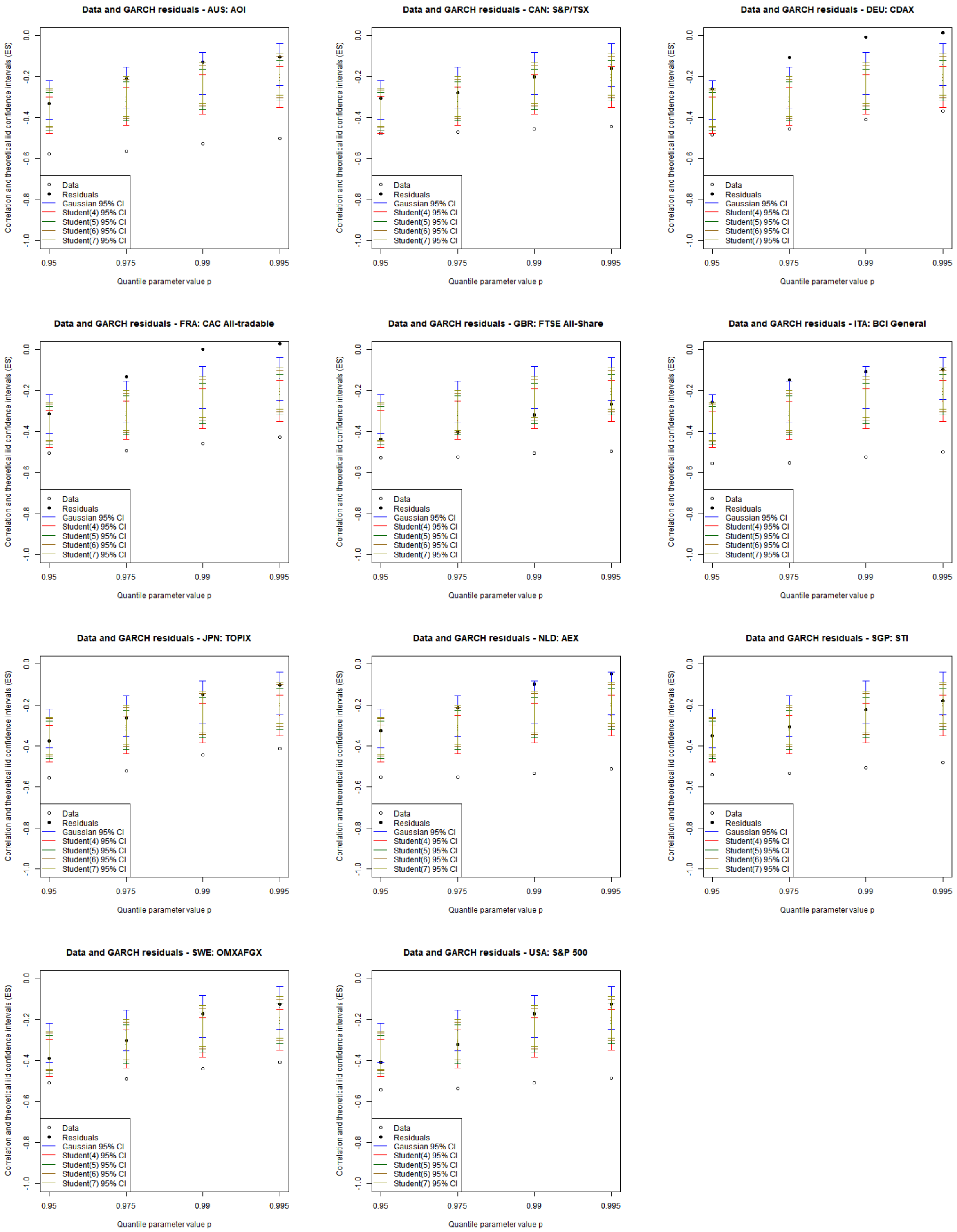

Figure 3: Comparison of pro-cyclicality for the real data (blank circle) with the pro-cyclicality for the GARCH(1, 1)-residuals (filled circle) for each index separately, when considering the ES as risk measure (estimator $\widetilde{\mathrm{ES}}_{n,\infty}$). Each plot contains the correlation for the four different quantile values p. For each of them, corresponding theoretical confidence intervals (for the sample correlation) assuming a specific underlying distribution (Gaussian or Student with different degrees of freedom) are plotted.

Appendix D Additional Tables of the Simulation Study in Section 4.1

D.1 With the VaR

As mentioned in Section 4.1, a far less general simulation study for the pro-cyclicality of the VaR exists in [2] that is not directly comparable to the setting used here.

Table 7: **Gaussian case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical VaR with the sample std, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Gaussian distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

VaR and std	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-37 (-81,27)	-35 (-81,26)	-38 (-83,29)	-37 (-83,28)	-39
25	-38 (-68,-3)	-38 (-69,-6)	-39 (-69,-2)	-39 (-67,-3)	-39
50	-39 (-60,-15)	-39 (-60,-15)	-39 (-62,-14)	-39 (-61,-16)	-39
100	-39 (-54,-22)	-39 (-54,-22)	-39 (-54,-22)	-39 (-54,-22)	-39
250	-39 (-48,-29)	-39 (-49,-28)	-39 (-49,-29)	-39 (-48,-29)	-39
500	-39 (-46,-32)	-39 (-46,-32)	-39 (-46,-31)	-39 (-46,-32)	-39
p=0.975					
10	-35 (-81,32)	-34 (-80,31)	-34 (-80,32)	-34 (-82,37)	-37
25	-37 (-68,1)	-37 (-67,-1)	-36 (-65,-2)	-37 (-65,1)	-37
50	-37 (-59,-13)	-37 (-59,-13)	-37 (-59,-13)	-37 (-58,-13)	-37
100	-38 (-53,-22)	-37 (-53,-20)	-36 (-52,-19)	-37 (-53,-21)	-37
250	-38 (-47,-27)	-37 (-47,-26)	-37 (-47,-26)	-37 (-46,-26)	-37
500	-38 (-45,-30)	-37 (-44,-30)	-37 (-44,-29)	-37 (-44,-30)	-37
p=0.99					
10	-31 (-80,35)	-30 (-79,32)	-31 (-80,30)	-28 (-80,39)	-31
25	-33 (-64,4)	-31 (-63,6)	-32 (-65,5)	-31 (-65,8)	-31
50	-33 (-56,-7)	-31 (-54,-5)	-32 (-55,-7)	-31 (-55,-6)	-31
100	-34 (-50,-16)	-32 (-49,-13)	-32 (-48,-15)	-31 (-48,-13)	-31
250	-34 (-44,-23)	-32 (-41,-21)	-32 (-43,-22)	-32 (-42,-21)	-31
500	-34 (-41,-26)	-32 (-40,-25)	-32 (-39,-24)	-32 (-39,-24)	-31
p=0.99855					
10	-26 (-74,43)	-23 (-77,45)	-17 (-74,47)	-18 (-72,52)	-19
25	-27 (-60,14)	-24 (-57,14)	-18 (-55,20)	-19 (-55,21)	-19
50	-28 (-51,-2)	-24 (-49,3)	-19 (-45,9)	-19 (-45,10)	-19
100	-28 (-45,-11)	-23 (-41,-5)	-19 (-37,0)	-20 (-38,0)	-19
250	-28 (-39,-17)	-23 (-34,-13)	-19 (-31,-7)	-20 (-31,-8)	-19
500	-28 (-37,-21)	-24 (-31,-16)	-19 (-27,-11)	-20 (-28,-11)	-19

Table 8: **Gaussian case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical VaR with the sample MAD, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Gaussian distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

VaR and MAD	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-33 (-79,33)	-30 (-80,35)	-33 (-83,30)	-32 (-80,37)	-34
25	-33 (-65,5)	-33 (-65,0)	-34 (-67,7)	-34 (-64,4)	-34
50	-34 (-56,-8)	-34 (-57,-10)	-34 (-57,-8)	-34 (-56,-9)	-34
100	-35 (-50,-16)	-34 (-50,-17)	-34 (-50,-17)	-34 (-50,-17)	-34
250	-34 (-44,-24)	-34 (-45,-23)	-34 (-45,-23)	-34 (-44,-23)	-34
500	-35 (-42,-27)	-34 (-42,-27)	-34 (-41,-26)	-34 (-42,-27)	-34
p=0.975					
10	-27 (-76,41)	-26 (-79,38)	-26 (-77,41)	-27 (-77,42)	-29
25	-29 (-63,10)	-29 (-62,8)	-28 (-63,9)	-29 (-60,10)	-29
50	-30 (-53,-4)	-29 (-56,-4)	-29 (-52,-4)	-29 (-52,-4)	-29
100	-30 (-47,-13)	-30 (-47,-10)	-29 (-45,-11)	-29 (-46,-11)	-29
250	-30 (-41,-20)	-30 (-40,-18)	-29 (-40,-17)	-29 (-39,-18)	-29
500	-30 (-37,-22)	-29 (-37,-22)	-29 (-37,-21)	-29 (-37,-22)	-29
p=0.99					
10	-23 (-77,43)	-21 (-76,43)	-22 (-74,41)	-20 (-75,50)	-22
25	-24 (-59,16)	-22 (-58,16)	-23 (-58,18)	-22 (-58,17)	-22
50	-24 (-48,4)	-22 (-47,5)	-23 (-50,4)	-22 (-46,5)	-22
100	-25 (-42,-7)	-23 (-41,-3)	-23 (-41,-4)	-22 (-39,-3)	-22
250	-25 (-36,-13)	-23 (-33,-11)	-23 (-35,-12)	-23 (-33,-11)	-22
500	-25 (-32,-17)	-23 (-30,-15)	-23 (-31,-14)	-23 (-30,-14)	-22
p=0.99855					
10	-18 (-72,51)	-14 (-72,50)	-10 (-72,55)	-11 (-71,54)	-11
25	-18 (-53,24)	-15 (-51,25)	-11 (-49,30)	-11 (-50,29)	-11
50	-18 (-44,9)	-15 (-42,12)	-11 (-38,16)	-12 (-38,18)	-11
100	-19 (-38,1)	-15 (-33,4)	-11 (-30,9)	-12 (-32,9)	-11
250	-19 (-31,-6)	-15 (-26,-3)	-11 (-23,1)	-12 (-24,0)	-11
500	-19 (-27,-11)	-15 (-23,-6)	-11 (-20,-3)	-12 (-21,-3)	-11

Table 9: **Student(5) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical VaR with the sample std, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(5) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

VaR and std	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-29 (-82,36)	-29 (-80,36)	-28 (-79,38)	-28 (-78,41)	-27
25	-30 (-61,9)	-31 (-65,10)	-31 (-64,11)	-29 (-62,10)	-27
50	-30 (-53,-4)	-31 (-54,-2)	-30 (-52,-1)	-29 (-53,-2)	-27
100	-31 (-48,-12)	-31 (-48,-10)	-29 (-45,-10)	-29 (-46,-9)	-27
250	-31 (-42,-18)	-30 (-41,-18)	-29 (-40,-16)	-29 (-40,-17)	-27
500	-31 (-39,-22)	-30 (-38,-22)	-29 (-38,-20)	-29 (-37,-20)	-27
p=0.975					
10	-34 (-81,34)	-32 (-81,35)	-30 (-80,34)	-30 (-80,37)	-29
25	-34 (-65,5)	-33 (-64,9)	-32 (-65,8)	-31 (-63,7)	-29
50	-34 (-58,-7)	-33 (-56,-4)	-32 (-54,-6)	-31 (-54,-6)	-29
100	-34 (-51,-14)	-33 (-50,-14)	-32 (-49,-13)	-31 (-48,-14)	-29
250	-34 (-45,-21)	-33 (-43,-20)	-32 (-43,-20)	-31 (-42,-19)	-29
500	-34 (-42,-25)	-32 (-40,-24)	-32 (-40,-23)	-31 (-39,-22)	-29
p=0.99					
10	-37 (-82,32)	-35 (-82,31)	-32 (-81,41)	-33 (-81,34)	-30
25	-37 (-66,3)	-35 (-67,4)	-34 (-66,7)	-33 (-64,8)	-30
50	-37 (-59,-11)	-35 (-58,-7)	-34 (-57,-7)	-33 (-57,-7)	-30
100	-37 (-53,-17)	-35 (-52,-15)	-33 (-50,-14)	-33 (-49,-13)	-30
250	-36 (-47,-24)	-34 (-45,-23)	-33 (-44,-21)	-33 (-43,-20)	-30
500	-36 (-45,-27)	-34 (-42,-26)	-33 (-41,-24)	-33 (-41,-23)	-30
p=0.99855					
10	-39 (-85,29)	-35 (-85,35)	-33 (-82,38)	-30 (-82,34)	-29
25	-40 (-72,-2)	-37 (-70,0)	-36 (-69,6)	-32 (-65,6)	-29
50	-41 (-63,-13)	-38 (-62,-11)	-36 (-60,-9)	-33 (-57,-5)	-29
100	-41 (-57,-23)	-38 (-55,-18)	-36 (-54,-17)	-32 (-49,-12)	-29
250	-40 (-51,-27)	-38 (-50,-26)	-36 (-49,-24)	-32 (-43,-20)	-29
500	-41 (-49,-31)	-38 (-47,-29)	-37 (-45,-27)	-32 (-41,-24)	-29

Table 10: **Student(5) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical VaR with the sample MAD, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(5) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

VaR and MAD	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-33 (-80,36)	-33 (-81,36)	-34 (-81,36)	-35 (-80,33)	-36
25	-35 (-65,0)	-35 (-66,4)	-36 (-68,1)	-36 (-68,4)	-36
50	-35 (-57,-11)	-35 (-57,-8)	-36 (-58,-11)	-36 (-58,-11)	-36
100	-36 (-52,-19)	-36 (-52,-17)	-36 (-51,-20)	-36 (-52,-17)	-36
250	-36 (-46,-27)	-36 (-46,-25)	-36 (-46,-25)	-36 (-46,-25)	-36
500	-36 (-44,-29)	-36 (-43,-29)	-36 (-43,-29)	-36 (-43,-29)	-36
p=0.975					
10	-33 (-80,33)	-31 (-80,37)	-32 (-78,33)	-32 (-82,39)	-33
25	-34 (-62,4)	-33 (-65,8)	-33 (-66,4)	-33 (-66,7)	-33
50	-34 (-57,-10)	-33 (-56,-9)	-33 (-55,-8)	-34 (-56,-8)	-33
100	-34 (-51,-15)	-34 (-49,-16)	-33 (-49,-15)	-34 (-50,-17)	-33
250	-34 (-45,-23)	-34 (-44,-23)	-33 (-43,-23)	-33 (-44,-23)	-33
500	-34 (-42,-27)	-34 (-41,-26)	-33 (-41,-26)	-33 (-41,-25)	-33
p=0.99					
10	-29 (-79,34)	-28 (-79,38)	-28 (-77,40)	-28 (-78,46)	-28
25	-31 (-62,8)	-29 (-62,8)	-29 (-60,8)	-29 (-61,9)	-28
50	-31 (-54,-3)	-28 (-52,-3)	-29 (-52,-3)	-29 (-53,-4)	-28
100	-31 (-47,-12)	-29 (-45,-10)	-29 (-46,-11)	-29 (-45,-10)	-28
250	-31 (-41,-19)	-29 (-40,-18)	-29 (-40,-19)	-29 (-40,-18)	-28
500	-31 (-38,-23)	-29 (-37,-21)	-29 (-37,-22)	-29 (-36,-21)	-28
p=0.99855					
10	-26 (-76,42)	-20 (-75,47)	-17 (-71,50)	-17 (-75,47)	-18
25	-26 (-61,12)	-21 (-57,18)	-19 (-54,22)	-18 (-54,18)	-18
50	-26 (-52,0)	-21 (-47,4)	-19 (-44,7)	-19 (-45,8)	-18
100	-27 (-44,-9)	-22 (-40,-3)	-19 (-37,0)	-19 (-38,1)	-18
250	-27 (-38,-15)	-22 (-33,-10)	-19 (-31,-7)	-19 (-31,-7)	-18
500	-27 (-35,-19)	-22 (-31,-14)	-19 (-27,-10)	-19 (-28,-11)	-18

Table 11: **Student(3) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical VaR with the sample std, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(3) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

VaR and std	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-22 (-75,45)	-19 (-73,54)	-15 (-73,54)	-14 (-74,54)	NaN
25	-22 (-57,19)	-18 (-56,23)	-16 (-52,26)	-14 (-51,29)	NaN
50	-20 (-47,10)	-18 (-46,10)	-15 (-43,13)	-13 (-41,18)	NaN
100	-20 (-39,3)	-17 (-37,3)	-14 (-34,6)	-12 (-31,10)	NaN
250	-19 (-32,-4)	-16 (-30,0)	-14 (-27,0)	-11 (-24,2)	NaN
500	-18 (-28,-7)	-15 (-25,-5)	-13 (-23,-2)	-11 (-21,-1)	NaN
p=0.975					
10	-27 (-78,39)	-22 (-76,44)	-21 (-78,50)	-18 (-75,49)	NaN
25	-27 (-61,14)	-21 (-58,20)	-20 (-58,22)	-18 (-56,25)	NaN
50	-25 (-51,4)	-21 (-47,11)	-19 (-45,11)	-17 (-42,13)	NaN
100	-24 (-43,0)	-21 (-41,1)	-18 (-37,5)	-16 (-35,6)	NaN
250	-23 (-36,-7)	-19 (-33,-4)	-17 (-31,-2)	-14 (-28,1)	NaN
500	-22 (-32,-9)	-19 (-29,-8)	-16 (-27,-5)	-13 (-24,-2)	NaN
p=0.99					
10	-31 (-83,37)	-30 (-80,38)	-25 (-79,46)	-22 (-78,46)	NaN
25	-32 (-67,11)	-28 (-63,15)	-23 (-60,20)	-22 (-59,21)	NaN
50	-31 (-56,3)	-27 (-54,5)	-22 (-49,9)	-21 (-46,9)	NaN
100	-30 (-50,-5)	-26 (-46,-2)	-22 (-42,1)	-19 (-39,3)	NaN
250	-29 (-43,-12)	-25 (-39,-9)	-20 (-35,-5)	-18 (-32,-2)	NaN
500	-28 (-38,-14)	-24 (-34,-11)	-19 (-30,-6)	-17 (-27,-5)	NaN
p=0.99855					
10	-40 (-86,34)	-41 (-87,31)	-39 (-87,33)	-33 (-83,38)	NaN
25	-42 (-74,5)	-42 (-75,2)	-41 (-74,5)	-31 (-68,12)	NaN
50	-42 (-66,-4)	-42 (-66,-8)	-41 (-66,-7)	-30 (-56,1)	NaN
100	-42 (-60,-15)	-41 (-60,-14)	-41 (-60,-16)	-29 (-51,-4)	NaN
250	-41 (-54,-21)	-40 (-53,-21)	-40 (-53,-19)	-28 (-42,-9)	NaN
500	-40 (-51,-24)	-40 (-50,-22)	-39 (-50,-21)	-26 (-38,-11)	NaN

Table 12: **Student(3) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical VaR with the sample MAD, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(3) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '$(n \to \infty)$' are provided as benchmark in the last column.

VaR and MAD	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value $(n \to \infty)$
p=0.95					
10	-33 (-80,33)	-31 (-80,38)	-31 (-79,37)	-31 (-81,39)	-34
25	-34 (-64,5)	-33 (-65,6)	-33 (-64,6)	-33 (-65,7)	-34
50	-34 (-58,-7)	-34 (-56,-8)	-33 (-57,-8)	-33 (-56,-6)	-34
100	-34 (-49,-14)	-34 (-50,-16)	-33 (-50,-13)	-33 (-49,-15)	-34
250	-34 (-44,-21)	-33 (-45,-21)	-33 (-44,-22)	-34 (-44,-22)	-34
500	-34 (-41,-24)	-34 (-41,-25)	-33 (-40,-25)	-34 (-41,-26)	-34
p=0.975					
10	-34 (-82,29)	-31 (-80,35)	-32 (-82,40)	-32 (-79,33)	-33
25	-35 (-66,2)	-33 (-65,4)	-33 (-66,5)	-34 (-63,8)	-33
50	-34 (-57,-6)	-33 (-55,-8)	-34 (-57,-7)	-34 (-56,-8)	-33
100	-34 (-50,-15)	-34 (-50,-15)	-34 (-49,-15)	-34 (-49,-16)	-33
250	-34 (-44,-21)	-34 (-44,-21)	-33 (-44,-21)	-34 (-43,-22)	-33
500	-34 (-42,-25)	-34 (-41,-25)	-33 (-41,-25)	-33 (-41,-26)	-33
p=0.99					
10	-32 (-82,31)	-31 (-81,37)	-31 (-80,36)	-31 (-80,39)	-31
25	-33 (-64,7)	-32 (-64,8)	-31 (-64,9)	-33 (-65,5)	-31
50	-33 (-56,-4)	-32 (-55,-7)	-32 (-55,-7)	-32 (-55,-7)	-31
100	-33 (-50,-14)	-33 (-50,-15)	-32 (-47,-13)	-32 (-48,-12)	-31
250	-34 (-45,-21)	-32 (-43,-21)	-32 (-42,-20)	-32 (-42,-20)	-31
500	-33 (-41,-24)	-32 (-40,-24)	-32 (-40,-23)	-32 (-39,-23)	-31
p=0.99855					
10	-31 (-81,40)	-29 (-81,41)	-25 (-82,46)	-26 (-80,46)	-25
25	-33 (-65,7)	-31 (-67,9)	-27 (-63,15)	-26 (-60,13)	-25
50	-34 (-57,-7)	-31 (-57,-2)	-28 (-53,0)	-27 (-51,0)	-25
100	-34 (-51,-15)	-32 (-50,-11)	-28 (-47,-10)	-27 (-44,-8)	-25
250	-34 (-47,-22)	-31 (-44,-18)	-29 (-41,-17)	-27 (-38,-15)	-25
500	-34 (-43,-25)	-32 (-40,-23)	-29 (-37,-21)	-27 (-34,-18)	-25

D.2 With the ES

Table 13: **Gaussian case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical ES with the sample std, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Gaussian distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

ES and std	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-40 (-84,28)	-39 (-84,24)	-40 (-85,21)	-38 (-82,31)	-42
25	-41 (-72,-3)	-42 (-69,-6)	-41 (-70,-8)	-41 (-71,-5)	-42
50	-42 (-62,-19)	-42 (-63,-18)	-42 (-63,-18)	-42 (-62,-19)	-42
100	-42 (-57,-26)	-42 (-56,-25)	-42 (-56,-26)	-42 (-57,-26)	-42
250	-42 (-52,-33)	-42 (-51,-32)	-42 (-52,-32)	-42 (-51,-33)	-42
500	-43 (-49,-36)	-42 (-48,-35)	-42 (-48,-35)	-42 (-48,-35)	-42
p=0.975					
10	-35 (-82,35)	-36 (-82,28)	-35 (-81,29)	-33 (-80,31)	-37
25	-38 (-70,-1)	-37 (-68,0)	-36 (-66,1)	-36 (-68,1)	-37
50	-38 (-60,-13)	-37 (-58,-12)	-37 (-59,-13)	-37 (-59,-11)	-37
100	-38 (-54,-22)	-37 (-52,-20)	-37 (-51,-20)	-37 (-53,-19)	-37
250	-39 (-48,-29)	-38 (-47,-27)	-37 (-47,-27)	-37 (-46,-27)	-37
500	-39 (-46,-32)	-38 (-44,-31)	-37 (-43,-30)	-37 (-44,-30)	-37
p=0.99					
10	-31 (-78,37)	-30 (-80,37)	-29 (-80,37)	-27 (-77,44)	-29
25	-33 (-64,8)	-31 (-63,9)	-30 (-63,9)	-29 (-65,9)	-29
50	-33 (-55,-7)	-31 (-54,-4)	-31 (-53,-5)	-30 (-55,-2)	-29
100	-33 (-49,-16)	-31 (-47,-13)	-31 (-47,-13)	-30 (-47,-11)	-29
250	-33 (-43,-22)	-31 (-41,-21)	-31 (-41,-20)	-30 (-40,-20)	-29
500	-33 (-41,-26)	-31 (-38,-24)	-31 (-38,-23)	-30 (-37,-23)	-29
p=0.99855					
10	-26 (-74,43)	-23 (-77,45)	-17 (-74,47)	-16 (-73,54)	-16
25	-27 (-60,14)	-24 (-57,14)	-18 (-55,20)	-18 (-55,21)	-16
50	-28 (-51,-2)	-24 (-49,3)	-19 (-45,9)	-18 (-45,11)	-16
100	-28 (-45,-11)	-23 (-41,-5)	-19 (-37,0)	-19 (-37,0)	-16
250	-28 (-39,-17)	-23 (-34,-13)	-19 (-31,-7)	-19 (-30,-6)	-16
500	-28 (-37,-21)	-24 (-31,-16)	-19 (-27,-11)	-18 (-27,-9)	-16

Table 14: **Gaussian case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical ES with the sample MAD, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Gaussian distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

ES and MAD	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-30 (-79,40)	-29 (-79,35)	-30 (-79,35)	-29 (-78,39)	-32
25	-32 (-66,6)	-32 (-65,3)	-32 (-64,5)	-32 (-65,7)	-32
50	-32 (-56,-7)	-32 (-56,-5)	-32 (-55,-7)	-32 (-55,-7)	-32
100	-33 (-49,-15)	-32 (-48,-15)	-32 (-48,-15)	-32 (-49,-14)	-32
250	-33 (-44,-22)	-32 (-42,-20)	-32 (-43,-21)	-32 (-42,-21)	-32
500	-33 (-40,-25)	-32 (-39,-25)	-32 (-39,-24)	-32 (-39,-25)	-32
p=0.975					
10	-25 (-78,46)	-25 (-77,39)	-24 (-75,44)	-23 (-75,44)	-26
25	-27 (-62,10)	-27 (-61,11)	-26 (-61,14)	-26 (-61,13)	-26
50	-28 (-52,-1)	-27 (-51,1)	-26 (-51,-1)	-26 (-50,2)	-26
100	-28 (-45,-10)	-27 (-44,-7)	-26 (-42,-8)	-26 (-44,-6)	-26
250	-28 (-39,-17)	-27 (-37,-16)	-26 (-38,-15)	-26 (-37,-15)	-26
500	-28 (-36,-21)	-27 (-34,-19)	-26 (-34,-18)	-26 (-34,-19)	-26
p=0.99					
10	-21 (-74,49)	-20 (-75,45)	-19 (-73,50)	-17 (-75,51)	-19
25	-22 (-57,20)	-21 (-56,20)	-20 (-55,23)	-19 (-58,20)	-19
50	-22 (-48,5)	-21 (-46,8)	-21 (-46,6)	-20 (-45,8)	-19
100	-23 (-41,-4)	-21 (-39,-1)	-20 (-38,-2)	-20 (-38,1)	-19
250	-23 (-34,-11)	-21 (-31,-9)	-21 (-32,-9)	-20 (-31,-8)	-19
500	-23 (-31,-15)	-21 (-28,-12)	-20 (-28,-12)	-20 (-28,-11)	-19
p=0.99855					
10	-18 (-72,51)	-14 (-72,50)	-10 (-72,55)	-10 (-71,57)	-9
25	-18 (-53,24)	-15 (-51,25)	-11 (-49,30)	-10 (-50,28)	-9
50	-18 (-44,9)	-15 (-42,12)	-11 (-38,16)	-11 (-37,19)	-9
100	-19 (-38,1)	-15 (-33,4)	-11 (-30,9)	-11 (-30,9)	-9
250	-19 (-31,-6)	-15 (-26,-3)	-11 (-23,1)	-11 (-23,1)	-9
500	-19 (-27,-11)	-15 (-23,-6)	-11 (-20,-3)	-11 (-19,-1)	-9

Table 15: **Student(5) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical ES with the sample std, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(5) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

ES and std	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-45 (-86,22)	-44 (-87,20)	-43 (-85,23)	-44 (-87,18)	-48
25	-46 (-72,-8)	-46 (-74,-8)	-46 (-73,-9)	-45 (-73,-10)	-48
50	-46 (-65,-19)	-46 (-66,-21)	-46 (-65,-22)	-46 (-65,-21)	-48
100	-46 (-60,-28)	-46 (-61,-28)	-46 (-60,-30)	-46 (-60,-29)	-48
250	-46 (-56,-33)	-46 (-55,-34)	-46 (-55,-34)	-46 (-55,-33)	-48
500	-46 (-53,-37)	-46 (-52,-37)	-45 (-52,-37)	-45 (-52,-37)	-48
p=0.975					
10	-44 (-87,21)	-43 (-87,21)	-43 (-84,25)	-43 (-87,19)	-48
25	-45 (-72,-8)	-45 (-74,-7)	-45 (-72,-8)	-45 (-72,-9)	-48
50	-45 (-65,-19)	-45 (-66,-19)	-45 (-65,-20)	-45 (-65,-23)	-48
100	-45 (-61,-27)	-45 (-60,-27)	-45 (-59,-29)	-45 (-60,-27)	-48
250	-45 (-55,-32)	-45 (-54,-33)	-45 (-55,-33)	-45 (-54,-33)	-48
500	-45 (-52,-36)	-45 (-52,-37)	-45 (-52,-36)	-45 (-52,-37)	-48
p=0.99					
10	-42 (-88,24)	-40 (-87,25)	-40 (-85,28)	-41 (-86,21)	-43
25	-43 (-73,-6)	-42 (-73,-4)	-43 (-72,-3)	-43 (-72,-6)	-43
50	-43 (-65,-18)	-42 (-65,-17)	-43 (-64,-17)	-43 (-64,-17)	-43
100	-43 (-59,-25)	-42 (-59,-23)	-43 (-58,-26)	-43 (-59,-23)	-43
250	-43 (-53,-30)	-43 (-53,-31)	-43 (-53,-32)	-43 (-53,-30)	-43
500	-43 (-51,-34)	-43 (-50,-34)	-43 (-50,-35)	-43 (-50,-35)	-43
p=0.99855					
10	-39 (-85,29)	-35 (-85,35)	-33 (-82,38)	-35 (-85,32)	-29
25	-40 (-72,-2)	-37 (-70,0)	-36 (-69,6)	-36 (-70,4)	-29
50	-41 (-63,-13)	-38 (-62,-11)	-36 (-60,-9)	-37 (-63,-8)	-29
100	-41 (-57,-23)	-38 (-55,-18)	-36 (-54,-17)	-37 (-55,-16)	-29
250	-40 (-51,-27)	-38 (-50,-26)	-36 (-49,-24)	-37 (-48,-25)	-29
500	-41 (-49,-31)	-38 (-47,-29)	-37 (-45,-27)	-37 (-45,-29)	-29

Table 16: **Student(5) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical ES with the sample MAD, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(5) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

ES and MAD	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-37 (-83,32)	-35 (-82,32)	-35 (-81,31)	-35 (-83,34)	-37
25	-38 (-68,2)	-37 (-68,3)	-37 (-68,0)	-37 (-67,3)	-37
50	-38 (-60,-13)	-37 (-58,-12)	-38 (-59,-14)	-38 (-57,-14)	-37
100	-38 (-54,-20)	-37 (-52,-21)	-37 (-52,-21)	-38 (-53,-22)	-37
250	-38 (-48,-28)	-37 (-47,-27)	-37 (-47,-28)	-38 (-48,-27)	-37
500	-38 (-45,-31)	-38 (-45,-31)	-38 (-44,-31)	-38 (-44,-30)	-37
p=0.975					
10	-34 (-80,34)	-31 (-79,36)	-31 (-79,36)	-30 (-81,37)	-32
25	-34 (-66,3)	-32 (-65,6)	-32 (-63,6)	-32 (-63,6)	-32
50	-35 (-57,-9)	-33 (-55,-8)	-33 (-54,-7)	-33 (-54,-9)	-32
100	-35 (-52,-17)	-33 (-49,-15)	-33 (-48,-15)	-33 (-49,-16)	-32
250	-35 (-45,-24)	-33 (-43,-23)	-33 (-43,-22)	-33 (-43,-22)	-32
500	-35 (-42,-27)	-33 (-41,-26)	-33 (-40,-26)	-33 (-40,-25)	-32
p=0.99					
10	-30 (-78,36)	-26 (-78,42)	-26 (-78,41)	-24 (-79,41)	-26
25	-30 (-63,9)	-27 (-62,13)	-27 (-59,11)	-26 (-59,12)	-26
50	-30 (-55,-4)	-27 (-53,-2)	-28 (-51,-2)	-27 (-50,-2)	-26
100	-31 (-47,-12)	-28 (-45,-9)	-28 (-43,-9)	-27 (-45,-8)	-26
250	-30 (-40,-19)	-28 (-38,-16)	-28 (-38,-17)	-27 (-38,-16)	-26
500	-31 (-38,-22)	-28 (-36,-20)	-28 (-36,-20)	-27 (-35,-19)	-26
p=0.99855					
10	-26 (-76,42)	-20 (-75,47)	-17 (-71,50)	-17 (-75,49)	-16
25	-26 (-61,12)	-21 (-57,18)	-19 (-54,22)	-18 (-54,24)	-16
50	-26 (-52,0)	-21 (-47,4)	-19 (-44,7)	-18 (-46,10)	-16
100	-27 (-44,-9)	-22 (-40,-3)	-19 (-37,0)	-18 (-37,1)	-16
250	-27 (-38,-15)	-22 (-33,-10)	-19 (-31,-7)	-18 (-30,-7)	-16
500	-27 (-35,-19)	-22 (-31,-14)	-19 (-27,-10)	-18 (-26,-10)	-16

Table 17: **Student(3) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical ES with the sample std, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(3) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

ES and std	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-42 (-87,31)	-42 (-88,27)	-41 (-86,30)	-41 (-86,28)	NaN
25	-44 (-74,3)	-42 (-75,4)	-42 (-73,2)	-42 (-73,0)	NaN
50	-43 (-65,-7)	-42 (-65,-10)	-42 (-65,-8)	-41 (-64,-8)	NaN
100	-42 (-60,-14)	-42 (-59,-16)	-41 (-59,-16)	-40 (-57,-12)	NaN
250	-42 (-55,-22)	-41 (-54,-21)	-40 (-54,-19)	-38 (-52,-16)	NaN
500	-41 (-51,-24)	-40 (-50,-23)	-39 (-50,-21)	-37 (-49,-20)	NaN
p=0.975					
10	-42 (-88,30)	-43 (-88,24)	-42 (-87,27)	-42 (-86,28)	NaN
25	-44 (-75,1)	-43 (-75,2)	-43 (-74,1)	-43 (-74,0)	NaN
50	-43 (-66,-6)	-43 (-67,-11)	-43 (-66,-8)	-42 (-66,-9)	NaN
100	-43 (-61,-15)	-43 (-60,-16)	-42 (-60,-17)	-41 (-59,-14)	NaN
250	-42 (-55,-22)	-41 (-55,-21)	-41 (-54,-19)	-40 (-53,-17)	NaN
500	-41 (-52,-24)	-41 (-51,-23)	-40 (-51,-22)	-39 (-50,-20)	NaN
p=0.99					
10	-42 (-87,32)	-43 (-87,25)	-43 (-87,31)	-43 (-87,24)	NaN
25	-44 (-74,4)	-43 (-75,1)	-44 (-74,0)	-44 (-74,1)	NaN
50	-43 (-66,-5)	-43 (-67,-11)	-43 (-66,-8)	-43 (-67,-7)	NaN
100	-43 (-60,-15)	-43 (-61,-16)	-43 (-61,-18)	-42 (-61,-14)	NaN
250	-42 (-55,-22)	-41 (-55,-22)	-41 (-55,-20)	-41 (-54,-18)	NaN
500	-41 (-51,-24)	-41 (-51,-23)	-40 (-51,-22)	-40 (-51,-22)	NaN
p=0.99855					
10	-40 (-86,34)	-41 (-87,31)	-39 (-87,33)	-42 (-87,27)	NaN
25	-42 (-74,5)	-42 (-75,2)	-41 (-74,5)	-42 (-73,3)	NaN
50	-42 (-66,-4)	-42 (-66,-8)	-41 (-66,-7)	-42 (-67,-6)	NaN
100	-42 (-60,-15)	-41 (-60,-14)	-41 (-60,-16)	-42 (-60,-14)	NaN
250	-41 (-54,-21)	-40 (-53,-21)	-40 (-53,-19)	-40 (-53,-17)	NaN
500	-40 (-51,-24)	-40 (-50,-22)	-39 (-50,-21)	-39 (-50,-21)	NaN

Table 18: **Student(3) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical ES with the sample MAD, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(3) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

ES and MAD	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-40 (-85,24)	-39 (-86,32)	-39 (-86,26)	-40 (-84,27)	-43
25	-42 (-71,-4)	-42 (-72,-5)	-42 (-72,-5)	-42 (-70,-9)	-43
50	-42 (-63,-15)	-42 (-64,-17)	-43 (-63,-17)	-43 (-63,-21)	-43
100	-42 (-58,-23)	-43 (-59,-26)	-43 (-58,-25)	-43 (-58,-26)	-43
250	-43 (-54,-30)	-43 (-53,-31)	-43 (-53,-32)	-43 (-53,-31)	-43
500	-43 (-50,-34)	-43 (-50,-35)	-43 (-50,-35)	-43 (-50,-35)	-43
p=0.975					
10	-37 (-84,25)	-37 (-85,33)	-36 (-85,27)	-38 (-84,32)	-39
25	-39 (-70,-1)	-39 (-71,0)	-39 (-70,-1)	-39 (-69,-4)	-39
50	-40 (-61,-13)	-40 (-62,-11)	-40 (-61,-13)	-40 (-61,-15)	-39
100	-40 (-56,-20)	-40 (-56,-23)	-40 (-55,-22)	-40 (-55,-23)	-39
250	-40 (-52,-28)	-40 (-51,-27)	-40 (-51,-28)	-40 (-50,-28)	-39
500	-40 (-48,-31)	-40 (-48,-32)	-40 (-47,-32)	-40 (-47,-32)	-39
p=0.99					
10	-34 (-83,35)	-33 (-83,34)	-33 (-84,32)	-33 (-81,37)	-35
25	-36 (-68,3)	-36 (-69,4)	-35 (-68,4)	-35 (-68,2)	-35
50	-37 (-59,-9)	-36 (-59,-7)	-36 (-59,-9)	-36 (-58,-11)	-35
100	-37 (-54,-17)	-36 (-53,-18)	-36 (-52,-18)	-36 (-53,-17)	-35
250	-37 (-49,-25)	-36 (-48,-23)	-36 (-47,-24)	-36 (-46,-24)	-35
500	-37 (-46,-28)	-36 (-45,-28)	-36 (-45,-29)	-36 (-44,-27)	-35
p=0.99855					
10	-31 (-81,40)	-29 (-81,41)	-25 (-82,46)	-26 (-78,46)	-26
25	-33 (-65,7)	-31 (-67,9)	-27 (-63,15)	-27 (-61,15)	-26
50	-34 (-57,-7)	-31 (-57,-2)	-28 (-53,0)	-28 (-52,0)	-26
100	-34 (-51,-15)	-32 (-50,-11)	-28 (-47,-10)	-28 (-46,-9)	-26
250	-34 (-47,-22)	-31 (-44,-18)	-29 (-41,-17)	-28 (-40,-15)	-26
500	-34 (-43,-25)	-32 (-40,-23)	-29 (-37,-21)	-28 (-37,-20)	-26

D.3 With the expectile

Table 19: **Gaussian case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical expectile with the sample std, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Gaussian distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

Expectile and std	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-31 (-80,35)	-32 (-80,32)	-32 (-81,32)	-29 (-79,45)	-33
25	-33 (-64,4)	-34 (-64,6)	-33 (-64,4)	-32 (-64,6)	-33
50	-33 (-56,-7)	-34 (-55,-8)	-33 (-55,-6)	-32 (-56,-6)	-33
100	-34 (-50,-16)	-34 (-49,-16)	-33 (-49,-16)	-33 (-49,-14)	-33
250	-34 (-44,-23)	-34 (-44,-23)	-33 (-44,-23)	-33 (-43,-22)	-33
500	-34 (-41,-26)	-34 (-41,-27)	-33 (-40,-25)	-33 (-40,-25)	-33
p=0.975					
10	-26 (-74,43)	-28 (-79,37)	-27 (-78,42)	-23 (-78,45)	-26
25	-27 (-60,14)	-29 (-62,11)	-27 (-60,16)	-26 (-62,15)	-26
50	-28 (-51,-2)	-29 (-52,0)	-27 (-51,0)	-26 (-52,2)	-26
100	-28 (-45,-11)	-29 (-45,-11)	-27 (-44,-9)	-27 (-45,-7)	-26
250	-28 (-39,-17)	-29 (-38,-18)	-27 (-38,-16)	-26 (-37,-15)	-26
500	-28 (-37,-21)	-29 (-36,-21)	-27 (-34,-19)	-26 (-34,-19)	-26
p=0.99					
10	-26 (-74,43)	-23 (-77,45)	-17 (-74,47)	-18 (-72,52)	-19
25	-27 (-60,14)	-24 (-57,14)	-18 (-55,20)	-19 (-55,21)	-19
50	-28 (-51,-2)	-24 (-49,3)	-19 (-45,9)	-19 (-45,10)	-19
100	-28 (-45,-11)	-23 (-41,-5)	-19 (-37,0)	-20 (-38,0)	-19
250	-28 (-39,-17)	-23 (-34,-13)	-19 (-31,-7)	-20 (-31,-8)	-19
500	-28 (-37,-21)	-24 (-31,-16)	-19 (-27,-11)	-20 (-28,-11)	-19
p=0.99855					
10	-26 (-74,43)	-23 (-77,45)	-17 (-74,47)	-14 (-72,53)	-8
25	-27 (-60,14)	-24 (-57,14)	-18 (-55,20)	-15 (-52,27)	-8
50	-28 (-51,-2)	-24 (-49,3)	-19 (-45,9)	-15 (-41,13)	-8
100	-28 (-45,-11)	-23 (-41,-5)	-19 (-37,0)	-15 (-34,5)	-8
250	-28 (-39,-17)	-23 (-34,-13)	-19 (-31,-7)	-15 (-27,-2)	-8
500	-28 (-37,-21)	-24 (-31,-16)	-19 (-27,-11)	-15 (-24,-6)	-8

Table 20: **Gaussian case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical expectile with the sample MAD, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Gaussian distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

Expectile and MAD	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-23 (-77,43)	-23 (-75,43)	-23 (-76,42)	-21 (-76,52)	-24
25	-24 (-59,16)	-25 (-57,15)	-24 (-60,15)	-24 (-58,19)	-24
50	-24 (-48,4)	-25 (-49,1)	-25 (-49,2)	-23 (-49,4)	-24
100	-25 (-42,-7)	-25 (-42,-6)	-24 (-41,-6)	-24 (-41,-4)	-24
250	-25 (-36,-13)	-25 (-35,-13)	-24 (-36,-12)	-24 (-35,-12)	-24
500	-25 (-32,-17)	-25 (-33,-17)	-24 (-32,-16)	-24 (-31,-16)	-24
p=0.975					
10	-18 (-72,51)	-19 (-75,49)	-18 (-74,49)	-15 (-74,52)	-17
25	-18 (-53,24)	-20 (-56,22)	-18 (-52,26)	-17 (-56,24)	-17
50	-18 (-44,9)	-20 (-45,8)	-18 (-43,10)	-17 (-45,12)	-17
100	-19 (-38,1)	-20 (-37,-1)	-18 (-36,0)	-18 (-36,2)	-17
250	-19 (-31,-6)	-20 (-31,-8)	-18 (-30,-7)	-17 (-28,-6)	-17
500	-19 (-27,-11)	-20 (-27,-11)	-18 (-26,-10)	-17 (-25,-10)	-17
p=0.99					
10	-18 (-72,51)	-14 (-72,50)	-10 (-72,55)	-11 (-71,54)	-11
25	-18 (-53,24)	-15 (-51,25)	-11 (-49,30)	-11 (-50,29)	-11
50	-18 (-44,9)	-15 (-42,12)	-11 (-38,16)	-12 (-38,18)	-11
100	-19 (-38,1)	-15 (-33,4)	-11 (-30,9)	-12 (-32,9)	-11
250	-19 (-31,-6)	-15 (-26,-3)	-11 (-23,1)	-12 (-24,0)	-11
500	-19 (-27,-11)	-15 (-23,-6)	-11 (-20,-3)	-12 (-21,-3)	-11
p=0.99855					
10	-18 (-72,51)	-14 (-72,50)	-10 (-72,55)	-8 (-69,57)	-4
25	-18 (-53,24)	-15 (-51,25)	-11 (-49,30)	-8 (-50,31)	-4
50	-18 (-44,9)	-15 (-42,12)	-11 (-38,16)	-8 (-35,20)	-4
100	-19 (-38,1)	-15 (-33,4)	-11 (-30,9)	-9 (-29,11)	-4
250	-19 (-31,-6)	-15 (-26,-3)	-11 (-23,1)	-9 (-20,3)	-4
500	-19 (-27,-11)	-15 (-23,-6)	-11 (-20,-3)	-8 (-17,1)	-4

Table 21: **Student(5) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical expectile with the sample std, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(5) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

Expectile and std	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-36 (-82,32)	-32 (-79,37)	-31 (-80,39)	-31 (-80,37)	-29
25	-35 (-66,5)	-33 (-64,7)	-33 (-65,6)	-32 (-63,6)	-29
50	-36 (-58,-8)	-34 (-56,-5)	-33 (-55,-7)	-32 (-55,-6)	-29
100	-35 (-51,-15)	-33 (-50,-14)	-32 (-49,-13)	-32 (-49,-14)	-29
250	-35 (-46,-21)	-33 (-44,-20)	-32 (-43,-20)	-32 (-43,-20)	-29
500	-35 (-43,-26)	-33 (-41,-24)	-32 (-40,-23)	-32 (-40,-22)	-29
p=0.975					
10	-37 (-82,32)	-35 (-82,31)	-33 (-81,34)	-32 (-81,33)	-30
25	-37 (-66,3)	-35 (-67,4)	-34 (-65,7)	-33 (-63,8)	-30
50	-37 (-59,-11)	-35 (-58,-7)	-34 (-58,-7)	-33 (-56,-7)	-30
100	-37 (-53,-17)	-35 (-52,-15)	-34 (-50,-15)	-33 (-49,-14)	-30
250	-36 (-47,-24)	-34 (-45,-23)	-33 (-44,-22)	-33 (-44,-20)	-30
500	-36 (-45,-27)	-34 (-42,-26)	-33 (-41,-24)	-33 (-41,-24)	-30
p=0.99					
10	-39 (-85,29)	-35 (-85,35)	-32 (-81,35)	-31 (-80,35)	-30
25	-40 (-72,-2)	-37 (-70,0)	-34 (-66,7)	-32 (-64,5)	-30
50	-41 (-63,-13)	-38 (-62,-11)	-35 (-58,-8)	-33 (-56,-6)	-30
100	-41 (-57,-23)	-38 (-55,-18)	-34 (-50,-16)	-32 (-50,-13)	-30
250	-40 (-51,-27)	-38 (-50,-26)	-34 (-45,-22)	-33 (-43,-21)	-30
500	-41 (-49,-31)	-38 (-47,-29)	-34 (-41,-25)	-33 (-41,-24)	-30
p=0.99855					
10	-39 (-85,29)	-35 (-85,35)	-33 (-82,38)	-33 (-84,36)	-26
25	-40 (-72,-2)	-37 (-70,0)	-36 (-69,6)	-33 (-68,6)	-26
50	-41 (-63,-13)	-38 (-62,-11)	-36 (-60,-9)	-34 (-60,-4)	-26
100	-41 (-57,-23)	-38 (-55,-18)	-36 (-54,-17)	-34 (-52,-14)	-26
250	-40 (-51,-27)	-38 (-50,-26)	-36 (-49,-24)	-34 (-46,-21)	-26
500	-41 (-49,-31)	-38 (-47,-29)	-37 (-45,-27)	-34 (-43,-26)	-26

Table 22: **Student(5) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical expectile with the sample MAD, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(5) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

VaR and MAD	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-32 (-79,33)	-30 (-80,36)	-31 (-78,36)	-31 (-80,40)	-32
25	-32 (-63,7)	-32 (-64,8)	-32 (-66,6)	-32 (-65,9)	-32
50	-33 (-55,-7)	-33 (-54,-7)	-32 (-54,-7)	-33 (-55,-7)	-32
100	-33 (-49,-14)	-33 (-49,-16)	-32 (-49,-14)	-32 (-49,-16)	-32
250	-33 (-44,-22)	-33 (-43,-22)	-32 (-43,-21)	-32 (-43,-21)	-32
500	-33 (-41,-26)	-33 (-40,-26)	-32 (-40,-25)	-32 (-40,-25)	-32
p=0.975					
-29 (-79,34)	-28 (-79,38)	-27 (-77,41)	-27 (-78,45)	-28	
25	-31 (-62,8)	-29 (-62,8)	-28 (-59,10)	-28 (-61,10)	-28
50	-31 (-54,-3)	-28 (-52,-3)	-28 (-51,-2)	-29 (-52,-4)	-28
100	-31 (-47,-12)	-29 (-45,-10)	-28 (-44,-10)	-29 (-46,-10)	-28
250	-31 (-41,-19)	-29 (-40,-18)	-28 (-39,-17)	-29 (-39,-18)	-28
500	-31 (-38,-23)	-29 (-37,-21)	-28 (-35,-21)	-28 (-36,-21)	-28
p=0.99					
10	-26 (-76,42)	-20 (-75,47)	-21 (-75,48)	-20 (-73,49)	-22
25	-26 (-61,12)	-21 (-57,18)	-22 (-55,19)	-22 (-55,18)	-22
50	-26 (-52,0)	-21 (-47,4)	-23 (-47,5)	-23 (-47,3)	-22
100	-27 (-44,-9)	-22 (-40,-3)	-22 (-39,-4)	-23 (-40,-4)	-22
250	-27 (-38,-15)	-22 (-33,-10)	-23 (-33,-11)	-23 (-34,-12)	-22
500	-27 (-35,-19)	-22 (-31,-14)	-23 (-31,-15)	-23 (-31,-15)	-22
p=0.99855					
10	-26 (-76,42)	-20 (-75,47)	-17 (-71,50)	-15 (-72,51)	-13
25	-26 (-61,12)	-21 (-57,18)	-19 (-54,22)	-16 (-53,28)	-13
50	-26 (-52,0)	-21 (-47,4)	-19 (-44,7)	-16 (-43,12)	-13
100	-27 (-44,-9)	-22 (-40,-3)	-19 (-37,0)	-16 (-34,4)	-13
250	-27 (-38,-15)	-22 (-33,-10)	-19 (-31,-7)	-16 (-27,-4)	-13
500	-27 (-35,-19)	-22 (-31,-14)	-19 (-27,-10)	-16 (-24,-7)	-13

Table 23: **Student(3) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical expectile with the sample std, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(3) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

Expectile and std	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-27 (-78,39)	-23 (-77,45)	-19 (-76,50)	-17 (-73,49)	NaN
25	-27 (-61,14)	-21 (-58,20)	-19 (-57,23)	-17 (-55,24)	NaN
50	-25 (-51,4)	-20 (-46,10)	-18 (-45,11)	-16 (-41,14)	NaN
100	-24 (-43,0)	-20 (-40,2)	-17 (-36,5)	-15 (-35,7)	NaN
250	-23 (-36,-7)	-19 (-32,-3)	-16 (-30,-2)	-13 (-27,1)	NaN
500	-22 (-32,-9)	-18 (-28,-7)	-15 (-25,-4)	-13 (-23,-2)	NaN
p=0.975					
10	-31 (-83,37)	-26 (-80,48)	-23 (-80,41)	-21 (-77,46)	NaN
25	-32 (-67,11)	-25 (-62,21)	-22 (-59,21)	-21 (-59,21)	NaN
50	-31 (-56,3)	-24 (-50,8)	-21 (-47,10)	-19 (-44,10)	NaN
100	-30 (-50,-5)	-24 (-45,-1)	-20 (-39,2)	-18 (-37,4)	NaN
250	-29 (-43,-12)	-23 (-37,-7)	-19 (-34,-3)	-16 (-30,-1)	NaN
500	-28 (-38,-14)	-22 (-32,-10)	-18 (-29,-5)	-15 (-26,-3)	NaN
p=0.99					
10	-40 (-86,34)	-32 (-84,38)	-30 (-79,39)	-26 (-80,48)	NaN
25	-42 (-74,5)	-32 (-65,12)	-29 (-65,10)	-25 (-60,16)	NaN
50	-42 (-66,-4)	-30 (-56,1)	-27 (-53,4)	-24 (-51,5)	NaN
100	-42 (-60,-15)	-30 (-50,-5)	-26 (-46,-3)	-22 (-42,1)	NaN
250	-41 (-54,-21)	-28 (-43,-11)	-24 (-38,-8)	-20 (-34,-4)	NaN
500	-40 (-51,-24)	-27 (-39,-13)	-23 (-35,-9)	-19 (-31,-6)	NaN
p=0.99855					
10	-40 (-86,34)	-41 (-87,31)	-39 (-87,33)	-40 (-87,30)	NaN
25	-42 (-74,5)	-42 (-75,2)	-41 (-74,5)	-41 (-73,4)	NaN
50	-42 (-66,-4)	-42 (-66,-8)	-41 (-66,-7)	-41 (-65,-4)	NaN
100	-42 (-60,-15)	-41 (-60,-14)	-41 (-60,-16)	-40 (-59,-13)	NaN
250	-41 (-54,-21)	-40 (-53,-21)	-40 (-53,-19)	-39 (-52,-17)	NaN
500	-40 (-51,-24)	-40 (-50,-22)	-39 (-50,-21)	-38 (-49,-20)	NaN

Table 24: **Student(3) case** - Average values from a 1'000-fold repetition. Comparing the sample Pearson correlations of the log-ratio of the historical expectile with the sample MAD, as a function of the sample size n on which the quantile is estimated (fixed length $l = 50$ of the bivariate sample used to estimate the correlation). We consider the thresholds $p = 0.95, 0.975, 0.99, 0.99855$. Underlying samples are simulated from a Student(3) distribution. Average empirical values are written first (with empirical 95% confidence interval in brackets). The theoretical correlation value in the asymptotic distribution '($n \to \infty$)' are provided as benchmark in the last column.

Expectile and MAD	$n = 126$	$n = 252$	$n = 504$	$n = 1008$	Theoretical value ($n \to \infty$)
p=0.95					
10	-34 (-82,29)	-32 (-81,33)	-32 (-82,34)	-33 (-79,32)	-34
25	-35 (-66,2)	-33 (-66,5)	-33 (-66,5)	-34 (-65,5)	-34
50	-34 (-57,-6)	-34 (-57,-8)	-34 (-56,-9)	-34 (-56,-8)	-34
100	-34 (-50,-15)	-34 (-50,-16)	-34 (-49,-16)	-34 (-49,-16)	-34
250	-34 (-44,-21)	-34 (-44,-21)	-34 (-44,-22)	-34 (-44,-22)	-34
500	-34 (-42,-25)	-34 (-41,-26)	-34 (-41,-25)	-34 (-41,-26)	-34
p=0.975					
10	-32 (-82,31)	-29 (-80,36)	-31 (-81,36)	-31 (-81,40)	-32
25	-33 (-64,7)	-32 (-65,8)	-32 (-65,6)	-33 (-63,4)	-32
50	-33 (-56,-4)	-32 (-55,-8)	-32 (-55,-7)	-33 (-55,-8)	-32
100	-33 (-50,-14)	-33 (-49,-16)	-32 (-47,-15)	-33 (-49,-14)	-32
250	-34 (-45,-21)	-33 (-44,-21)	-32 (-43,-20)	-32 (-42,-20)	-32
500	-33 (-41,-24)	-33 (-40,-24)	-32 (-40,-24)	-32 (-40,-24)	-32
p=0.99					
10	-31 (-81,40)	-29 (-80,38)	-29 (-80,36)	-29 (-82,40)	-30
25	-33 (-65,7)	-31 (-65,8)	-30 (-63,9)	-30 (-64,6)	-30
50	-34 (-57,-7)	-31 (-54,-3)	-31 (-54,-5)	-31 (-55,-5)	-30
100	-34 (-51,-15)	-32 (-49,-13)	-30 (-47,-11)	-30 (-46,-12)	-30
250	-34 (-47,-22)	-32 (-43,-20)	-30 (-41,-18)	-30 (-41,-18)	-30
500	-34 (-43,-25)	-32 (-40,-23)	-30 (-38,-22)	-30 (-38,-22)	-30
p=0.99855					
10	-31 (-81,40)	-29 (-81,41)	-25 (-82,46)	-24 (-76,47)	-23
25	-33 (-65,7)	-31 (-67,9)	-27 (-63,15)	-25 (-60,19)	-23
50	-34 (-57,-7)	-31 (-57,-2)	-28 (-53,0)	-25 (-51,2)	-23
100	-34 (-51,-15)	-32 (-50,-11)	-28 (-47,-10)	-26 (-44,-6)	-23
250	-34 (-47,-22)	-31 (-44,-18)	-29 (-41,-17)	-26 (-38,-12)	-23
500	-34 (-43,-25)	-32 (-40,-23)	-29 (-37,-21)	-26 (-35,-17)	-23

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