Proof of equation 10:

$$\beta_{cov} = \frac{\text{var}(Z_{ij})\,\text{cov}(X_{ij}, Y_{ij}) - \text{cov}(Z_{ij}, X_{ij})\,\text{cov}(Z_{ij}, Y_{ij})}{\text{var}(Z_{ij})\,\text{var}(X_{ij}) - \text{cov}(Z_{ij}, X_{ij})^2}$$

$$= \frac{\begin{array}{l}\text{var}(Z_{ij})\,\text{cov}\left(X_{ij}, \beta_{YX}X_{ij} + \beta_{YC}C_{ij} + \epsilon_{Y_{ij}}\right) \\ \quad - \text{cov}\left(\beta_{ZC}C_{ij} + \epsilon_{Z_{ij}}, \beta_{XC}C_{ij} + \epsilon_{X_{ij}}\right)\text{cov}\left(\beta_{ZC}C_{ij} + \epsilon_{Z_{ij}}, \beta_{YX}X_{ij} + \beta_{YC}C_{ij} + \epsilon_{Y_{ij}}\right)\end{array}}{\text{var}(Z_{ij})\,\text{var}(X_{ij}) - \text{cov}\left(\beta_{ZC}C_{ij} + \epsilon_{Z_{ij}}, \beta_{XC}C_{ij} + \epsilon_{X_{ij}}\right)^2}$$

$$= \frac{\begin{array}{l}\text{var}(Z_{ij})\left[\beta_{YX}\text{var}(X_{ij}) + \beta_{YC}\text{cov}(X_{ij}, C_{ij})\right] \\ \quad - \beta_{ZC}\beta_{XC}\text{var}(C_{ij})\left[\beta_{ZC}\beta_{YX}\text{cov}(C_{ij}, X_{ij}) + \beta_{ZC}\beta_{YC}\text{var}(C_{ij})\right]\end{array}}{\text{var}(Z_{ij})\,\text{var}(X_{ij}) - \left[\beta_{ZC}\beta_{XC}\text{var}(C_{ij})\right]^2}$$

$$= \frac{\begin{array}{l}\beta_{YX}\text{var}(Z_{ij})\,\text{var}(X_{ij}) + \beta_{YC}\text{var}(Z_{ij})\,\text{cov}\left(\beta_{XC}C_{ij} + \epsilon_{X_{ij}}, C_{ij}\right) \\ \quad - \beta_{ZC}\beta_{XC}\text{var}(C_{ij})\left[\beta_{ZC}\beta_{YX}\text{cov}\left(C_{ij}, \beta_{XC}C_{ij} + \epsilon_{X_{ij}}\right) + \beta_{ZC}\beta_{YC}\text{var}(C_{ij})\right]\end{array}}{\text{var}(Z_{ij})\,\text{var}(X_{ij}) - \left[\beta_{ZC}\beta_{XC}\text{var}(C_{ij})\right]^2}$$

$$= \frac{\begin{array}{l}\beta_{YX}\text{var}(Z_{ij})\,\text{var}(X_{ij}) + \beta_{YC}\beta_{XC}\text{var}(Z_{ij})\,\text{var}(C_{ij}) \\ \quad - \beta_{ZC}\beta_{XC}\text{var}(C_{ij})\left[\beta_{ZC}\beta_{YX}\beta_{XC}\text{var}(C_{ij}) + \beta_{ZC}\beta_{YC}\text{var}(C_{ij})\right]\end{array}}{\text{var}(Z_{ij})\,\text{var}(X_{ij}) - \left[\beta_{ZC}\beta_{XC}\text{var}(C_{ij})\right]^2}$$

$$= \frac{\beta_{YX}\text{var}(Z_{ij})\,\text{var}(X_{ij}) + \beta_{YC}\beta_{XC}\sigma_C^2\text{var}(Z_{ij}) - \beta_{YX}\beta_{ZC}^2\beta_{XC}^2(\sigma_C^2)^2 - \beta_{ZC}^2\beta_{XC}\beta_{YC}(\sigma_C^2)^2}{\text{var}(Z_{ij})\,\text{var}(X_{ij}) - \beta_{ZC}^2\beta_{XC}^2(\sigma_C^2)^2}$$

$$= \beta_{YX} + \frac{\beta_{YC}\beta_{XC}\sigma_C^2\text{var}(Z_{ij}) - \beta_{ZC}^2\beta_{XC}\beta_{YC}(\sigma_C^2)^2}{\text{var}(Z_{ij})\,\text{var}(X_{ij}) - \beta_{ZC}^2\beta_{XC}^2(\sigma_C^2)^2}$$

$$= \beta_{YX} + \frac{\beta_{YC}\beta_{XC}\sigma_C^2\left[\text{var}(Z_{ij}) - \beta_{ZC}^2\sigma_C^2\right]}{\text{var}(Z_{ij})\,\text{var}(X_{ij}) - \beta_{ZC}^2\beta_{XC}^2(\sigma_C^2)^2}$$

$$= \beta_{YX} + \frac{\beta_{YC}\beta_{XC}\sigma_C^2\left[\left(\beta_{ZC}^2\sigma_C^2 + \sigma_{\epsilon_Z}^2\right) - \beta_{ZC}^2\sigma_C^2\right]}{\left(\beta_{ZC}^2\sigma_C^2 + \sigma_{\epsilon_Z}^2\right)\left(\beta_{XC}^2\sigma_C^2 + \sigma_{\epsilon_X}^2\right) - \beta_{ZC}^2\beta_{XC}^2(\sigma_C^2)^2}$$

If $\text{var}(C_{ij}) = \text{var}(X_{ij}) = \text{var}(Z_{ij}) = 1$, then

$$\beta_{cov} = \beta_{YX} + \frac{\beta_{YC}\beta_{XC}\left(1 - \beta_{ZC}^2\right)}{1 - \beta_{ZC}^2\beta_{XC}^2}$$

Proof of equation 11:

In a linear model, β_B may be excluded from Eq. 3 without changing the value of $\beta_{W_{covstd}}$[1]:

$$E\left(Y_{ij} \mid X_{ij}, \bar{X}_i, Z_{ij}\right) = \beta_0 + \beta_{W_{covstd}}\left(X_{ij} - \bar{X}_i\right) + \beta_Z Z_{ij},$$

We can then modify this model as:

$$E\left(Y_{ij} \mid X_{ij}, \bar{X}_i, Z_{ij}\right) = \beta_0 + \beta^*_{W_{covstd}}\frac{1}{2}\left(X_{ij} - X_{ij'}\right) + \beta_Z Z_{ij} \text{ so that } \beta^*_{W_{covstd}} = \tfrac{\beta_{W_{covstd}}}{2}.$$

We then have:

$$\beta^*_{W_{covstd}} = \frac{\text{var}\left(Z_{ij}\right)\text{cov}\left[Y_{ij},\left(X_{ij}-X_{ij'}\right)\right] - \text{cov}\left[Z_{ij},\left(X_{ij}-X_{ij'}\right)\right]\text{cov}\left(Y_{ij},Z_{ij}\right)}{\text{var}\left(Z_{ij}\right)\text{var}\left(X_{ij}-X_{ij'}\right) - \text{cov}\left[Z_{ij},\left(X_{ij}-X_{ij'}\right)\right]^2}$$

Plugging in pieces from the derivations below gives:

$$\beta^*_{W_{covstd}} = \frac{\begin{array}{r}\left(\beta^2_{ZC}\sigma^2_C + \sigma_{\epsilon_Z}\right)\left[\beta_{YX}\left(1-\beta^2_{XC}\sigma^2_C\rho_C - \sigma^2_{\epsilon_X}\rho_{\epsilon_X}\right) + \beta_{YC}\beta_{XC}\sigma^2_C\left(1-\rho_C\right)\right] \\ - \beta_{ZC}\beta_{XC}\sigma^2_C\left(1-\rho_C\right)\left(\beta_{YX}\beta_{XC}\beta_{ZC}\sigma^2_C + \beta_{YC}\beta_{ZC}\sigma^2_C\right)\end{array}}{2\left(\beta^2_{ZC}\sigma^2_C + \sigma_{\epsilon_Z}\right)\left(1-\beta^2_{XC}\sigma^2_C\rho_C - \rho_{\epsilon_X}\sigma^2_{\epsilon_X}\right) - \left[\beta_{ZC}\beta_{XC}\sigma^2_C\left(1-\rho_C\right)\right]^2}$$

If $\text{var}\left(C_{ij}\right) = \text{var}\left(X_{ij}\right) = \text{var}\left(Z_{ij}\right) = 1$, then this simplifies to

$$\beta^*_{W_{covstd}} = \frac{\beta_{YX}\left(1-\beta^2_{XC}\rho_C - \sigma^2_{\epsilon_X}\rho_{\epsilon_X}\right) + \beta_{YC}\beta_{XC}\left(1-\rho_C\right) - \beta_{ZC}\beta_{XC}\left(1-\rho_C\right)\left(\beta_{YX}\beta_{XC}\beta_{ZC} + \beta_{YC}\beta_{ZC}\right)}{2\left(1-\beta^2_{XC}\rho_C - \rho_{\epsilon_X}\sigma^2_{\epsilon_X}\right) - \left[\beta_{ZC}\beta_{XC}\left(1-\rho_C\right)\right]^2}$$

Proof of equation 12:

Similarly, in a linear model, β_B may be excluded from Eq. 4 without changing the value of $\beta_{W_{cov}}$[1]:

$$E\left(Y_{ij} \mid X_{ij}, \bar{X}_i, Z_{ij}, \bar{Z}_i\right) = \beta_0 + \beta_{W_{cov}}\left(X_{ij} - \bar{X}_i\right) + \beta_Z\left(Z_{ij} - \bar{Z}_i\right)$$

We can then modify this model as:

$E\left(Y_{ij} \mid X_{ij}, \bar{X}_i, Z_{ij}, \bar{Z}_i\right) = \beta_0 + \beta^*_{W_{cov}}\frac{1}{2}\left(X_{ij} - X_{ij'}\right) + \beta^*_Z\frac{1}{2}\left(Z_{ij} - Z_{ij'}\right)$ so that $\beta^*_{W_{cov}} = \frac{\beta_{W_{cov}}}{2}$ and $\beta^*_Z = \frac{\beta_Z}{2}$.

We then have:

$$\beta^*_{W_{cov}} = \frac{\text{var}\left(Z_{ij}-Z_{ij'}\right)\text{cov}\left[Y_{ij},\left(X_{ij}-X_{ij'}\right)\right] - \text{cov}\left[\left(Z_{ij}-Z_{ij'}\right),\left(X_{ij}-X_{ij'}\right)\right]\text{cov}\left[Y_{ij},\left(Z_{ij}-Z_{ij'}\right)\right]}{\text{var}\left(Z_{ij}-Z_{ij'}\right)\text{var}\left(X_{ij}-X_{ij'}\right) - \text{cov}\left[\left(Z_{ij}-Z_{ij'}\right),\left(X_{ij}-X_{ij'}\right)\right]^2}$$

Plugging in pieces from the derivations below gives:

$$\beta^*_{W_{cov}} = \frac{\begin{aligned}&\left[2\left(1-\beta^2_{ZC}\sigma^2_C\rho_C-\rho_{\epsilon_Z}\sigma^2_{\epsilon_Z}\right)\right]\left[\beta_{YX}\left(1-\beta^2_{XC}\sigma^2_C\rho_C-\sigma^2_{\epsilon_X}\rho_{\epsilon_X}\right)+\beta_{YC}\beta_{XC}\sigma^2_C\left(1-\rho_C\right)\right]\\&-\left[2\beta_{ZC}\beta_{XC}\sigma^2_C\left(1-\rho_C\right)\right]\left[\left(\beta_{YX}\beta_{XC}\beta_{ZC}+\beta_{YC}\beta_{ZC}\right)\sigma^2_C\left(1-\rho_C\right)\right]\end{aligned}}{\left[2\left(1-\beta^2_{ZC}\sigma^2_C\rho_C-\rho_{\epsilon_Z}\sigma^2_{\epsilon_Z}\right)\right]\left[2\left(1-\beta^2_{XC}\sigma^2_C\rho_C-\rho_{\epsilon_X}\sigma^2_{\epsilon_X}\right)\right]-\left[2\beta_{ZC}\beta_{XC}\sigma^2_C\left(1-\rho_C\right)\right]^2}$$

If $\text{var}(C_{ij}) = \text{var}(X_{ij}) = \text{var}(Z_{ij}) = 1$, then this simplifies to

$$\beta^*_{W_{cov}} = \frac{\begin{aligned}&\left[2\left(1-\beta^2_{ZC}\rho_C-\rho_{\epsilon_Z}\sigma^2_{\epsilon_Z}\right)\right]\left[\beta_{YX}\left(1-\beta^2_{XC}\rho_C-\sigma^2_{\epsilon_X}\rho_{\epsilon_X}\right)+\beta_{YC}\beta_{XC}\left(1-\rho_C\right)\right]\\&-\left[2\beta_{ZC}\beta_{XC}\left(1-\rho_C\right)\right]\left[\left(\beta_{YX}\beta_{XC}\beta_{ZC}+\beta_{YC}\beta_{ZC}\right)\left(1-\rho_C\right)\right]\end{aligned}}{\left[2\left(1-\beta^2_{ZC}\rho_C-\rho_{\epsilon_Z}\sigma^2_{\epsilon_Z}\right)\right]\left[2\left(1-\beta^2_{XC}\rho_C-\rho_{\epsilon_X}\sigma^2_{\epsilon_X}\right)\right]-\left[2\beta_{ZC}\beta_{XC}\left(1-\rho_C\right)\right]^2}$$

Derivation pieces:

The components of the within-pair estimates above are derived as:

$$\begin{aligned}
\text{cov}\left[Y_{ij},\left(X_{ij}-X'_{ij}\right)\right] &= \text{cov}\left(Y_{ij},X_{ij}\right)-\text{cov}\left(Y_{ij},X_{ij'}\right)\\
&= \text{cov}\left(\beta_{YX}X_{ij}+\beta_{YC}C_{ij}+\epsilon_{Y_{ij}},X_{ij}\right)-\text{cov}\left(\beta_{YX}X_{ij}+\beta_{YC}C_{ij}+\epsilon_{Y_{ij}},X_{ij'}\right)\\
&= \beta_{YX}\text{var}\left(X_{ij}\right)+\beta_{YC}\text{cov}\left(C_{ij},X_{ij}\right)-\beta_{YX}\text{cov}\left(X_{ij},X_{ij'}\right)-\beta_{YC}\text{cov}\left(C_{ij},X_{ij'}\right)\\
&= \beta_{YX}\left[\text{var}\left(X_{ij}\right)-\text{cov}\left(X_{ij},X_{ij'}\right)\right]+\beta_{YC}\left[\text{cov}\left(C_{ij},X_{ij}\right)-\text{cov}\left(C_{ij},X_{ij'}\right)\right]\\
&= \beta_{YX}\left[\text{var}\left(\beta_{XC}C_{ij}+\epsilon_{X_{ij}}\right)-\text{cov}\left(\beta_{XC}C_{ij}+\epsilon_{X_{ij}},\beta_{XC}C_{ij'}+\epsilon_{X_{ij'}}\right)\right]\\
&\quad+\beta_{YC}\left[\text{cov}\left(C_{ij},\beta_{XC}C_{ij}+\epsilon_{X_{ij}}\right)-\text{cov}\left(C_{ij},\beta_{XC}C_{ij'}+\epsilon_{X_{ij'}}\right)\right]\\
&= \beta_{YX}\left(\beta^2_{XC}\sigma^2_C+\sigma^2_{\epsilon_X}-\beta^2_{XC}\sigma^2_C\rho_C-\sigma^2_{\epsilon_X}\rho_{\epsilon_X}\right)+\beta_{YC}\left(\beta_{XC}\sigma^2_C-\beta_{XC}\sigma^2_C\rho_C\right)\\
&= \beta_{YX}\left[\beta^2_{XC}\sigma^2_C\left(1-\rho_C\right)+\sigma^2_{\epsilon_X}\left(1-\rho_{\epsilon_X}\right)\right]+\beta_{YC}\left[\beta_{XC}\sigma^2_C\left(1-\rho_C\right)\right]\\
&= \beta_{YX}\beta^2_{XC}\sigma^2_C\left(1-\rho_C\right)+\beta_{YX}\sigma^2_{\epsilon_X}\left(1-\rho_{\epsilon_X}\right)+\beta_{YC}\beta_{XC}\sigma^2_C\left(1-\rho_C\right)\\
&= \beta_{YX}\left(1-\beta^2_{XC}\sigma^2_C\rho_C-\sigma^2_{\epsilon_X}\rho_{\epsilon_X}\right)+\beta_{YC}\beta_{XC}\sigma^2_C\left(1-\rho_C\right)
\end{aligned}$$

$$\text{cov}\left(Y_{ij}, Z_{ij}\right) = \text{cov}\left(\beta_{YX}X_{ij} + \beta_{YC}C_{ij} + \epsilon_{Y_{ij}}, Z_{ij}\right)$$

$$= \beta_{YX}\text{cov}\left(X_{ij}, Z_{ij}\right) + \beta_{YC}\text{cov}\left(C_{ij}, Z_{ij}\right)$$

$$= \beta_{YX}\text{cov}\left(\beta_{XC}C_{ij} + \epsilon_{X_{ij}}, \beta_{ZC}C_{ij} + \epsilon_{Z_{ij}}\right) + \beta_{YC}\text{cov}\left(C_{ij}, \beta_{ZC}C_{ij} + \epsilon_{Z_{ij}}\right)$$

$$= \beta_{YX}\beta_{XC}\beta_{ZC}\sigma_C^2 + \beta_{YC}\beta_{ZC}\sigma_C^2$$

$$\text{cov}\left[Z_{ij}, (X_{ij} - X_{ij'})\right] = \text{cov}\left(Z_{ij}, X_{ij}\right) - \text{cov}\left(Z_{ij}, X_{ij'}\right)$$

$$= \text{cov}\left(\beta_{ZC}C_{ij} + \epsilon_{Z_{ij}}, \beta_{XC}C_{ij} + \epsilon_{X_{ij}}\right) - \text{cov}\left(\beta_{ZC}C_{ij} + \epsilon_{Z_{ij}}, \beta_{XC}C_{ij'} + \epsilon_{X_{ij'}}\right)$$

$$= \beta_{ZC}\beta_{XC}\sigma_C^2 - \beta_{ZC}\beta_{XC}\sigma_C^2\rho_C$$

$$= \beta_{ZC}\beta_{XC}\sigma_C^2(1 - \rho_C)$$

$$\text{cov}\left[Y_{ij}, (Z_{ij} - Z_{ij'})\right] = \text{cov}\left(Y_{ij}, Z_{ij}\right) - \text{cov}\left(Y_{ij}, Z_{ij'}\right)$$

$$= \text{cov}\left(\beta_{YX}X_{ij} + \beta_{YC}C_{ij} + \epsilon_{Y_{ij}}, Z_{ij}\right) - \text{cov}\left(\beta_{YX}X_{ij} + \beta_{YC}C_{ij} + \epsilon_{Y_{ij}}, Z_{ij'}\right)$$

$$= \beta_{YX}\text{cov}\left(X_{ij}, Z_{ij}\right) + \beta_{YC}\text{cov}\left(C_{ij}, Z_{ij}\right) - \beta_{YX}\text{cov}\left(X_{ij}, Z_{ij'}\right) - \beta_{YC}\text{cov}\left(C_{ij}, Z_{ij'}\right)$$

$$= \beta_{YX}\left[\text{cov}\left(X_{ij}, Z_{ij}\right) - \text{cov}\left(X_{ij}, Z_{ij'}\right)\right] + \beta_{YC}\left[\text{cov}\left(C_{ij}, Z_{ij}\right) - \text{cov}\left(C_{ij}, Z_{ij'}\right)\right]$$

$$= \beta_{YX}\left[\text{cov}\left(\beta_{XC}C_{ij} + \epsilon_{X_{ij}}, Z_{ij}\right) - \text{cov}\left(\beta_{XC}C_{ij} + \epsilon_{X_{ij}}, Z_{ij'}\right)\right]$$

$$+ \beta_{YC}\left[\text{cov}\left(C_{ij}, \beta_{ZC}C_{ij} + \epsilon_{Z_{ij}}\right) - \text{cov}\left(C_{ij}, \beta_{ZC}C_{ij'} + \epsilon_{Z_{ij'}}\right)\right]$$

$$= \beta_{YX}\left[\beta_{XC}\text{cov}\left(C_{ij}, \beta_{ZC}C_{ij} + \epsilon_{Z_{ij}}\right) - \beta_{XC}\text{cov}\left(C_{ij}, \beta_{ZC}C_{ij'} + \epsilon_{Z_{ij'}}\right)\right]$$

$$+ \beta_{YC}\left(\beta_{ZC}\sigma_C^2 - \beta_{ZC}\sigma_C^2\rho_C\right)$$

$$= \beta_{YX}\left(\beta_{XC}\beta_{ZC}\sigma_C^2 - \beta_{XC}\beta_{ZC}\sigma_C^2\rho_C\right) + \beta_{YC}\left(\beta_{ZC}\sigma_C^2 - \beta_{ZC}\sigma_C^2\rho_C\right)$$

$$= \beta_{YX}\beta_{XC}\beta_{ZC}\sigma_C^2\left(1 - \rho_C\right) + \beta_{YC}\beta_{ZC}\sigma_C^2\left(1 - \rho_C\right)$$

$$= \left(\beta_{YX}\beta_{XC}\beta_{ZC} + \beta_{YC}\beta_{ZC}\right)\sigma_C^2\left(1 - \rho_C\right)$$

$$
\begin{aligned}
\text{cov}\left[\left(Z_{ij}-Z_{ij'}\right),\left(X_{ij}-X_{ij'}\right)\right] &= \text{cov}\left(Z_{ij},X_{ij}\right)-\text{cov}\left(Z_{ij},X_{ij'}\right)-\text{cov}\left(Z_{ij'},X_{ij}\right)+\text{cov}\left(Z_{ij'},X_{ij'}\right) \\
&= \text{cov}\left(\beta_{ZC}C_{ij}+\epsilon_{Z_{ij}},X_{ij}\right)-\text{cov}\left(\beta_{ZC}C_{ij}+\epsilon_{Z_{ij}},X_{ij'}\right) \\
&\quad -\text{cov}\left(\beta_{ZC}C_{ij'}+\epsilon_{Z_{ij'}},X_{ij}\right)+\text{cov}\left(\beta_{ZC}C_{ij'}+\epsilon_{Z_{ij'}},X_{ij'}\right) \\
&= \beta_{ZC}\text{cov}\left(C_{ij},X_{ij}\right)-\beta_{ZC}\text{cov}\left(C_{ij},X_{ij'}\right)-\beta_{ZC}\text{cov}\left(C_{ij'},X_{ij}\right)+\beta_{ZC}\left(C_{ij'},X_{ij'}\right) \\
&= \beta_{ZC}\text{cov}\left(C_{ij},\beta_{XC}C_{ij}+\epsilon_{X_{ij}}\right)-\beta_{ZC}\text{cov}\left(C_{ij},\beta_{XC}C_{ij'}+\epsilon_{X_{ij'}}\right) \\
&\quad -\beta_{ZC}\text{cov}\left(C_{ij'},\beta_{XC}C_{ij}+\epsilon_{X_{ij}}\right)+\beta_{ZC}\text{cov}\left(C_{ij'},\beta_{XC}C_{ij'}+\epsilon_{X_{ij}}\right) \\
&= \beta_{ZC}\beta_{XC}\sigma_C^2-\beta_{ZC}\beta_{XC}\sigma_C^2\rho_C-\beta_{ZC}\beta_{XC}\sigma_C^2\rho_C+\beta_{ZC}\beta_{XC}\sigma_C^2 \\
&= 2\beta_{ZC}\beta_{XC}\sigma_C^2-2\beta_{ZC}\beta_{XC}\sigma_C^2\rho_C \\
&= 2\beta_{ZC}\beta_{XC}\sigma_C^2\left(1-\rho_C\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{var}\left(X_{ij}-X_{ij'}\right) &= 2\left[\text{var}\left(X_{ij}\right)-\text{cov}\left(X_{ij},X_{ij'}\right)\right] \\
&= 2\left(1-\beta_{XC}^2\sigma_C^2\rho_C-\rho_{\epsilon_X}\sigma_{\epsilon_X}^2\right)
\end{aligned}
$$

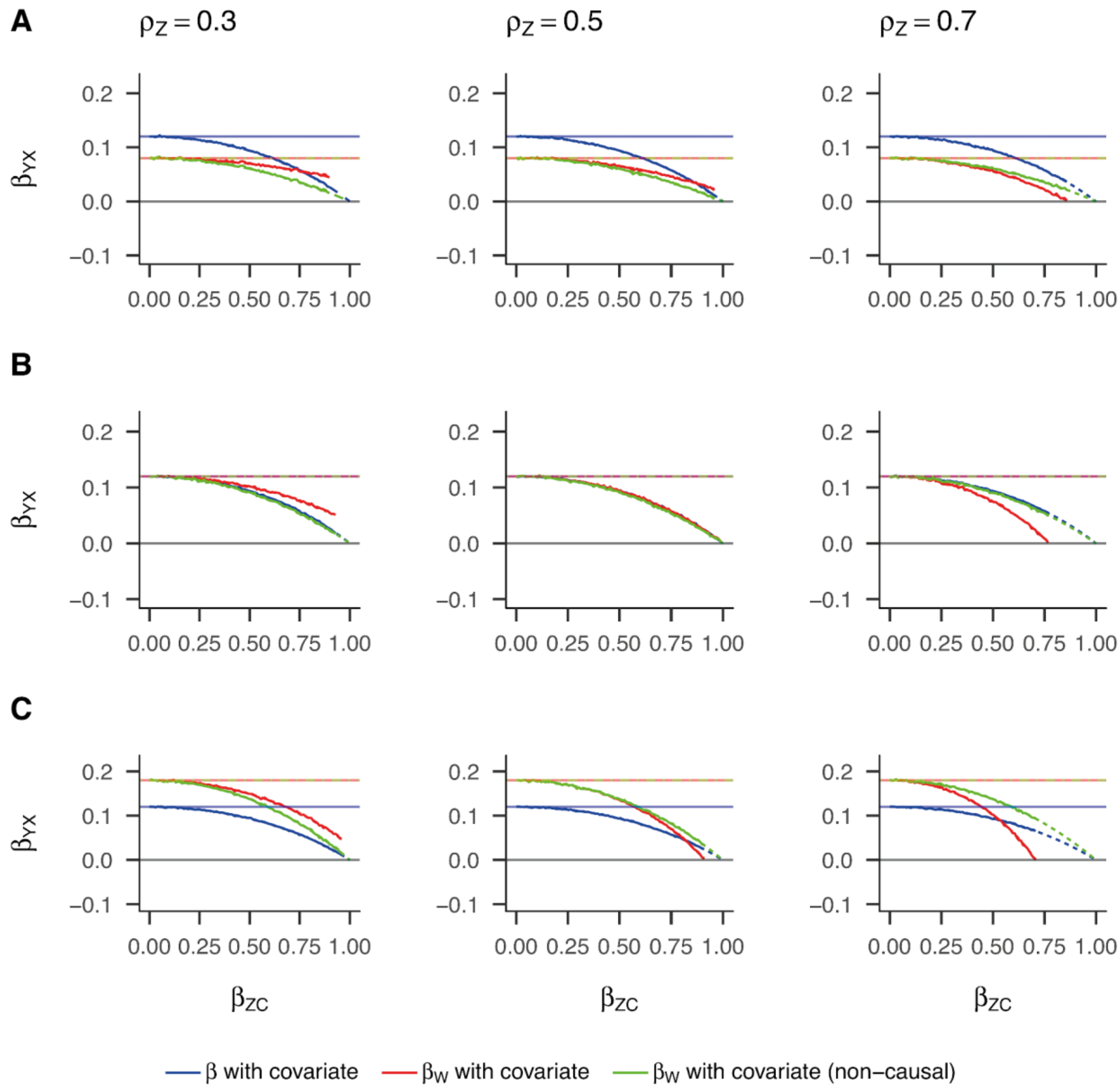
$$
\begin{aligned}
\text{var}\left(Z_{ij}-Z_{ij'}\right) &= 2\left[\text{var}\left(Z_{ij}\right)-\text{cov}\left(Z_{ij},Z_{ij'}\right)\right] \\
&= 2\left(1-\beta_{ZC}^2\sigma_C^2\rho_C-\rho_{\epsilon_Z}\sigma_{\epsilon_Z}^2\right)
\end{aligned}
$$

R code to replicate findings

```
## Estimate byx with confounding
##
##  Arguments
##    byx, effect of x on y
##    byc, effect of c on y
##    bxc, effect of c on x
##    bzc, effect of c on z
##    rho.X, twin correlation in X
##    rho.C, twin correlation in C
##    rho.Z, twin correlation in Z
##    n, number of twin pairs per replication
##    reps, number of replications
##
##  Returns crude = (y ~ x)
##          crude.cov = (y ~ x + c)
##          within = (y ~ (x - x.mean))
##          within.cov = (y ~ (x - x.mean) + (z - z.mean))
##          within.cov2 = (y ~ (x - x.mean) + z)

sim.confounding.lm = function(byx, byc, bxc, bzc, rho.X, rho.C, rho.Z, n, reps)
{
  # Define rho.Y
  rho.Y = 0.2

  # Error variances
  var.epsY = 1 - (byx ^ 2 + byc ^ 2 + 2 * byx * byc * bxc)
  var.epsX = 1 - bxc ^ 2
  var.epsZ = 1 - bzc ^ 2
  var.C = 1.0

  # Correlations (sibling)
  rho.epsY = (rho.Y - byx ^ 2 * rho.X - byc ^ 2 * rho.C - 2 * byx * byc *
              bxc * rho.C) / var.epsY
  rho.epsX = (rho.X - bxc ^ 2 * rho.C) / var.epsX
  rho.epsZ = (rho.Z - bzc ^ 2 * rho.C) / var.epsZ

  # Covariances (sibling)
  cov.epsY = var.epsY * rho.epsY
  cov.epsX = var.epsX * rho.epsX
  cov.epsZ = var.epsZ * rho.epsZ
  cov.C = rho.C

  # Error covariance matrix
  A = matrix(c(var.epsY,0,0,0,
             0,var.epsX,0,0,
             0,0,var.epsZ,0,
             0,0,0,1), nrow=4, byrow=T)

  B = matrix(c(cov.epsY,0,0,0,
             0,cov.epsX,0,0,
             0,0,cov.epsZ,0,
```

```r
             0,0,0,cov.C), nrow=4, byrow=T)

M = rbind(cbind(A,B),
          cbind(B,A))

# Cholesky decomposition
# Prints NA for all estimates when M matrix is singular
if (is.na(tryCatch(chol(M), error=function(err) NA))){
  crude.est = NA
  crude.cov.est = NA
  within.est = NA
  within.cov.est = NA
  within.cov.est2 = NA
} else {
  L = chol(M)
  nvars = dim(L)[1]

  # Random variables that follow an M error covariance matrix
  r = t(L) %*% matrix(rnorm(nvars*n), nrow=nvars, ncol=n)
  r = t(r)
  rdata = as.data.frame(r)
  names(rdata) = c('epsY1','epsX1','epsZ1','epsC1','epsY2','epsX2','epsZ2','epsC2')
  round(cov(rdata), 2)

  # Generate Y,X,Z,C for each member of a twin pair
  C1 = rdata$epsC1
  C2 = rdata$epsC2

  Z1 = bzc*C1 + rdata$epsZ1
  Z2 = bzc*C2 + rdata$epsZ2

  X1 = bxc*C1 + rdata$epsX1
  X2 = bxc*C2 + rdata$epsX2

  Y1 = byx*X1 + byc*C1 + rdata$epsY1
  Y2 = byx*X2 + byc*C2 + rdata$epsY2

  # Set up data
  id <- c(1:n, 1:n)
  y <- c(Y1,Y2)
  x <- c(X1,X2)
  c <- c(C1,C2)
  z <- c(Z1,Z2)
  mx = with(data.frame(cbind(X1,X2)), (X1+X2)/2)
  x.mean <- c(mx,mx)
  mz = with(data.frame(cbind(Z1,Z2)), (Z1+Z2)/2)
  z.mean <- c(mz,mz)
  dat <- data.frame(cbind(id, y, x, c, z, x.mean, z.mean))

  # The crude estimate w/o covariate
  crude = summary(tryCatch(lm(y ~ x, data=dat), error=function(err) NA))
  crude.est = crude$coef[2,1]
```

```r
    # The within-pair estimate w/o covariate
    within = summary(tryCatch(lm(y ~ I(x - x.mean), data=dat), error=function(err) NA))
    within.est = within$coef[2,1]

    # The crude estimate w/covariate
    crude.cov = summary(tryCatch(lm(y ~ x + z, data=dat), error=function(err) NA))
    crude.cov.est = crude.cov$coef[2,1]

    # The within-pair estimate w/covariate
    within.cov = summary(tryCatch(lm(y ~ I(x - x.mean) + I(z - z.mean),
                                  data=dat), error=function(err) NA))
    within.cov.est = within.cov$coef[2,1]
    within.cov2 = summary(tryCatch(lm(y ~ I(x - x.mean) + z, data=dat),
                                 error=function(err) NA))
    within.cov.est2 = within.cov2$coef[2,1]
  }
  return(list(crude=crude.est, crude.cov=crude.cov.est,
              within=within.est,  within.cov=within.cov.est,
              within.cov2=within.cov.est2))
}
```



Supplemental Figure. Exposure effect estimates with the inclusion of a covariate from individual-level and within-pair models when A) the within-pair correlation in the exposure is less than the within-pair correlation in the confounder; B) the within-pair correlation in the exposure equals the within-pair correlation in the confounder; C) the within-pair correlation in the exposure is more than the within-pair correlation in the confounder. For each scenario ρ_C = 0.5, while ρ_X varies between 0.3, 0.5, and 0.7 (consistent with Figure 2). Additionally, each column represents a different value of ρ_Z, the within-pair correlation in the covariate. β_{ZC} is the effect of the confounder on the covariate. Blue lines denote the exposure estimate from individual-level models, red lines denote the exposure estimate from CTC models as specified in equation 4, and green lines denote the exposure estimate from CTC models as specified in equation 3. Simulation results (solid lines) are overlaid on the derivation results (dashed lines) to show their concordance. Darker lines denote the exposure effect estimate with covariate inclusion while lighter shaded lines denote the same estimate without covariate inclusion. The true causal exposure effect is zero ($\beta_{YX} = 0$).