Appendices

A Power Accumulation before the Designation

We begin to discuss a variation of the model, in which the potential candidate may accumulate power before he is designated the successor. We start with the one-candidate case. The timing of this model in a generic period is as follows. At period t, if no successor exists, then the ruler decides whether or not to designate a successor. If she does, then the game follows the main model. If she does not, then the candidate may choose whether or not to increase his owner power. If the candidate chooses to do so, then his power becomes $S_t = S_{t-1} + L_t$ by increasing L_t units. Otherwise, his power remains at the same level, i.e., $S_t = S_{t-1}$. To simplify the analysis, we assume the following

Assumption 3. *$\{L_t\}$ is a positive sequence with $L > L_{t-1} > L_t > 0$ such that*

$$\widetilde{S} + \lim_{t\to\infty} \sum_{t=0}^{\infty} L_t = \bar{\bar{S}} \equiv \frac{b}{R+b}.$$

This assumption indicates that the increase in the candidate's power is a marginal decreasing process and that the maximal power he can accumulate is limited by $\bar{\bar{S}}$. This assumption implies that as an ordinary subordinate (not a successor) of the ruler, the candidate's power cannot be allowed to reach a certain level, which may threaten the ruler's throne.

In stage 3, the candidate chooses whether to challenge the ruler. If he does, then he will take the throne with probability $\min\{S_t, 1\}$. If he does not, then ruler may decide whether to exile the candidate in the last stage of this period. The payoffs remain as the main model.

The difference between this variation and our main model is that the candidate may accumulate power before the designation. His power increase is zero if he chooses not to do so. This difference implies that conflict is not necessary if the candidate is never designated as the successor, because the candidate may halt the increment of his power.

From the ruler's perspective, leaving the successor position unfilled is a dominated strategy, which is the same as the main model. When a successor is designated at time t^a, the strategies in this sub-game is the same as that in the main model when the initial power of the successor becomes S_{t^a}.

Before the designation, the candidate has no incentive to challenge the ruler by Assumption 3, because when $\widetilde{S}_t < \bar{\bar{S}}$, challenging the ruler gives the candidate a worse expected payoff than being a loyal subordinate. Meanwhile, the ruler will not exile the candidate before the designation when the candidate's power is less than $\bar{\bar{S}}$, because the candidate poses no threat to the ruler's throne. Considering that the candidate may control the power increase, the optimal strategy is to increase his power in each period, because he knows that he will become the successor sooner or later. Thus, an increase in power gives him more advantage to hold the throne if the ruler dies. Now we summarize these results as follows:

Proposition A.1. (i) *the candidate chooses to increase his power in each period before he becomes the successor, and chooses not to challenge the ruler.*

(ii) *The ruler does not exile the candidate in any period before she designates the successor.*

When the ruler chooses the optimal time of designation, she faces a candidate whose power is changing. Considering that the candidate's power is limited by $\bar{\bar{S}}$, the optimal designation time should not be later than when $\widetilde{S} = \bar{\bar{S}}$. In the main model, we know that the lower bound of optimal designation times is weakly increasing in $\widetilde{S}$. Therefore, the ruler should designate the successor no earlier than the corresponding time in the main model, because the candidate's power is increasing with time before the designation. These results are summarized in the next proposition.

Proposition A.2. *Given an initial power $\widetilde{S}$, there exist two periods—$\hat{t}'$ and $\hat{t}''$, with $1 \leq \hat{t}'(\widetilde{S}) \leq \hat{t}''(\widetilde{S})$—such that the ruler will not designate a successor sooner than $\hat{t}'$ or later than $\hat{t}''$. Both $\hat{t}'$ and $\hat{t}''$ are weakly increasing functions of $\widetilde{S}$.*

Now we discuss the multi-candidate case. When there are two candidates with initial powers of $\widetilde{S}^1$ and $\widetilde{S}^2$ respectively. Like the main model, the ruler needs to designate candidate 1 before candidate 2. We simply assume that $L_t^i < L_t^j$ is for all t when $\widetilde{S}^i < \widetilde{S}^j$. This assumption indicates that the candidate with a higher initial power has a larger marginal power increase in each period.

Before the designation of the first successor, the relative position of these two candidates' powers will not change, and the power gap between them increases with time. Therefore, all the intuition and results in the main model can be preserved in this situation (Proposition 7 and Proposition 8). We simply repeat them as follows:

Proposition A.3. *If candidate 1 has lower initial power than candidate 2 ($\widetilde{S}^1 < \widetilde{S}^2$), then there exists a $\hat{d}'_2 \geq 0$ such that the ruler and his first successor will not immediately conflict only if $\widetilde{S}^1 \geq \widetilde{S}^2 - \hat{d}'_2$.*

Proposition A.4. (i) *If $\widetilde{S}^1 > \widetilde{S}^2$, then both the lower and the upper bounds of the first successor's designation interval are weakly greater than their counterparts in the single-candidate case.*

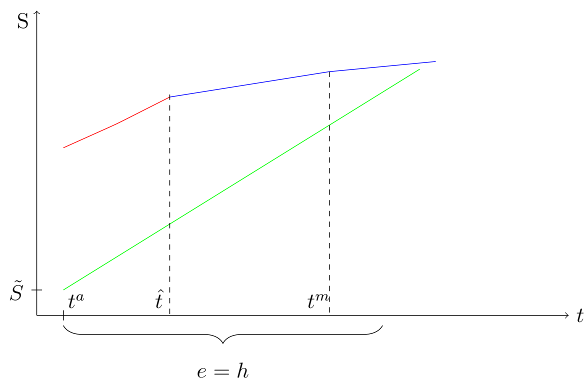
(ii) *If the optimal designation time for the first successor does not differ from that in the single-candidate case, then the duration of this successor–ruler's honeymoon phase will be weakly lower than that in the single-candidate case. Moreover, this honeymoon phase will be weakly less than that in Proposition 8 of the main model.*

Proposition A.3 is similar to Proposition 7. It indicates that the ruler has the incentive to strip the first successor as soon as possible when he is too weak to seize the power. Proposition A.4 is similar to Proposition 8. It indicates that the first successor is replaceable, and the ruler will prefer to choose any successor in a later period, thereby avoiding potential challenges. Moreover, the probability of conflict also increases in the presence of a backup candidate. The last part of this position shows that the probability of conflict in this situation is higher than in the main model with two candidates, because the backup candidate's power increases with time. Thus, the ruler is more likely to replace the successor with another candidate.

B Simulations

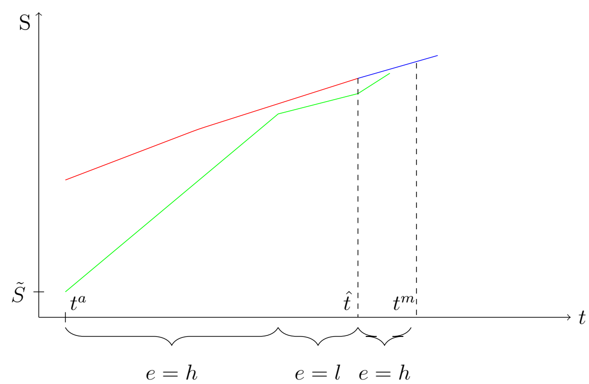
The conflict between the two parties is unlikely to occur under certain degenerated situations. Figure A.1 represents the scenario in which the ruler's health rapidly deteriorates, and the honeymoon phase is thus directly connected with the power transition phase. In this situation, the gap between the successor's power and the monitoring threshold is sufficient

Figure A.1



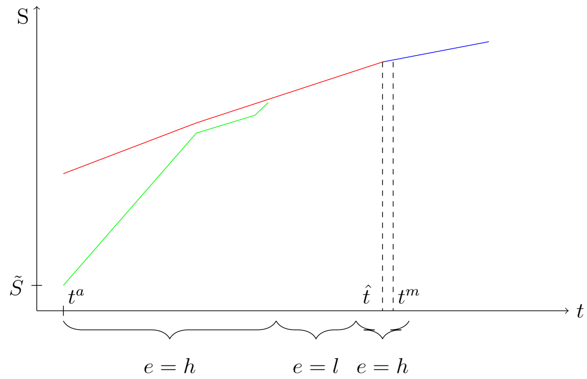
Notes: This figure represents the scenario that the ruler's health deteriorates quickly such that the honeymoon phase is connected with the power transition phase directly. This case indicate that the peaceful power transition is highly possible or the successor challenges the ruler with a high chance to win. The red curve represents the change of the monitoring thresholds $\bar{s}_t^m$ with time. The blue curve represents the change of the challenge thresholds $\bar{s}_t^c$ with time. When $t \leq \hat{t}$, $\bar{s}_t^m = \bar{s}_t^c$; when $\hat{t} < \bar{t}^m$, $\bar{s}_t^c < \bar{s}_t^m$; when $\bar{t}^m \leq t$, $\bar{s}_t^m$ does not exists. The green line represents the change of the expected power accrued by the successor with time. Since the successor always chooses high effort, the average power increase rate is $p_h(H - L) + L$. The parameters are chosen as follows: $b = 10$, $\delta = 0.7$, $R = 10$, $r = 0.1$, $L = 0.001$, $H = 0.01$, $p_h = 0.5$, $w = 0.05$, $\eta = 0.7$ $p_t = p_{t-1} + 0.01$, $p_0 = 0$, $\tilde{S} = 0.01$. Thresholds are calculated as: $\hat{t} = 6$, $t^m = 60$.

Figure A.2



Notes: This figure represents the scenario that the ruler's health deteriorates moderately such that after the honeymoon phase the successor has to keep a low profile to avoid the conflict and wait for the deterioration of the ruler's health. This situation indicates that the successor has a high chance to succeeds the throne. The red curve represents the change of the monitoring thresholds $\bar{s}_t^m$ with time. The blue curve represents the change of the challenge thresholds $\bar{s}_t^c$ with time. When $t \leq \hat{t}$, $\bar{s}_t^m = \bar{s}_t^c$; when $\hat{t} < \bar{t}^m$, $\bar{s}_t^c < \bar{s}_t^m$; when $\bar{t}^m \leq t$, $\bar{s}_t^m$ does not exists. The green line represents the change of the expected power accrued by the successor with time. When the successor chooses high effort, the average power increase rate is $p_h(H - L) + L$. When the successor chooses low effort, the power increase rate is L. The parameters are chosen as follows: $b = 10$, $\delta = 0.7$, $R = 10$, $r = 0.1$, $L = 0.001$, $H = 0.01$, $p_h = 0.5$, $w = 0.05$, $\eta = 0.7$ $p_t = p_{t-1} + 0.005$, $p_0 = 0$, $\tilde{S} = 0.29$. Thresholds are calculated as: $\hat{t} = 81$, $t^m = 119$.

Figure A.3



Notes: This figure depicts the scenario under which a ruler remains healthy for a relatively long time. This situation indicates that the successor has a high chance to conflict with the ruler and low chance to win. The red curve represents the change of the monitoring thresholds $\bar{s}_t^m$ with time. The blue curve represents the change of the challenge thresholds $\bar{s}_t^c$ with time. When $t \leq \hat{t}$, $\bar{s}_t^m = \bar{s}_t^c$; when $\hat{t} < \bar{t}^m$, $\bar{s}_t^c < \bar{s}_t^m$; when $\bar{t}^m \leq t$, $\bar{s}_t^m$ does not exists. The parameters are chosen as follows: $b = 10$, $\delta = 0.7$, $R = 10$, $r = 0.1$, $L = 0.001$, $H = 0.01$, $p_h = 0.5$, $w = 0.05$, $\eta = 0.7$ $p_t = p_{t-1} + 0.001$, $p_0 = 0$, $\tilde{S} = 0.01$. Thresholds are calculated as: $\hat{t} = 404$, $t^m = 592$.

to avoid the conflict during the period $\hat{t}$ so that the successor does not worry about being stripped of his title. Thereafter, the successor will challenge the throne only if his power reaches his challenge threshold (blue line in Figure A.1). This scenario can be observed in reality: if a weak (or young) successor is designated by an old ruler, then a conflict is unlikely to occur. Figure A.2 presents another scenario in which the ruler's health worsens at a moderate speed. Although the power increase of the successor is sufficiently large to raise the suspicion of the ruler, the successor can maintain his position until period $\hat{t}$ by keeping a low profile and lowering the power increase. Figure A.3 presents the scenario in which the ruler's health worsens slowly, then as the result in Proposition 5, the successor needs to maintain a low profile after the honeymoon phase, however if the ruler lives long enough, the conflict is unavoidable.

C Math Proofs

First we need to introduce some notations: $V^g_{t,j}(S)$ denotes the value function of player g at stage j in period t with status variable S. Furthermore, to simplify the notation for the case that the power of the successor is no greater than 1, let $V^g_{t,j}(S) = V^g_{t,j}(1)$ if $S > 1$. When $j = 1$, we sometime drop j to represent the value function of that period, i.e. $V^g_t \equiv V^g_{t,1}$.

For the ruler, in period t, her value function at stage 1 of this period is

$$V^m_{t,1}(S_{t-1}) = p_t \min(S_{t-1} + w, 1)\eta R + (1 - p_t)V^m_{t,2}(S_{t-1}). \tag{3}$$

The value function at the beginning of the stage 2 is

$$V^m_{t,2}(S_{t-1}) = \begin{cases} p_h V^m_{t,3}(S_{t-1} + H) + (1 - p_h)V^m_{t,3}(S_{t-1} + L) & \text{if } e = h, \\ V^m_{t,3}(S_{t-1} + L) & \text{if } e = l. \end{cases} \tag{4}$$

The value function at the beginning of the stage 3 is

$$V^m_{t,3}(S_t) = \begin{cases} -(b + k_t) \min(S_t, 1) + k_t & \text{if being challenged}, \\ V^m_{t,4}(S_t) & \text{if not}, \end{cases} \tag{5}$$

where $k_t \equiv r + r \sum_{i=t+1}^{\infty} \delta^{i-t} \prod_{j=t+1}^{i} (1 - p_j)$.

The value function at the beginning of the stage 4 is

$$V^m_{t,4}(S_t) = \begin{cases} r + \delta V^m_{t+1,1}(S_t) & \text{if keep the successor}, \\ k_t & \text{if strip the successor} \end{cases} \tag{6}$$

Proof of Lemma 1. This lemma is equivalent that maintaining the successor's position is a dominant strategy for the ruler in any subgame-perfect equilibrium of the subgame starting in period $t \geq \bar{t}^m$.

At stage 4 of any t, if the ruler keeps the successor, the value function has the following property:

$$\begin{aligned} V^m_{t,4}(S_t) &= r + \delta[p_{t+1}(\min(S_t + w, 1))\eta R + (1 - p_{t+1})V^m_{t+1,1}(S_t)], && (7) \\ &\geq r + \delta[p_{t+1}(\min(\widetilde{S} + w, 1)\eta R + b) - b]. && (8) \end{aligned}$$

The inequality comes from that $V^m_{t+1,1}(S_t) > -b$ and $S_t \geq \widetilde{S}$. If the ruler strips the successor, his expected payoff is k_t.

We denote the right hand side of (8) as f_t. It is an increasing function of t, and $\lim_{t\to\infty} f_t = r + \delta \min(\widetilde{S} + w, 1)\eta R$. We also know k_t is decreasing with t and $\lim_{t\to\infty} k_t = r$. Therefore, there exists t^1 such that $f_t \geq k_t$ when $t > t^1$. Furthermore, there exists $\bar{t}^m$ with $\bar{t}^m \leq t^1$ such that $V^m_{t,4}(S) \geq k_t$ for any $S \in [0,1]$ when $t > \bar{t}^m$. So the ruler will not strip the successor after period $\bar{t}^m$. $\square$

For the successor, in period t, his value function at stage 1 of this period is

$$V^c_{t,1}(S_{t-1}) = p_t(\min(S_{t-1} + w, 1)(R + b) - b) + (1 - p_t)V^c_{t,2}(S_{t-1}). \tag{9}$$

The value function at the beginning of the stage 2 is

$$V^c_{t,2}(S_{t-1}) = \begin{cases} p_h V^c_{t,3}(S_{t-1} + H) + (1 - p_h)V^c_{t,3}(S_{t-1} + L) & \text{if } e = h, \\ V^c_{t,3}(S_{t-1} + L) & \text{if } e = l. \end{cases} \tag{10}$$

The value function at the beginning of the stage 3 is

$$V^c_{t,3}(S_t) = \begin{cases} \min(S_t, 1)(R + b) - b & \text{if challenge,} \\ V^c_{t,4}(S_t) & \text{if not.} \end{cases} \tag{11}$$

The value function at the beginning of the stage 4 is

$$V^c_{t,4}(S_t) = \begin{cases} \delta V^c_{t+1,1}(S_t) & \text{if not being stripped,} \\ -b & \text{if being stripped} \end{cases} \tag{12}$$

Proof of Proposition 1. In period t with $t \geq \bar{t}^m$, since the successor will not be stripped by the ruler, he always chooses h to increase the power at stage 2.

At stage 3, if he challenges the ruler, his payoff is

$$(R + b)S_t - b. \tag{13}$$

If he remains loyal, his payoff is

$$\delta[p_{t+1}(\min(S_t + w, 1)(R + b) - b) + (1 - p_{t+1})(p_h V^c_{t+1,2}(S_t + H) + (1 - p_h)V^c_{t+1,2}(S_t + L))]. \tag{14}$$

Since $lim_{t\to\infty}p_t = 1$, for a given very small $\epsilon > 0$, there exists a t^b such that equation (14) is less than $\delta(\min(S_t + w, 1)(R + b) - b) + \epsilon$ when $t > t^b$.

Since the successor's strategies after t^b all follow the same equilibrium strategy when $t = t^b$, we only need to consider the strategies of the successor when $t \leq t^b$. We will use the induction in the backward induction procedure. First, we will prove there is an unique cut-off strategy for the successor to challenge the ruler in period t^b. Then we will show that if this cut-off rule exists in any $t' < t^b$ uniquely, then it is also true for period $t' - 1$.

When $t = t^b$, without loss of generality, equation (14) can be rewritten as $\delta(\min(S_t + w, 1)(R + b) - b)$ by ignoring the error term.

Consider equation

$$(R + b)S - b = \delta((S + w)(R + b) - b), \tag{15}$$

the root is $S^2 = \frac{b}{R+b} + \frac{\delta w}{1-\delta}$.

For any $\bar{t}^m < t' < t^b$ (t' always exists because we can let t^b sufficiently large such that $t^b > \bar{t}^m$), we use the induction to characterize $V^c_{t',3}$. We claim $V^c_{t',3}$ satisfies the following properties:

C1, $V^c_{t',3}$ is a continuous piecewise linear function when $S \in [0, 1]$, i.e. $\{a^c_j S + d^c_j | q^c_j \leq S < q^c_{j+1}\}$ with $q^c_1 = 0 < q^c_2 < \cdots < q^c_n < q^c_{n+1} = 1$ and $a^c_j \geq 0$;

C2, $a^c_j < a^c_{j'}$ with $j < j'$, except when $a^c_{j'} = 0$,

C3, the last segment of $V^c_{t',3}$ is $S(R + b) - b$.

If C1-C3 are true, then the starting point of the last segment is the cut-off the successor adopts to challenge the throne. Now we begin to use the induction to prove the claim. The intuition of this proof is the payoff of remaining loyal is a continuous piecewise linear function and the slope of each segment is less than the slope of the payoff function when the successor challenges the ruler. Then there will be a intersection.

Two cases need to discuss. When δ is small, the payoff function of remaining loyal will directly intersect the payoff function of challenging; when δ is large, the former will intersect with the constant R before it intersects with the latter, and this situation needs to be discussed separately.

Let $\bar{\delta} \equiv 1 - w - wb/R$. Then we consider two cases. When $\delta > \bar{\delta}$, if the successor remains loyal, his expected payoff is $\delta(\min(S + w, 1)(R + b) - b)$, which intersects with $(R + b)S - b$ at $S^3 \equiv (\delta R + b)/(R + b) < S^2$. Therefore, the successor will challenge the ruler if $S > S^3$, and remain loyal otherwise. $V^c_{t',3} = \delta(S(R + b) - b)$ if $S < 1 - w$, $V^c_{t',3} = \delta R$ if $1 - w \leq S < S^3$, and $V^c_{t',3} = S(R+b) - b$ if $S^3 \leq S$. So it satisfies C1-C3. We also know the first segment of $V^c_{t',3}$ has the linear form $a^c_1 S + d^c_j$ and it intersects with δR at a point that is less than S^2 when $\delta > \bar{\delta}$.

Then, for any t', if the successor remains loyal, his expected payoff at stage 3 is $\min(\delta[p_{t'}((S_{t'-1} + w)(R+b) - b) + (1-p_{t'})(aS_{t'-1} + p_h(H-L) + L + b)], \delta R)$. Then $\delta[p_{t'}(S_{t'-1} + w(R+b) - b) + (1-p_{t'})(aS_{t'-1} + p_h(H-L) + L + b)$ intersects $(R+b)S - b$ at $S'_2 > S^2$ and intersects δR before S^3. Therefore in period t', the payoff of remaining loyal is $\delta[p_{t'}(\min(S_{t'-1} + w, 1)(R + b) - b) + (1-p_{t'})(a\min(S_{t'-1} + p_h(H-L) + L, 1) + b)]$.

So $V^c_{t',2}$ also satisfies C1-C3, and the cutoff threshold to challenge the ruler is still S^3. This situation represents the case that the successor is patient, he will not challenge the ruler unless his power is sufficient large.

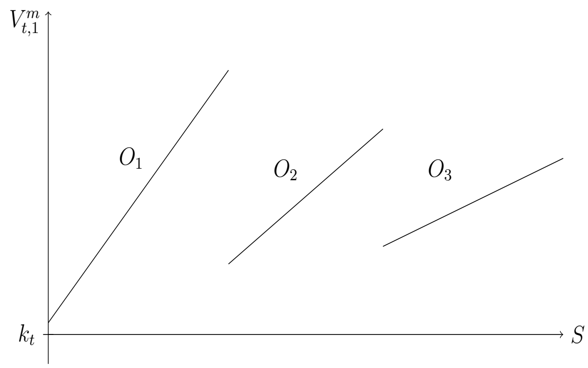
When $\delta < \bar{\delta}$, we have $S^2 < S^3$. In period t', let $\bar{s}^c_{t'} = S^2$, and the successor should challenge if $S_{t'} > \bar{s}^c_{t'}$ and remain loyal otherwise. Therefore $V^c_{t',3} = \delta(S_{t'}(R+b) - b)$ if $S_{t'} < S^2$, and $V^c_{t',3} = S(R+b) - b$ if $S^2 \leq S_{t'}$. So it satisfies C1-C3.

Assume $V^c_{t',2}$ satisfies C1-C3 at t'. At $t'-1$, if the successor remains loyal, then his expected payoff is $\delta(p_{t'}((S_{t'}+w)(R+b)-b)) - (1-p_{t'})(p_h V^c_{t',3}(S_{t'}+H) + (1-p_h)(V^c_{t',3}(S_{t'}+L))))$. This function is still piecewise linear. Also, the slope of each segment of this function satisfies C2, and the last segment $\delta(R+b)(S_{t'-1} + p_{t'}w + (1-p_{t'})(p_h(H-L)+L)) - b$ has the largest slope, and it intersects $S_{t'-1}(R+b) - b$ at $\bar{s}^c_{t'-1} \equiv b/(R+b) + (p_{t'-1}w + (1-p_{t'-1})(p_h(H-L) + L))\delta/(1-\delta)$. By assumption 1, we also have $\bar{s}^c_{t'-1} < \bar{s}^c_{t'}$ and less than S^2 and S^3. Therefore the successor will challenge the ruler if $S_{t'-1} > \bar{s}^c_{t'-1}$, and not otherwise. Furthermore, $\bar{s}^c_{t'-1}$ approaches $\bar{s}_{lim} = b/(R+b) + \delta(p_h(H-L) + L + w)/(1-\delta)$ when p_t goes to 0. Therefore, no $\bar{s}^c_{t'}$ is less than $\bar{s}_{lim}$.

Moreover $V^c_{t'-1,3}(S) = \delta(p_{t'}((S+w)(R+b)-b)) - (1-p_{t'})(p_h V^c_{t',3}(S+H) + (1-p_h)(V^c_{t'-1,3}(S + L))))$ if $S_{t'-1} \leq \bar{s}^c_{t'-1}$, and $V^c_{t'-1,3} = S(R+b) - b$ if $S > \bar{s}^c_{t'-1}$. Therefore, $V^c_{t'-1,3}(S)$ satisfies C1-C3. □

Proof of Proposition 2. We need several steps to finish this proof. First, from Proposition 1, we have characterized $V^c_{t,3}$ when $t \geq \bar{t}^m$, we also need to characterize the value function of the ruler when $t \geq \bar{t}^m$. This procedure still needs to use the induction in the backward induction. After that, since the equilibrium is unique after $\bar{t}^m$, then we can use $\bar{t}^m$ as the new starting point to use the backward induction to characterize the equilibrium strategies before $\bar{t}^m$, of course we need to use the induction procedure again, i.e. find the general strategy in

Figure A.4



period t, then prove it is also true in period $t-1$.

The technological intuition of this proof is that if the ruler keeps the successor in $t < \bar{t}^m$, the ruler's value function will increases with the successor's power S first and it is always larger than the return of stripping the successor. When S is large, this value function begins to decrease with S and when $S = 1$ the value function is less than the return of stripping the successor. So there is a unique cutoff for the ruler to strip the successor. The difficulty of this proof is the ruler's value function is a combination of a set of piecewise linear segments. The construction of the entire proof is to show that different linear combinations of these segments will not affect the uniqueness of the cutoff.

We begin to consider the ruler's strategy. In the first intermediate result, we will show that $V^m_{t,1}(S)$ is composed of a set of piecewise linear functions(M1); and the slopes of these linear segments is decreasing(M2); the end point of each segment is greater than the starting point of the next segment (M3); the intercepts of these linear segments is decreasing (M4);

Claim 1. *When $t \geq \bar{t}^m$, the ruler's value function at stage 1 of period t, $V^m_{t,1}(S)$, satisfies the following properties (Figure A.4).*
M1, $V^m_{t,1}$ is composed of a set of piecewise linear functions, i.e. $V^m_{t,1} = \{a^m_j S + d^m_j | q^m_j \leq S < q^m_{j+1}\}$ with $q_1 = 0 < q_2 < \cdots < q_n < q_{n+1} = 1$. We use $O_j \equiv a^m_j S + d^m_j$ with $q^m_j \leq S < q^m_{j+1}$ to denote each segment of $V^m_{t,1}$, i.e. $V^m_{t,1}(S) = \cup O_j(S)$
M2, $a^m_j > a^m_{j+1}$;
M3, $a^m_j q^m_{j+1} + d^m_j > a^m_{j+1} q^m_{j+1} + d^m_{j+1}$;
M4, $d^m_j > d^m_{j+1}$.

Proof of Claim 1. In period t, if the successor is not stripped, the ruler's expected payoff at stage 4 is

$$r + \delta[p_{t+1}(S_t + w)\eta R + (1 - p_{t+1})V^m_{t+1,1}(S_t)] \tag{16}$$

Let's start in a sufficient large period $t^b > \bar{t}^m$ such that equation (16) can be rewritten as $r + \delta(S_{t^b} + w)\eta R + \epsilon$ and simply denoted as $r + \delta(S_{t^b} + w)\eta R$ by dropping the error term. Since the successor always chooses the high effort level after period $\bar{t}^m$, and he challenges the ruler at stage 3 if $S_{t^b} > \bar{s}^c_{t^b}$, then the value function of ruler at stage 2 of period t^b is

$$V^m_{t^b,2}(S_{t^b-1}) = \begin{cases} r + \delta(S_{t^b-1} + p_h(H-L) + L + w)\eta R & \text{if } S_{t^b-1} < \bar{s}^c_{t^b} - H, \\ p_h(r + \delta(S_{t^b-1} + w)\eta R) + (1-p_h)(-(b+k_t)S_{t^b-1} + k_t) & \text{if } \bar{s}^c_{t^b} - H \le S_{t^b-1} < \bar{s}^c_{t^b} - L, \\ -(b+k_t)(S_{t^b-1} + p_h(H-L) + L) + k_t & \text{if } \bar{s}^c_{t^b} - L \le S_{t^b-1}. \end{cases} \tag{17}$$

Then at stage 1, $V^m_{t^b,1}(S_{t^b-1}) = p_t(S_{t^b-1} + w)\eta R + (1-p_t)V^m_{t^b,2}(S_{t^b-1})$. It is easy to check $V^m_{t^b,1}(S_{t^b-1})$ satisfies M1-M4.

At stage 4 of period $t-1$, $V^m_{t-1,4}(S) = r + \delta k_t$ if the ruler strips the successor; and $V^m_{t-1,4}(S) = r + \delta V^m_{t,1}(S)$ otherwise. At stage 3, since the successor challenges the ruler if $S > \bar{s}^c_{t-1}$, then $V^m_{t-1,3} = r + \delta V^m_{t,1}(S)$ if $S \le \bar{s}^c_{t-1}$, and $-(b + r + \delta k_t)S + r + \delta k_t$ if $S > \bar{s}^c_{t-1}$.

$V^m_{t-1,2} = p_h V^m_{t-1,3}(S+H) + (1-p_h)V^m_{t-1,3}(S+L)$. It is easy to find $V^m_{t-1,2}$ is a piecewise linear function (M1). Any segment O_v of $V^m_{t-1,2}$ can be written into a form: $p_h(a_j(S+H) + d_j) + (1-p_h)(a_{j'}(S+L) + d_{j'})$. The next segment is either $O_{v+1}(S) \equiv p_h(a_{j''}(S+H) + d_{j''}) + (1-p_h)(a_{j'}(S+L) + d_{j'})$ with $j \ge j''$ (case 1) or $O_{v+1}(S) \equiv p_h(a_j(S+H) + d_j) + (1-p_h)(a_{j'''}(S+L) + d_{j'''})$ with $j''' \ge j'$ (case 2).

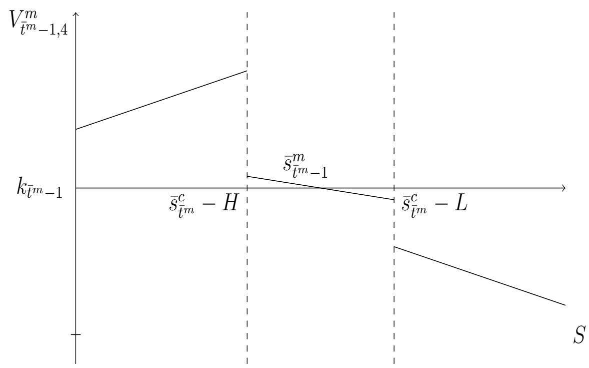
In either case, the slope of O_v is greater than the slope of O_{v+1} (M2). Then the end point of O_v must be either $q_{j+1} - H$ or $q_{j'+1} - L$. When it is the former, $O_v(q_{j+1} - H) > O_{v+1}(q_{j+1} - H)$ (case 1). When it is the latter, we have $O_v(q_{j'+1} - L) > O_{v+1}(q_{j'+1} - L)$ (case 2). Therefore M3 is satisfied. The intercept of O_v is $p_h(a_j H + d_j) + (1-p_h)(a_{j'}L + d_{j'})$, and the intercept of O_{v+1} is either $p_h(a_{j''}H + d_{j''}) + (1-p_h)(a_{j'}L + d_{j'})$ or $p_h(a_j H + d_j) + (1-p_h)(a_{j'''}L + d_{j'''})$, then $V^m_{t-1,2}(S)$ also satisfied M4. Since $V^m_{t-1,1}(S) = p_{t-1}(S+w)\eta R + (1-p_{t-1})V^m_{t-1,2}$, it preserves all four properties. □

Claim 2. *At $\bar{t}^m - 1$, a tolerance threshold $\bar{s}^m_{\bar{t}_m - 1}$ exists for the ruler. (Figure A.5)*

Proof of Claim 2. For any $t \ge \bar{t}^m$, the last segment of $V^m_{t-1,1}(S)$ is

$$p_{t-1}(S+w)\eta R + (1-p_{t-1})[-(b+k_{t-1})(S + p_h(H-L) + L) + k_{t-1}], \tag{18}$$

Figure A.5



and the segment before the last is

$$p_{t-1}(S+w)\eta R + (1-p_{t-1})[p_h(-(b+k_{t-1})(S+H)+k_{t-1}) + (1-p_h)V^m_{t-1,3}(S+L)]. \quad (19)$$

We know $V^m_{t-1,3}(S+L) \geq -(b+k_{t-1})(S+H)+k_{t-1}$, it is because $V^m_{t-1,3}(S+L)$ is either $-(b+k_{t-1})(S+L)+k_{t-1}$, which is greater than $-(b+k_{t-1})(S+H)+k_{t-1}$, or $V^m_{t-1,4}(S+L)$ which is greater or equal to k_{t-1} due to the ruler can strip the successor at stage 4. Therefore equation (19) is greater or equal to $p_{t-1}(S+w)\eta R + (1-p_{t-1})[(-(b+k_{t-1})(S+H)+k_{t-1})]$. By Assumption 2, equation (19) is greater than $(1-p_{t-1})k_{t-1}$. Then all other segments' intercept is greater than $(1-p_{t-1})k_{t-1}$, except the last one.

In each period t, the last segment's slope is increasing with t, it is because p_t and $-(b+k_t)$ is increasing with t. Similarly, for any segment of $V^m_{t,1}$ between $\bar{s}^c_t - H$ and $\bar{s}^c_t - L$ we have the form $p_t(S+w)\eta R + (1-p_t)[p_h(-(b+k_t)(S+H)+k_t) + (1-p_h)(\delta V^m_{t+1,1}(S+L)+r)]$. Since $p_{t+1} > p_t$, the segment between $\bar{s}^c_t - H$ and $\bar{s}^c_t - L$ has a smaller slope in period t than the slope of the corresponding part in $t+1$. Therefore, when t decreases, we can find the ruler should strip the successor if $S_t > \bar{s}^m_{\bar{t}^m-1}$ in period $\bar{t}^m - 1$. □

Claim 3. *A $\hat{t}$ exists with $\hat{t} \leq \bar{t}^m - 1$, such that when $\hat{t} \leq t \leq \bar{t}^m$, the successor always chooses h, $\bar{s}^m_t > \bar{s}^c_t$ and $\bar{s}^m_t - \bar{s}^m_{t-1} \leq \bar{s}^m_{t-1} - \bar{s}^m_{t-2}$.*

Proof of Claim 3. Now we begin to consider the period before $\bar{t}^m$. Since the successor will not set up the challenge threshold greater than the tolerance threshold, i.e $\bar{s}^m_t \geq \bar{s}^c_t$ when $t < \bar{t}^m$. If $\bar{s}^m_t > \bar{s}^c_t$, the successor still choose high effort at t. Since $\bar{s}^m_t > \bar{s}^c_t$, only the successor's challenging threshold matters, so we can follow the similar procedure in the

Figure A.6

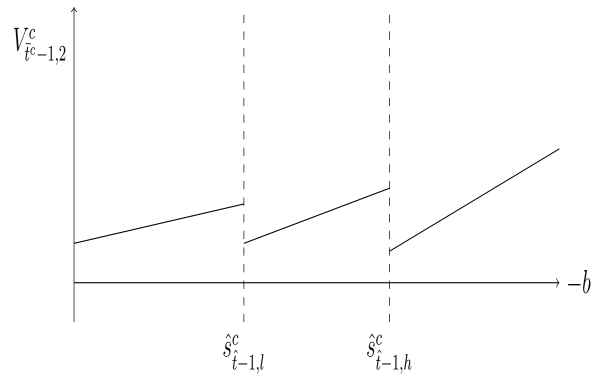
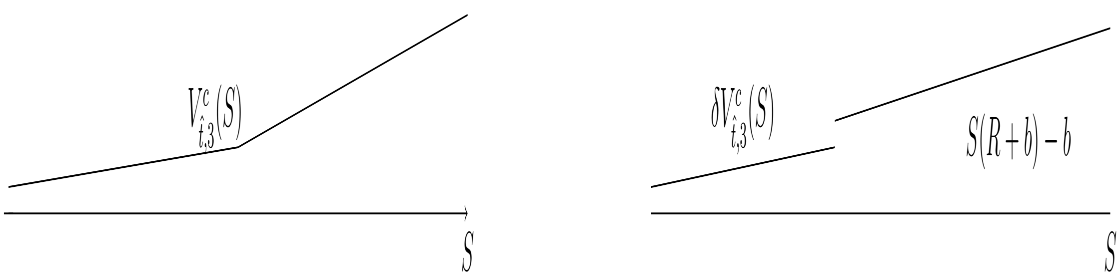


Figure A.7



proof of Claim 2. Since the slope of the segment of $V^m_{t,1}$ becomes smaller when t decreases, so $\bar{s}^m_t - \bar{s}^m_{t-1} \leq \bar{s}^m_{t-1} - \bar{s}^m_{t-2}$. When the successor does not considering the risk of being stripped, then his challenging threshold is determined by the intersect of $(R+b)S_{t-1} - b$ and $\delta[p_t((S_t + w)(R+b) - b) + (1-p_t)(p_h V^c_{t,2}(S_{t-1} + H) + (1-p_h)V^c_{t,2}(S_{t-1} + L))$. This intersection is bounded by $\bar{s}_{\lim}$ from below. Moreover, since $p_{t-1} < p_t$, the gap between the successive these intersections decreases as t decreases. On the contrary, since the gap between the successive tolerance thresholds increases as t deceases, therefore, a $\hat{t}$ exists such that $\bar{s}^m_{\hat{t}} = \bar{s}^c_{\hat{t}}$, otherwise let $\hat{t} = 0$. □

Claim 4. *(i) In period $\hat{t}-1$, two thresholds, $\hat{s}^c_{\hat{t}-1,h}$ and $\hat{s}^c_{\hat{t}-1,l}$, exist for the successor, such that the successor chooses h when $S_{\hat{t}-1} \leq \hat{s}^c_{\hat{t}-1,h}$ or $S_{\hat{t}-1} > \hat{s}^c_{\hat{t}-1,l}$, and chooses l otherwise (Figure A.6).*

(ii) There is one challenge threshold for the successor, $\bar{s}^c_{\hat{t}-1} = \bar{s}^m_{\hat{t}-1}$

(iii) The ruler's value function $V^m_{\hat{t}-1,1}$ still satisfies M1-M4.

Proof of Claim 4. Next we focus on the equilibrium strategies when $t \le \hat{t}$, and assume $\delta < \bar{\delta}$. In period $\hat{t}$, stage 1, the value function $V^c_{\hat{t},3}(S_{\hat{t}})$ is a continuous piecewise linear function with property C1-C3 (similar as the proof of Proposition 1).

At stage 4 of period $\hat{t}-1$, if the successor is stripped, then his payoff is $-b$. Therefore, at stage 3, he cannot remain loyal if his power $S_{\hat{t}-1} > \bar{s}^m_{\hat{t}-1}$, instead he should challenge. So the challenge threshold for the successor is $\bar{s}^c_{\hat{t}-1} = \bar{s}^m_{\hat{t}-1}$, his value function can be written as

$$V^c_{\hat{t}-1,3}(S_{\hat{t}-1}) = \begin{cases} \delta V^c_{\hat{t},1}(S_{\hat{t}-1}) & \text{if} \quad S_{\hat{t}-1} \le \bar{s}^c_{\hat{t}-1}, \\ S_{\hat{t}-1}(R+b)-b & \text{if} \quad S_{\hat{t}-1} > \bar{s}^c_{\hat{t}-1} \end{cases} \tag{20}$$

This value function is not continuous at $\bar{s}^c_{\hat{t}-1}$, because $\delta V^c_{\hat{t},1}(\bar{s}^c_{\hat{t}-1}) > \bar{s}^c_{\hat{t}-1}(R+b)-b$. Define $\Delta \equiv \delta V^c_{\hat{t},1}(\bar{s}^c_{\hat{t}-1}) - \bar{s}^c_{\hat{t}-1}(R+b)+b$ (See Figure A.7).

At stage 2 of period $\hat{t}-1$, if the successor chooses l, then his power becomes $S_{\hat{t}-1} = S_{\hat{t}-2}+L$. So his value function is $V^c_{\hat{t}-1,3}(S_{\hat{t}-2}+L)$. When he chooses h, his value function is $p_h V^c_{\hat{t}-1,3}(S_{\hat{t}-2}+H)+(1-p_h)V^c_{\hat{t}-1,3}(S_{\hat{t}-2}+L)$. When $S_{\hat{t}-2} < \bar{s}^c_{\hat{t}-1}-H$, we have $V^c_{\hat{t}-1,3}(S_{\hat{t}-2}+H) > V^c_{\hat{t}-1,3}(S_{\hat{t}-2}+L)$. Then there exists a $\hat{s}^c_{\hat{t}-1,l}$ such that $V^c_{\hat{t}-1,3}(S_{\hat{t}-2}+H) < V^c_{\hat{t}-1,3}(S_{\hat{t}-2}+L)$ when $\bar{s}^c_{\hat{t}-1}-H < S_{\hat{t}-2} \le \hat{s}^c_{\hat{t}-1,l}$, and $V^c_{\hat{t}-1,3}(S_{\hat{t}-2}+H) > V^c_{\hat{t}-1,3}(S_{\hat{t}-2}+L)$ when $S_{\hat{t}-2} > \hat{s}^c_{\hat{t}-1,l}$. Intuitively, the existence of $\hat{s}^c_{\hat{t}-1,2}$ comes from that (i) at point $\bar{s}^c_{\hat{t}-1}-H$, $\delta V^c_{\hat{t}-1,3}(S+H) > \delta V^c_{\hat{t}-1,3}(S+L) > (S+H)(R+b)-b$, it guarantees the payoff from the low effort is large than the high effort level due to the gap at point $\bar{s}^c_{\hat{t}-1}-H$; (ii) because the slope of $(S+H)(R+b)-b$ is greater than the slope of any segment of $\delta V^c_{\hat{t}-1,3}(S+L)$ when $S \in [\bar{s}^c_{\hat{t}-1}-H, \hat{s}^c_{\hat{t}-1,l}]$, we have the payoff from the high effort level is increasing faster than the payoff from the low effort level; (iii), when $S > \hat{s}^c_{\hat{t}-1,l}$, $(S+H)(R+b)-b > (S+L)(R+b)-b$, which can make sure that when $S > \hat{s}^c_{\hat{t}-1,l}$, the payoff from the high effort is greater than that from the low effort. Now we define $\hat{s}^c_{\hat{t}-1,h} \equiv \bar{s}^c_{\hat{t}-1}-H$. □

Now, we are ready to prove the general case for $t < \hat{t}$. The value function $V^m_{\hat{t},1}$ preserves the properties M1-M4, because the successor always chooses high effort in each period since $\hat{t}$. Consider the ruler's strategy at period $\hat{t}-2$, stage 3. If he strips the successor, the expected payoff is $r+\delta k_{\hat{t}-2}$. If he keeps the successor, the expected payoff is $r+\delta V^m_{\hat{t}-1,1}$. So we focus on $V^m_{\hat{t}-1,1}$. At period $\hat{t}-1$, stage 2, the successor challenges the ruler if $S_{\hat{t}-1} > \bar{s}^c_{\hat{t}-1} = \bar{s}^m_{\hat{t}-1}$. Since, in stage 1, the successor chooses high effort if $S_{\hat{t}-1} \ge \bar{s}^c_{\hat{t}-1}-H$, therefore the segment of $V^m_{\hat{t}-1,1}$ before $\bar{s}^c_{\hat{t}-1}-H$ is $r+\delta(S+w+p_h(H-L)+L)\eta R$. Since the successor chooses low effort if $\bar{s}^c_{\hat{t}-1}-H \le S_{\hat{t}-1} < \hat{s}^c_{\hat{t}-1,h}$, then the segment of $V^m_{\hat{t}-1,1}$ is $r+\delta(S+w+L)\eta R$

between $\bar{s}^c_{\hat{t}-1} - H$ and $\hat{s}^c_{\hat{t}-1,h}$. Due to M2-M4, any segments of $r + \delta V^m_{\hat{t},1}$ moved paralleled into the interval $[\bar{s}^c_{\hat{t}-1} - H, \bar{s}^c_{\hat{t}-1} - L]$ has smaller slope than $r + \delta(S + w + L)\eta R$ and lower value. Therefore the segments of $V^m_{\hat{t}-1,2}$ after $\hat{s}^c_{\hat{t}-1,h}$ are $(r + \delta(p_h \delta V^m_{\hat{t},3}(S+H)) + (1-p_h)V^m_{\hat{t},3}(S+L))$. Now $V^m_{\hat{t}-1,2}$ satisfies M1-M4. Finally $V^m_{\hat{t}-1,1} = p_t((S_{\hat{t}-1} + w)\eta R) + (1-p_t)V^m_{\hat{t}-1,2}(S_{\hat{t}-1})$ satisfies M1-M4.

Now we restate the assumption for the successor as follows for any $t < \hat{t}$:

C1', $V^c_{t',2}$ is a piecewise linear function when $S \in [0,1]$, i.e. $\{a^c_j S + d^c_j | q^c_j \leq S < q^c_{j+1}\}$ with $q^c_1 = 0 < q^c_2 < \cdots < q^c_n < q^c_{n+1} = 1$;

C2', the last segment of $V^c_{t',2}$ is $S(R+b) - b$.

C3', At stage 2, there exists a $\bar{s}^c_t$ such that the successor challenges the ruler if $S_t > \bar{s}^c_t$, and not otherwise. At stage 1, there exist $\hat{s}^c_{t,h}$ and $\hat{s}^c_{t,l}$, with $\hat{s}^c_{t,h} = \bar{s}^c_t - H \leq \hat{s}^c_{t,l} \leq \bar{s}^c_t - L$, such that the successor chooses h when $S_{t-1} \leq \hat{s}^c_{t,l}$, low effort when $\hat{s}^c_{t,l} < S_{t-1} \leq \hat{s}^c_{t,h}$, and high effort when $\hat{s}^c_{t,h} < S_{t-1}$.

In period $t-1$, stage 3, if the successor remain loyal, his payoff if $\delta V^c_{t,1}(S)$ if the ruler does not strip him and $-b$ if he is stripped. He needs to compare this payoff with the payoff if he challenges the ruler now $(S(R+b) - b)$. Since $\delta V^c_{t,1}(S)$'s last segment is $\delta(p_t((S+w)(R+b) - b) + (1-p_t)((S + p_h(H-L) + L)(R+b) - b))$ which intersects with $S(R+b) - b$ after $\bar{s}^m_{t-1}$, then $\bar{s}^c_{t-1} = \bar{s}^m_{t-1} \in [\bar{s}^c_t - H, \bar{s}^c_t - L]$.

At stage 3, once the successor observes S_{t-1}, his value function at that moment can be written as

$$V^c_{t-1,3}(S) = \begin{cases} \delta(p_t((S+w)(R+b) - b) + (1-p_t)V^c_{t,1}(S)) & \text{if} \quad S \leq \bar{s}^c_{t-1}, \\ S(R+b) - b & \text{if} \quad S > \bar{s}^c_{t-1} \end{cases} \tag{21}$$

To calculate the strategy, we need to compare $V^c_{t-1,3}(S+H)$ and $V^c_{t-1,3}(S+L)$. First we have $V^c_{t-1,3}(S+H) > V^c_{t-1,3}(S+L)$ when $S < \bar{s}^c_t - H$. Then let $\hat{s}^c_{t-1,h} \equiv \bar{s}^c_t - H$, and we know the successor should choose high effort, if $S_t < \hat{s}^c_{t-1,h}$. When $S \geq \bar{s}^c_t - H$, we have $V^c_{t-1,2}(S+H) \geq (S+H)(R+b) - b$. Also we have $\delta V^c_{t,1}(\bar{s}^c_t - H + H) > \delta V^c_{t,1}(\bar{s}^c_t - H + L) > (\bar{s}^c_t - H + H)(R+b) - b$. Moreover, we have $R + b > p_h(a^c_i) + (1-p_h)(a^c_j)$ for any i and j when $S < \bar{s}^c_{t-1} - L$. Furthermore, when $S > \bar{s}^c_{t-1} - L$, we have $(S+H)(R+b) - b > (S+L)(R+b) - b$. Then there exists a $\hat{s}_{t-1,l}$, such that $V^c_{t-1,2}(S_{t-2} + H) < V^c_{t-1,2}(S_{t-2} + L)$ when $\hat{s}_{t-1,h} < S_{t-2} < \hat{s}_{t-1,l}$, and $V^c_{t-1,2}(S_{t-2} + H) > V^c_{t-1,2}(S_{t-2} + L)$ when $S_{t-2} > \hat{s}_{t-1,l}$. In other word, the successor should chooses high effort when $S_{t-2} < \hat{s}_{t-1,h}$, low effort when $\hat{s}_{t-1,h} < S_{t-2} < \hat{s}_{t-1,l}$, and switch to high effort again when $S_{t-2} > \hat{s}_{t-1,l}$.

It is easy to find that $V^c_{t-1,2}(S_{t-2})$ is still a function consist of a set of linear segments, with positive slope for each segment. Also, the last segment is $(S_{t-2}+p_h(H-L)+L)(R+b)-b$ when $S_{t-2} > \bar{s}^c_{t-1} - L$, and $V^c_{t-1,2}(S_{t-2}) > -b$. In summary $V^c_{t-1,2}$ satisfies C1'-C3'.

For the case when $\delta \geq \bar{\delta}$, for any $t \geq \bar{t}^m$, the ruler does not strip the successor, so $\bar{s}^c_t = S^3$ always (S^3 is defined in the proof of proposition 1). For any $t < \bar{t}^m$, we follow the same procedure as the case when $\delta > \bar{\delta}$, and can get the same results. □

Proof of Proposition 3. For a given $\widetilde{S}$, if $\widetilde{S} \geq \bar{s}^c_1$, then let $t' = 1$. Otherwise, since $\bar{s}^c_t$ is an increasing function of t, we can always find $t' > 1$ such that $\widetilde{S} \geq \bar{s}^c_{t'}$. Then $\widetilde{S} \geq \bar{s}^c_t$ for $t < t'$. Since the appointed successor always challenge the ruler right after the appointment, the ruler's value function is increasing function of t when $t < t'$, therefore the optimal choosing time must greater or equal to t'. From the proof of proposition 1, we the value function of the ruler is pairwise linear function, it implies the slope of the last segment of the value function is negative when t is small, and this slope is an increasing function of time. When t is large, the slope becomes positive. So for a given $\widetilde{S}$, we can find a t' such that the intersection of the last segment of ruler's value function and $(1-p_{t'}k_{t'})$ is less than $\widetilde{S}$, but that intersection is greater than $\widetilde{S}$ in period $t'+1$. So t' increases with $\widetilde{S}$.

From the proof of proposition 2, we know in a sufficient large period t^b, the ruler's value function at stage 1 can be written as $(S_t + w)\eta R$ for $S_t \in [0,1]$. Furthermore, the value function $V^m_{t^b-1,1}(S)$ in period t^b-1 is a piecewise linear function and its first segment has the form $a_1 S + b_1$ with $a_1 > 0$ for any $S \in [1, \bar{s}^c_{t^b-1} - H]$. Therefore, for any given initial power $\widetilde{S}$, there exists a period $t'' \leq t^b$ such that $\widetilde{S}$ belongs to the first segment of $V^m_{t'',1}(S)$ and this segment has the positive slope, moreover $\widetilde{S}$ belongs to the first segment of $V^m_{t^{cc},1}(S)$ and the segment has the positive slope for any $t^b \geq t^{cc} > t''$.

If the successor is appointed in $t''+1$, and the first segment of $V^m_{t'',1}(S)$ is denoted as $aS+d$, then the value function in period t'' is $(1-p_{t''})(r+\delta(aS+b))$. If the successor is appointed in t'', the ruler's value function is $p_{t''-1}(S+w)\eta R + (1-p_{t''-1})(r+\delta(S+p_h(H-L)+L)+b)$. Therefore, appointing the successor in period t'' is better than $t''+1$ for the ruler. Also, it is easy to show that postponing the appointment will make the ruler worse off for any $t > t''$. Therefore, the optimal choosing time should be less than or equal to t''. When $\widetilde{S}$ increase, using the backward induction to calculate the value function of $V^m_{t,1}$ from t^b, then $t''(\widetilde{S})$ is increasing with $\widetilde{S}$. □

Corollary A.1. *If a candidate is designated as successor in period t^a, then there exists a time $\bar{t}^c(t^a, \tilde{S})$ such that the successor will challenge the ruler no later than $\bar{t}^c$.*

Proof of Corollary A.1. There exists $\widetilde{t}$ such that $\widetilde{t}L \geq 1$, then $S_{\widetilde{t}} \geq 1$. When the successor can win the fight with the ruler, challenge the ruler is a dominant strategy. Therefore $\bar{t}^c$ exists and $\bar{t}^c \leq \widetilde{t}$. □

Proof of Proposition 4. We prove the second part of the proposition first. Since $1 - p_t$ converge to zeros when t goes to infinite, therefore for a given ϵ, there exist t^ϵ such that $1 - p_t < \epsilon$. Then we have $CP_t < \epsilon$ by the definition.

When the initial power $\widetilde{S} = 1$, then the chosen successor will challenge the ruler immediately, therefore the ruler will not designate the heir apparent until his life close to the end. Therefore we can find CP_{t^a} must be less than a given ϵ, with $t^a \geq t'$ (proposition 3). Furthermore, there exists $\widetilde{S}_1 \in [0, 1]$ such that the successor always challenge the successor right after his designation. Also,

When $\widetilde{S} < \widetilde{S}_1$, from the proof of proposition 3, we know the ruler also will not designate the successor in any period in which the successor will challenge him when the successor receives a high outcome in this period. It means that the optimal choosing period must satisfy the condition that $\widetilde{S} < \bar{s}^c_{t^a} - H$. Also we can find another $\widetilde{S}_2$ such that when $\widetilde{S} < \widetilde{S}_2$, there exists a period t if the successor is designated in this period, then $\widetilde{S} < \bar{s}^c_t - H$. Now let $\hat{S} = \min\{\widetilde{S}_1, \widetilde{S}_2\}$. Then we know in the optimal designation time $\widetilde{S} < \bar{s}^c_{t^a} - H$. Therefore we know t^h exists with $t^h \geq t^a$. This result implies that t^h is the minimal period such that $\widetilde{S} + (t - t^a + 1)H < \widetilde{s}^c_t$, in other words, the last period that the upper bound of the successor's power is less than the challenge threshold. □

Proof of Proposition 5.

$$CP_t \equiv Pr(S_t \geq \bar{s}^c_t \cap (\cap_{t'=1}^{t-1} S_{t'} < \bar{s}^c_{t'})) \prod_{i=1}^{t} (1 - p_i).$$

Since the tolerance threshold $\bar{s}^m_t$ will vanish after period $\bar{t}^m$, while $\bar{t}^m$ is determined by the ruler's health. Both $\bar{t}^m$ and $\bar{s}^m_t$ are independent with $\widetilde{S}$. Also $t''(\widetilde{S})$ increases with $\widetilde{S}$. Therefore we can find constants p' and δ' and $\widetilde{S}^v$ such that when $p_1 < p'$, $p_t - p_{t-1} < \delta'$, and $\widetilde{S} \leq \widetilde{S}^v$, $\bar{t}^m - t''$ is greater than $\bar{t}^c - t^a$. This set up is not trivial because when $\widetilde{S}^v = 0$, we can find p' and δ' such that $\bar{t}^m$ is sufficiently large. Furthermore, from the proof of proposition 2, when p_t increases in a sufficient small scale in each period, there exists a t^u such that

$\bar{s}^m_t = \bar{s}^c_t$ when $t \leq t^u$ and $\bar{s}^m_t = \bar{s}^m_{t+1} - H$. Therefore, we can choose a even smaller p' and δ' such that $t'' \leq t^u$.

The chosen $\widetilde{S}^v$, p' and δ' is to guarantee that the successor is designated in a early period, and there exist sufficiently long periods after the honeymoon phase and before the ruler's health gets worse.

At the end of the honeymoon phase ($t = t^h$), the possible outcome of the successor's power is between $[\widetilde{S} + t^h L, \widetilde{S} + t^h H]$, since the increase rate of tolerance thresholds $\bar{s}^m_t$ converges to zeros before $\bar{t}^m$ and the increase rate of the successor's power is at least L, therefore there exists a period such that the increase rate of $\bar{s}^m_{t-1}$ is less than L after this period. Furthermore, it implies that a period t^β exists such that the smallest possible $S_{t^\beta - 1}$ is greater than $\bar{s}^c_{t^\beta, l}$, and $S_{t^\beta - 2}$ is less than $\bar{s}^c_{t^\beta - 1, l}$. It means, the successor will choose high effort in period t^β, and once the outcome is high, then he will challenge the ruler. Then conditional on no conflict in period $t^\beta - 1$, the conflict probability in t^β is $p_h(1 - p_{t^\beta})$. From the perspective of period t^a, the conflict probability in period t^β is $CP_{t^\beta} = p_h(1 - p_{t^\beta})(1 - p_h)^{t^\beta - t^a} \prod_{i=1}^{t^\beta - 1}(1 - p_i)$. The conflict probability of the next period, $CP_{t^\beta + 1}$, is either $p_h(1 - p_{t^\beta + 1})(1 - CP_{t^\beta})$ or $(1 - p_{t^\beta + 1})(1 - CP_{t^\beta})$. Then choose a small enough constant $0 < \phi < CP_{t^\beta}$ and let t^γ be the maximal period such that $CP_t > \phi$ for all consecutive $t > t^\alpha$, and $\bar{c}$ is the minimal CP_t with $t \in [t^\beta, t^\gamma]$. □

Proof of Proposition 6. For a given initial power $\widetilde{S}$, let $\widetilde{w} = 1 - \widetilde{S}$. When $w < \widetilde{w}$, the ruler's value function can be written as $\min(S + w, 1)\eta R$ as proof of Proposition 2 when he designate a successor at stage 1 of a sufficient large period t^b. Therefore, when $S > 1 - w$, the value function is ηR, which is the last segment of $V^m_{t^a, 1}$. Furthermore, the last segment of any $V^m_{t,1}$ can be written as $p_t(\eta R) + (1 - p_t)(-(b + r + k_{t+1})S + r + k_{t+1})$ when $\bar{s}^c_t > 1 - w$. It implies $\bar{s}^m_t$ is not affected by w when $\bar{s}^c_t > 1 - w$.

When $\widetilde{S} < 1 - w$, from the proof of proposition 3, we know t'' is the first period such that $\widetilde{S} > \bar{s}^c_t$. Therefore, when w increases, both S^1 and S^2 in Proposition 2 will decrease. It indicates the successor is less likely to challenge the ruler when the ruler does not have incentive to strip him. Following the same induction procedure, we can find the thresholds $\bar{s}^c_t$, $\bar{s}^m_t$ in each period will increase with w. Then following the proof in proposition 3 and the increase of $\bar{s}^c_t$, we have both $t'(\widetilde{S})$ and $t''(\widetilde{S})$ will decrease.

Given w and an optimal designation period t^a, when w increases, the optimal designation period will not increase. It comes from the proof that t' decreases with w such that the optimal designation time cannot increase when w increases. If the optimal designation time does not change, since $\bar{s}_t^c$ weakly increases with w, it indicates the gap between $\widetilde{S}$ and $\bar{s}_{t^a}^c$ increases. Therefore, the length of the honeymoon phase will weakly increase.

From the proof of part 1, we know when w increase, t^h weakly increase with w, which indicates the gap between $\bar{s}_t^c$ and S_t in the last period of the honeymoon phase before will increase with w. It also indicate that t^β weakly increases with w, i.e. the successor need more time to reach period t^β. Therefore, from the proof of proposition 5, we know CP_{t^β} will decreases with w. Since the conflict probability in each period decreases in the interval $[t^\beta, t^\gamma]$, $\bar{c}$ will decreases with w.

For part 3, assume that the optimal designation time is t^a under w_1 (this game is referred as game 1), and when w_1 is increased to w_2 the optimal designation time becomes $t^a - 1$. Under w_2, it is equivalent to consider a new game (game 2) that the ruler designates the crown prince with $\widetilde{S} + H$ at time t^a. This is because the ruler will not designate the crown prince who will challenge him in the first period. In this new game, we can find a sufficient small H such that the honeymoon phase is the same as $t^h - t^a$ under the game with w_1, then we follow the proof in Proposition 5, we can find $t^{\beta'}$ for this new game which is weakly less than t^β under the game with w_1, it is because the crown prince's power is weakly larger in game 2 than that in game 1 in each period. Therefore, $t^{\beta'}$ is weakly less than t^β. Meanwhile, the conflict probability in $t^{\beta'}$ is $p_h(1 - p_{t^{\beta'}})$ that is weakly greater than that in game 1. Then use the same procedure, we can find that the lower bound of the conflict probability in game 2 in higher than that in game 1. Finally we let the upper bound of H that satisfies this procedure as $\overline{H}$. □

Proof of Corollary 1. At stage 4 of period 1, if the successor has received a power such that he will receive the power exceed 1 in the next period, then he will challenge the ruler in period two. In this situation, if the ruler keeps the successor, his expected payoff is $r + \delta(p_2\eta R - (1 - p_2)b)$, and if he strips the successor, his expected payoff is k_1. Then if we have $k_1 > r + \delta(p_2\eta R - (1 - p_2)b)$, i.e. $((k_1 - r)/\delta + b)/(\eta R + b) > p_2$, when we know $\bar{s}_1^m$ exists because the ruler has an incentive to strip the successor. Let $d_1 = p_2 - p_1$, then when $p_1 < ((k_1 - r)/\delta + b)/(\eta R + b) - d_1$, $\bar{s}_1^m$ exists. This result indicates when p_1 is not sufficiently large, then $t_1 < \bar{t}^m$. In other words, the ruler's health is not worse enough such that he gives up the chance to strip the successor.

For given $p_1, p_2, ...$ with $p_1 < ((k_1 - r)/\delta + b)/(\eta R + b) - d_1$, if $bars_1^c \leq \widetilde{S}$, then our proposition holds immediately. If not, then let $d_t = p_t - p_{t-1}$ and $d_m = sup_t(d_t)$. In the first case that $\bar{s}_1^m = \bar{s}_1^c$, then we fix p_1 and let d_m decreases, it implies the except p_1, all other p_t decreases. From the proof of proposition 2, we know the value function of ruler at stage 1 $V_{t,1}^m(S)$ is a pairwise linear function and its the last segment is $p_t(S_{t-1}+w)+(1-p_t)(-(b+k_t)(S+p_h(H-L)+L)+k_t)$ when $S \leq \bar{s}_t^c$, and it is decreasing with p_t. Therefore, repeat the induction in the proof of proposition 2, we have $\bar{s}_t^m$ decreases with p_t. Therefore, for given $\widetilde{S}$, when there exist a $\widetilde{d}$ such that when $d_m < \widetilde{d}$, then $\bar{s}_1^m \leq \widetilde{S}$.

In the second case that $\bar{s}_1^m > \bar{s}_1^c$, when d_m decreases, though the proof of the existence of $\hat{t}$ in proposition 2. Then $\bar{t}$ is increasing with d_m, it implies that a d' exists such that when $d_m < d'$, $\bar{s}_1^c = \bar{s}_1^m$. Then further decreases d_m may lead $\bar{s}_1^m \leq \widetilde{S}$. This procedure indicates the existence of $\widetilde{d}$ in the second case such that the conflict will occur in the first period, when $p_t - p_{t-1} < \widetilde{d}$. □

Proof of Corollary 2. Agnatic seniority in this setup implies a large $\widetilde{S}$ and primogeniture implies a small $\widetilde{S}$. When the succession order and all other parameters are fixed. Suppose we begins to increase $\widetilde{S}$ from zero, for a given health sequence $\{p_t\}$, if the successor is designated at time 1, then there is no conflict in the first period when $\widetilde{S} = 0$, therefore the conflict may happen when these parameters are satisfied the conditions in Proposition 5. When $\widetilde{S}$ increases, we can always find a large $\widetilde{\hat{S}}$ such that the conflict happens right after the designation for any $\widetilde{\hat{S}} > \widetilde{\hat{S}}$.

□

Lemma A.1. *For any period t, suppose that candidate 1 has been designated as successor. Then a tolerance threshold $\bar{s}_{t,1}^m$ exists such that the ruler will strip this successor's title if that threshold is exceeded by the successor's power—that is, if $S_t^1 > \bar{s}_{t,1}^m$.*

Proof of Lemma A.1. The proof of this lemma can be found in the proof of part 1 of Proposition A.5. □

In the next proposition, I calculate the equilibrium strategies in any subgame when the successor has been chosen.

Proposition A.5. *Given $\widetilde{S}_2$, in any period t when Candidate 1 has been designated as the successor,* *1. At stage 4, a threshold $\bar{s}^m_{t,1}$ exists such that the ruler will strip the successor when $S^1_t \leq \bar{s}^m_{t,1}$. There exists a time period $\bar{t}^m$, when $t < \bar{t}^m$, a threshold $\bar{s}^m_{t,2}$ exists such that the ruler will strip the the successor when $S^1_t > \bar{s}^m_{t,2}$, and $\bar{s}^m_{t,2}$ weakly increases with time.*

2. At stage 3, two thresholds $\bar{s}^c_{t,1}$ and $\bar{s}^c_{t,2}$ exist such that the successor challenges the ruler if $S_t \leq \bar{s}^c_{t,1}$ or $S_t > \bar{s}^c_{t,2}$; otherwise, he remains loyal. Moreover, a $\hat{t}$ exists with $\hat{t} \leq \bar{t}^m$ such that $\bar{s}^c_{t,2} < \bar{s}^m_{t,2}$ when $\hat{t} \leq t \leq \bar{t}^m$, and $\bar{s}^c_{t,2} = \bar{s}^m_{t,2}$ when $t < \hat{t}$.

3. At stage 2, then $\bar{s}^c_t = \bar{s}^m_t$, two thresholds, $\hat{s}^c_{t,h}$ and $\hat{s}^c_{t,l}$, exist such that the successor will choose a low effort level if $\hat{s}^c_{t,h} \leq S_{t-1} < \hat{s}^c_{t,l}$; he will then switch back to a high effort level if $\hat{s}^c_{t,l} \leq S_{t-1}$.

Proof of Proposition A.5. The proof of Proposition A.5 is similar as the proof of Proposition 2. The difference between these two propositions will be highlighted. Assume Candidate 1 has been designated as the successor. I use $V^g_{i,j}(S, \widetilde{S}^2)$ to denote player g's expected payoff when the first successor's power is S and Candidate 2's initial power is $\widetilde{S}^2$ at the beginning of stage j of period i, meanwhile, $V^g_{i,j}(S)$ is denoted player g's expected payoff when the second successor's power is S, which is equivelent to the case when there is only one candidate.

Start in a very large t, similar as the proof of Proposition 2, such that the ruler's expected payoff when he keeps Candidate 1 as the successor at stage 4 can be rewritten as $r + \delta(S^1_t + w)\eta R$. If the ruler strip the successor, then he should appoint Candidate 2 as the successor at stage 1 of period $t + 1$. The ruler's payoff is denoted as $V^m_{t,4}(\widetilde{S}^2)$ which can be written as $r + \delta(\widetilde{S}^2 + w)\eta R$. Therefore, Candidate 1 should be stripped if $S^1_t < \widetilde{S}^2$. So we have $\bar{s}^m_{t,1} = \widetilde{S}^2$.

At stage 3, When the successor challenge the ruler, his expected payoff is $(R + b)S^1_t - b$ (hereafter we drop the super- and sub-script for the existing successor), and when he remains loyal, his expect payoff is $\delta((S + w)(R + b) - b)$ when $\delta > \bar{\delta} \equiv R/((1 + w)(R + b) - b)$, or $\delta((S + w)(R + b) - b)$ if $S < (\delta R + b)/(R + b)$ and δR if $S \geq (\delta R + b)/(R + b)$ when $\delta > \bar{\delta} \equiv R/((1 + w)(R + b) - b)$. Therefore, he challenges the ruler if $S < \bar{s}^c_{t,1} \equiv \bar{s}^m_{t,1}$ or $S \geq \bar{s}^c_{t,2}$, where $\bar{s}^c_{t,2} = (\delta R + b)/(R + b)$.

At stage 2, in the case that $\delta < \bar{\delta}$, we have $\delta((S+w)(R+b)-b)$ intersects $\delta((S+w)(R+b)-b)$ less than $S = 1$. It implies $V^m_{t,3}(S + H, \widetilde{S}^2) > V^m_{t,3}(S + L, \widetilde{S}^2)$ for all $S \leq 1$. Therefore, the successor should choose high effort at this stage, and $V^m_{t,2}(S, \widetilde{S}^2) = p_h V^m_{t,3}(S + H, \widetilde{S}^2) + (1-p_h)V^m_{t,3}(S+L, \widetilde{S}^2)$. In the case that $\delta \geq \bar{\delta}$, we will have the same result that the successor will choose the high effort level. In the special case that $\widetilde{S}^2 > \bar{s}^c_{t,1}$, we define $\bar{s}^c_{t,1} = \bar{s}^c_{t,2}$. Now the expected payoff function of the successor can be denoted as a piecewise linear function. Moreover, the last segment is $(R+b)S-b$ with starting point $\bar{s}^c_{t,2}$. Hereafter we focus on the cast that $\delta < \bar{\delta}$. The case that $\delta < \bar{\delta}$ is similar as the analysis in Proposition 2.

At stage 1, the ruler's expected payoff is denoted as $V^m_{t,1}(S,\widetilde{S}^2) = p_t(S+w)\eta R + (1-p_t)V^m_{t,2}(S,\widetilde{S}^2)$. We make the following claim

Claim 5. *When $S < \widetilde{S}^2$, $V^m_{t,1}(S,\widetilde{S}^2) < V^m_{t,1}(\widetilde{S}^2,\widetilde{S}^2) = p_t(\widetilde{S}^2+w)\eta R + (1-p_t)V^m_{t,2}(\widetilde{S}^2,\widetilde{S}^2)$.*

Proof of Claim 5. For $V^m_{t,2}(S,\widetilde{S}^2)$, if $\widetilde{S}^2 - L < \bar{s}^c_{t,2} - H$, then $V^m_{t,1}(\widetilde{S}^2,\widetilde{S}^2) = p_t(\widetilde{S}^2+w)\eta R + (1-p_t)(r+\delta(\widetilde{S}^2 + p_h(H-L)+L)\eta R)$. If $\widetilde{S}^2 - L < S \geq \widetilde{S}^2 - L$, then we have $V^m_{t,1}(\widetilde{S}^2,\widetilde{S}^2) > p_t(S+w)\eta R + (1-p_t)(r+\delta(S+p_h(H-L)+L)\eta R)$. If $S \geq \widetilde{S}^2 - L$, then $V^m_{t,2}(S,\widetilde{S}^2)$ is either $p_t(S+w)\eta R + (1-p_t)(r+\delta(p_h(-(b+V^m_t(\widetilde{S}^2))S+b)+(1-p_h)(S+L)\eta R))$ or $p_t(S+w)\eta R + (1-p_t)(-(b+V^m_t(\widetilde{S}^2))(S+p_h(H-L)+L)+b)$. In either case, we have $V^m_{t,1}(S,\widetilde{S}^2) < p_t(\widetilde{S}^2+w)\eta R + (1-p_t)V^m_{t,2}(\widetilde{S}^2,\widetilde{S}^2)$.

If $\widetilde{S}^2 - L \geq \bar{s}^c_{t,2} - H$, then $V^m_{t,1}(\widetilde{S}^2,\widetilde{S}^2) = p_t(S+w)\eta R + (1-p_t)(r+\delta(p_h(-(b+V^m_t(\widetilde{S}^2))\widetilde{S}^2 + b)+(1-p_h)(\widetilde{S}^2+L)\eta R))$. For any $S < \widetilde{S}^2$, $V^m_{t,2}(S,\widetilde{S}^2)$ is either $p_t(S+w)\eta R + (1-p_t)(r+\delta(p_h(-(b+V^m_t(\widetilde{S}^2))S+b)+(1-p_h)(S+L)\eta R))$ or $p_t(S+w)\eta R + (1-p_t)(-(b+V^m_t(\widetilde{S}^2))(S+p_h(H-L)+L)+b)$. In either case, we have $V^m_{t,1}(S,\widetilde{S}^2) < p_t(\widetilde{S}^2+w)\eta R + (1-p_t)V^m_{t,2}(\widetilde{S}^2,\widetilde{S}^2)$. □

We simply assume that $\widetilde{S}^2 - L < \bar{s}^c_{t,2} - H$, and discuss the case that $\widetilde{S}^2 - L \geq \bar{s}^c_{t,2} - H$ later. After calculate the expected payoffs in period t, we know $\widetilde{S}^2$ belongs to the segment of $V^m_{t,1}(S,\widetilde{S}^2)$ with the largest slope and the ruler will not be challenged by the successor when the successor's power S^1_{t-1} at stage 1 of period t belongs to this segment because of the assumption. We name this segment the "untouched segment" $O_{t,u}$. This segment's starting and end points are denoted as $S_{t,s}$ and $S_{t,e}$ respectively.

Claim 6. *At stage 1 of period $t-1$, if $S < S_{t,e} - \bar{s}^c_{t,2} - 2H$, then for any S that belongs to the untouched segment $O_{t,u}$, we have $V^m_{t-1,1}(S) > (1-p_{t-1})(r+\delta V^m_{t,1}(S))$.*

Proof of Claim 6. Since S belongs to the untouched segment in period t, then the ruler also does not face any challenge when the successor's power is S at stage 1 of period $t-1$ if $S < S_{t,e} - \bar{s}^c_{t,2} - 2H$. Therefore, we have $V^m_{t-1,1}(S) = p_{t-1}(S+w)\eta R + (1-p_{t-1})(r+\delta V^m_{t,1}(S + p_h(H-L)+L))$, which is greater than $(1-p_{t-1})(r+\delta V^m_{t,1}(S))$. □

Now we can extend the definite of untouched segment to any period: For any given period t', the untouched segment at stage one of period t', $O_{u_{t'}}$, consists of the S such that the ruler has zero probability of being challenged in the period. It is easy to show if this segment is not empty, then this is continuous inside with starting and end points, $S_{t',s}$ and $S_{t',e}$ respectively. It is worth to note that the untouched segment may be an empty set in some periods.

The ruler's value function, $V^m_{t,1}(S, \widetilde{S}^2)$, at stage 1 of period t is a piecewise linear function, $\{O_{i_t} | i_t = 1, \cdots, n_t\}$. For the untouched segment of period t, we have O_{u_t} with $u_t \in \{1, \cdots, n_t\}$ if it is not an empty set.

Let's consider the situation in period $t-1$. Assume $\widetilde{S}^2 < \bar{s}^c_{t,2} - 2H$, at stage 4 of period $t-1$, the ruler's payoff is $r + \delta V^m_{t,1}(\widetilde{S}^2)$ and $\widetilde{S}^2$ is on the untouched segment of period t. It implies that the current $V^m_{t,1}(\widetilde{S}^2)$ provides the ruler the optimal payoff in period t if Candidate 1 has been stripped (or equivelent to the case that Candidate 2 is the unique candidate). If the ruler keeps Candidate 1 at stage 4 of period $t-1$, then from Claim 5, we know any S less than $\widetilde{S}^2$ cannot give the ruler higher payoff than $r + \delta V^m_{t,1}(\widetilde{S}^2)$. Therefore the ruler strips Candidate 1 if the successor's power at this moment is less than $\widetilde{S}^2$, and we have $\bar{s}^m_{t-1,1} = \widetilde{S}^2$.

The ruler's expected payoff when he strips the current successor is $r + \delta V^m_{t,1}(\widetilde{S}^2)$, and this payoff does not satisfies Assumption 2. So for any segment O_{i_t} with $i_t > u_t$, it is not guaranteed that O_{i_t} does not intersect with $V^m_{t,1}(\widetilde{S}^2)$ if the slope of O_{i_t} is positive, but we do know the last segment has the form $O_{n_t} = p_t(S + w)\eta R + (1 - p_t)(-(b + V^m_{t,1}(\widetilde{S}^2))(S + p_h(H - L) + L) + V^m_{t,1}(\widetilde{S}^2))$. Then we put all the intersect points of $V^m_{t,1}(S, \widetilde{S}^2)$ and $V^m_{t,1}(\widetilde{S}^2)$ together with all the start points of each segment except 0, then delete the repeated points. This set is denoted as $\{\bar{s}^{m,t-1}_j | j = 1, \cdots, k\}$ with $\bar{s}^{m,t-1}_j < \bar{s}^{m,t-1}_{j'}$ when $j < j'$. Then the ruler's strategy is described as follow: Given a sufficient small number ϵ, for each $\bar{s}^{m,t-1}_j$, consider a point $\bar{s}^{m,t-1}_j + \epsilon$. If the slope of the segment of $V^m_{t,1}(S, \widetilde{S}^2)$ at this point is positive, then the ruler keeps the successor when $\bar{s}^{m,t-1}_j < S \leq \bar{s}^{m,t-1}_{j+1}$. If this slope is negative, then the ruler strips the successor when $\bar{s}^{m,t-1}_j < S \leq \bar{s}^{m,t-1}_{j+1}$. Consider a point $\bar{s}^{m,t-1}_j - \epsilon$, if the slope of the segment of $V^m_{t,1}(S, \widetilde{S}^2)$ at this point is positive, then the ruler keeps the successor when $\bar{s}^{m,t-1}_{j-1} < S \leq \bar{s}^{m,t-1}_j$. If this slope is negative, then the ruler strips the successor when $\bar{s}^{m,t-1}_{j-1} < S \leq \bar{s}^{m,t-1}_j$. Now define a new set SW^m_{t-1} that contains the points in $\bar{s}^{m,t-1}_j + \epsilon$ and the ruler switches the strategy across these points. So SW^m_{t-1} is set of all thresholds. Moreover, if the last segment, O_{n_t}'s slope is negative and it intersects with $V^m_{t,1}(S, \widetilde{S}^2)$, then let $\bar{s}^m_{t,2}$ be this intersect and it must be $\bar{s}^{m,t-1}_k$. If O_{n_t}'s starting point is less than $V^m_{t,1}(\widetilde{S}^2)$, find the large S such that $V^m_{t,1}(S', \widetilde{S}^2) \geq V^m_{t,1}(\widetilde{S}^2)$ with $S' \geq S$, then let $\bar{s}^m_{t-1,2}$ be this S, and also $\bar{s}^m_{t-1,2} \in SW^m_{t-1}$. Otherwise $\bar{s}^m_{t-1,2} = \emptyset$.

At stage 3 of period $t-1$, the last segment of $V^c_{t,1}(S, \widetilde{S}^2)$ equals $p_t((S+w)(R+b)-b)+(1-p_t)((R+b)(S+p_h(H-L)+L)-b)$, which is denoted as $O^c_{t,l}$. If $\delta(O^c_{t,l})$ intersects $(R+b)S-b$ before $\bar{s}^m_{t-1,2}$ or $\bar{s}^m_{t-1,2} = \emptyset$, then let $\bar{s}^c_{t-1,2}$ be the intersection. If $\delta(O^c_{t,l})$ intersects $(R+b)S-b$ after $\bar{s}^m_{t-1,2}$, then $\bar{s}^c_{t-1,2} = \bar{s}^m_{t-1,2}$. Let $SW^c_{t-1} = SW^m_{t-1} \cup \{\bar{s}^m_{t-1,2}\}$, then delete any points in SW^m_{t-1} but greater than $\bar{s}^m_{t-1,2}$ from SW^c_{t-1}. Then SW^c_{t-1} is denoted as $\{\bar{s}^{c,t-1}_j | j = 1, \cdots, k'\}$. The successor's equilibrium strategy is described as follow: If $k' > 2$, for any threshold, $\bar{s}^{c,t-1}_j$ with $1 < j \leq k'$, if ruler strips the successor when $S \in (\bar{s}^{c,t-1}_{j-1}, \bar{s}^{c,t-1}_j]$, then the successor should challenge the ruler is this interval. If ruler keeps the successor in this interval, then the successor should remain loyal. When $j = k'$, the successor should challenge the ruler when $S > \bar{s}^{c,t-1}_{k'} = \bar{s}^c_{t-1,2}$. If $k' = 2$, then the successor challenges the ruler when $S > \bar{s}^{c,t-1}_{k'}$ or $S \leq \bar{s}^{c,t-1}_{k'-1}$, and remain loyal otherwise.

At stage 2 of period $t-1$, since we know at each S, the successor makes the decision by comparing $V^c_{t-1,3}(S+H, \widetilde{S}^2)$ and $V^c_{t-1,3}(S+L, \widetilde{S}^2)$; and both these value functions are piecewise linear functions. Therefore there is a unique optimal action for the successor at each S. When $S < \bar{s}^c_{t-1,1}$, all segments of $V^c_{t-1,3}(S, \widetilde{S}^2)$ have the property that the maximal value of each segment is less than the minimal value of its next segment and they all have positive slope. Therefore, for any given S, we have $V^c_{t-1,3}(S+H, \widetilde{S}^2) > V^c_{t-1,3}(S+L, \widetilde{S}^2)$. So the successor always choose high effort. For the segments that contain S that larger than $\widetilde{S}^2$, we know the last segment, always has the form $(R+b)S-b$. If the last segment and the one before the last segment are continuous at threshold $\bar{s}^c_{t,2}$, then if $S > S_s - L$, the successor will high effort, where S_s is the start point of the last segment. If the last segment and the one before the last segment are not continuous at threshold $\bar{s}^c_{t,2}$, then we must have the value of the start point in the last segment is less than the value of the end point in the segment before the last. Similar as Proposition 2, there exist an interval such that the successor chooses the low effort in this interval, and high effort when S is larger than the upper bound of this interval. Also there must exists a adjunct interval such that the successor chooses the high effort also. Then the lower bound and upper bound of this interval are denoted as $\hat{s}^c_{t,h}$ and $\hat{s}^c_{t,l}$ respectively.

Now we can repeat the induction with same procedure in the proof of Proposition 2 to get the conclusion.

It is worth to mention that the existence of $\bar{t}^m$ is guaranteed because the last segment of

$V^m_{t,1}$ always has the form $p_t(S_{t-1}+w)+(1-p_t)(-(b+\hat{V}^m_{t,1}(\bar{S}^2))(S+p_h(H-L)+L)+\hat{V}^m_{t,1}(\bar{S}^2))$, where $\hat{V}^m_{t,1}(\bar{S}^2))$ is the ruler's maximal expected payoff at stage 1 of period t when Candidate 2 is the unique candidate.

The existence of $\hat{t}$ is also guaranteed. From the proof of Proposition 2, we know when t is large, $\bar{s}^c_{t,2}$ is either fixed when $\delta \geq \bar{\delta}$ or decreases but bounded from below when $\delta \geq \bar{\delta}$. Also $\bar{s}^c_{t,2}$ cannot greater than $\bar{s}^m_{t,2}$. When $\bar{s}^m_{t,2}$ decreases when t becomes small, there is not lower bound for $\bar{s}^m_{t,2}$. Hence there exists a $\hat{t}$ such that $\bar{s}^c_{t,2} = \bar{s}^m_{t,2}$ when $t < \hat{t}$. It is worth to mention, it is possible that $\bar{s}^c_{t,2} = \bar{s}^m_{t,2}$ for all t when $\bar{t}^m$ exists. In this case, $\hat{t} = \bar{t}^m$.

It is also worth to point out, when $\widetilde{S}^2$ is in the untouched segment of $V^m_{t,1}(S, \widetilde{S}^2)$, $\hat{V}^m_{t,1}(\bar{S}^2))$ is always $V^m_{t,1}(\bar{S}^2)$ because of Claim 6. Once $t < t^*(\widetilde{S}^2)$, where $t^*(\widetilde{S}^2)$ is the optimal time to choose Candidate 2 as the successor in the signal candidate case. Then $\hat{V}^m_{t,1}(\widetilde{S}^2) = r(1-\delta^{t^*-t})/(1-\delta) + \delta^{t^*-t}V^m_{t^*,1}(\widetilde{S}^2)$. Also if $t^*(\widetilde{S}^2)$ is not unique, we can use any optimal time without changing $\hat{V}^m_{t,1}(\bar{S}^2))$. Furthermore, when t decrease, there must exist a time $\hat{t}'$ such that when $t < \hat{t}'$, the untouched segment is empty. In the previous proof when we assume that $\widetilde{S}^2 - L < \bar{s}^c_{t,2} - H$. If $\widetilde{S}^2 - L \geq \bar{s}^c_{t,2} - H$, then this is a special case that the untouched segment becomes empty in period $t-1$. Also if $\widetilde{S}^2 \geq \bar{s}^c_{t,2} - 2H$, this is another special case that the untouched segment becomes empty in period $t-1$. Both these two situations do not affect any previous proof. □

Proof of Proposition 7. Part 1: In any period t, when $\widetilde{S}^2$ is in the untouched segment, we know if the successor's power $S_{t-1} < \widetilde{S}^2$, then the ruler strips the successor at stage 4 of period $t-1$ (Claim 5 and Claim 6). It implies the ruler will not appoint Candidate 1 as the successor in any these period. Since as long as $\widetilde{S}^2$ is in the untouched segment of period t, $V^m_{t,1}(\bar{S}^2)$ increases with decreasing t. From the proof of Proposition A.5, there exists period $\hat{t}_2 \geq \hat{t}$ such that $\widetilde{S}^2$ is not on the untouched segment when $t < \hat{t}_2$, where $\hat{t}$ is the time that the untouched segment becomes an empty set when $t < \hat{t}$.

If $\hat{t}_2$ is the optimal time to appoint Candidate 2 as the successor in the single candidate case, then let $d' = L - p_h(H-L)$. It implies if the successor's power, $S_{\hat{t}_2-1}$, is greater $\widetilde{S}^2 - d'$, then the ruler's expected payoff is greater than stripping the successor and appoint Candidate 2 as the successor in period $\hat{t}_2$. However, for any successor whose power cannot reach $\widetilde{S}^2 - d'$ in period $S_{\hat{t}_2-1}$, he cannot provide better expected payoff to the ruler than choosing Candidate 2. It means for any Candidate 1, if his initial power is less than $\widetilde{S}^2 - (p_h(H-L)+L)\hat{t}_2\hat{t}_2$, then his expected power increase cannot exceeds $\widetilde{S}^2$ in period $\hat{t}_2$. Therefore, to make the space to Candidate 2, an appointed the successor should be removed immediately, otherwise, from the ruler's perspective, the probability of losing the conflict will increase. Then $d_2 = \widetilde{S}^2 - (p_h(H-L)+L)\hat{t}_2$.

In the case that $\hat{t}_2$ is not the optimal time to appoint Candidate 2 as the successor in the single candidate, it implies that the optimal time $t^* < \hat{t}_2$. Let O_v be the segment contains $\widetilde{S}^2$ in period t^*, this segment cannot be the untouched segment. Then there exist at least one segment before O_v, since these segments' slopes must be positive, then the set of S_{t^*-1} such that $V^m_{t^*,1}(S_{t^*-1}, \bar{S}^2) > V^m_{t^*,1}(\bar{S}^2)$ must non-empty. Then let the distance between the lower bound of this set and $\bar{S}^2$ be d'. After that we can repeat the procedure in the previous paragraph to find d_2. □

Proof of Proposition 8. Part 1: In the single candidate case with only Candidate 1, the proof of Proposition 3 indicates $t^{''}-1$ is the first period that the untouched segment does not includes $\widetilde{S}^1$. In the two candidate case, whenever there is a conflict between the ruler and the first successor, the ruler's expected payoff is $-(b+\hat{V}^m_{t,1}(\widetilde{S}^2))S^1_t + \hat{V}^m_{t,1}(\widetilde{S}^2)$, where $\hat{V}^m_{t,1}(\widetilde{S}^2)$ is the ruler's maximal expected payoff in period t when there is one candidate, Candidate 2. We have $\hat{V}^m_{t,1}(\widetilde{S}^2) \leq k_t$. Therefore from the proof of Proposition A.5, we know $\widetilde{S}^1$ will be excluded from the untouched segment with weakly large t. Therefore we have $t^{''}_1 \leq t^{''}$.

Similarly t' is the time that $\widetilde{S}^1$ is included into the segment with from $-(b+\hat{V}^m_{t,1}(\widetilde{S}^2))S + \hat{V}^m_{t,1}(\widetilde{S}^2)$. By the proof of Proposition 3, t'_1 exists, since $-(b+\hat{V}^m_{t,1}(\widetilde{S}^2))S^1_t + \hat{V}^m_{t,1}(\widetilde{S}^2)$ has a more negative slope than $-(b+k_t)S^1_t + k_t$, then we have $t'_1 \leq t'$.

Part 2: Assume t^a is the optimal time to designate Candidate 1 successor in the single candidate case, then if now add Candidate 2 into the game dose not change the optimal designation time t^a. Then we can just follow the proof of Proposition 4, the difference is the lowest challenge threshold $\bar{s}^c_t$ in each period is weakly less than the challenge threshold in the single candidate case. □

Proof of Corollary 3. This corollary can be derived from the proof of Proposition 7. To choose Candidate 2 as the first as successor, first Candidate 2's initial power cannot less than d_2. The proof of Proposition 7 indicates that if Candidate 2 is chosen first, his expected power increase needs to exceed Candidate 1's initial power at the optimal time t^* of choosing Candidate 1 in the single candidate case. If choosing Candidate 1 as the first successor gives the ruler less expected utility at time t^*, then choosing Candidate 2 is still better than choosing 1. If choosing Candidate 1 as the first successor gives the ruler higher expected utility at time t^*, then it implies Candidate 2 has to obtain additional power to making choosing Candidate 2 first the optimal choice. So we can obtain another $\hat{d}$ with the same procedure in the proof of Proposition 8, which is weakly less than d_2. When $\widetilde{S}^1 - \widetilde{S}^2 > \hat{d}$, Candidate 2 has no change to acquire enough power before t^*. □

Proof of Proposition A.1. The proof is straightforward by Assumption 3 □

Proof of Proposition A.2. For a given $\widetilde{S}$, suppose in the main model, the lower bound of the optimal designation time is t', then in the new model, at time t', the candidate's power has became $\widetilde{S} + \sum_{t=1}^{t'} L_t$, therefore the optimal designation time must be later than t'. Therefore we denote the lower bound in the new model as $\hat{t}'$.

Since the candidate's power will not exceeds $\bar{\bar{S}}$, therefore the ruler will designate the successor before the upper bound when $\widetilde{S} = \bar{\bar{S}}$, then we denote this upper bound as $\hat{t}''$. □

Proof of Proposition A.3 and Proposition A.4. The proofs of Proposition A.3 and A.4 are straightforward from the proofs of Proposition 7 and 8. □

D Historical Data

D.1 Sample Selection

The data set covers the Han Dynasty (206 BC), the second empire of Imperial China, until the Qing Dynasty (1911 AD).[4] A total of 37 regimes are included in the data set. Only regimes with at least two monarchs and at least one crown prince are incorporated into the data set.[5]

[4]Fusu in Qin Dynasty (221 BC) was not officially designated the crown prince.

[5]The list of the regimes are: Western Han Dynasty (202 BC-AD 8); Eastern Han Dynasty (AD 25-AD 220); Wei (220-265); Shu (221-263); Wu (229-280); Western Jin Dynasty (AD 265-AD 317); Eastern Jin Dynasty (AD 317-AD 420); Cheng Han (AD 303-AD 346); Former Zhao (AD 304-AD 329); Later Zhao (AD 319-AD 350); Former Yan(AD 337-AD 370); Former Qin (AD 351-AD 394); Later Yan (AD 384-AD 407); Later Qin (AD 384-AD 417); South Yan (AD 398-AD 410); Xia (AD 407-AD 431); Northern Yan (AD 407-AD 439); Northern Wei (AD 386-AD 534); Liu Song (AD 420-AD 479); Southern Qi (AD 479-AD 502); Liang (AD 502-AD 557); Chen (AD 557-AD 589); Northern Qi (AD 550-AD 577); Northwen Zhou (AD 557-AD 581); Sui Dynasty (AD 581-AD 618); Tang Dynasty (AD 618-AD 907); Min (AD 909-AD 945); Former Shu (AD 907-AD 925); Southen Tang (AD 937-AD 975); Later Shu (AD 934-AD 965); Northern

Among these regimes, I excluded the designated crown princes in the following cases. First, some crown princes, such as Zhu Yuyi of the Ming Dynasty, were symbolically bestowed the title posthumously. I excluded such princes because the scenario does not reflect the true relationship between the monarch and the crown prince. Second, certain princes, such as Li Shimin of the Tang Dynasty, headed coups or rebellions to grab power and force the then monarchs to bestow the title of crown prince upon them. Third, some crown princes were neither chosen by monarchs nor through a selection process that was not manipulated. For example, several crown princes were designated during the War of the Eight Princes in the Jin Dynasty. As their puppet, Emperor Hui was controlled by different princes. I excluded these two types of crown princes because they were not selected through the will of the monarchs.

In the selection of their successor, monarchs were influenced by different political coalitions, such as powerful officials or empresses. I have no solid means of determining whether the decision of the monarch is influenced or fully manipulated (as with the second and third cases above). I presume that the crown princes were selected by the monarchs unless official historical records explicitly indicate that the monarch was forced or manipulated by other people in choosing a crown prince.[6]

D.2 Variables

Son of the empress: The status of an empress may evolve. For example, the empress may die before her son assumes the throne, and the monarch may select another consort as the new empress. Therefore, the son of the empress was defined as whether the crown prince is the son of the then empress when the position of the crown prince ends (either by taking the throne, being stripped of the title, or death).

Song Dynasty (AD 960-AD 1127); Southen Song Dynasty (AD 1127-AD 1279); Liao Dynasty (AD 916-AD 1125); Jin Dynasty (AD 1115-AD 1234); Yuan Dynasty (AD 1271-AD 1368); Ming Dynasty (AD 1368-AD 1644); Qing Dynasty (AD 1636-AD 1912).

[6]The official historical record used in the project includes: Twenty-Four Histories, "Zizhi Tongjian", "Spring and Autumn of the Ten States" and "Draft History of Qing".

Adult son of the monarch: Several standards are used to calculate the number of adult sons of the monarch. For sons with the exact birth and death years in the historical records, I excluded princes younger than 10 years old when the position of crown prince became available. I also omitted princes who died before the then-crown prince was designated. For other princes, those who were recorded as deceased before adulthood in the historical record regardless of the precise age were disregarded too.

Table 2: Descriptive Statistics

	Observation	Percentage	Mean(years)	s.d.
Crown Princes	180	100%		
Length of tenure			8.64	7.71
Age of being chosen as crown princes			16.88	10.94
Son of empress	45			
Age gap with monarchs			23.68	12.04
Relationship with Monarchs:				
Sons	152	84.44%		
Brothers	9	5%		
Others	19	10.56%		
Ending:				
Throne	109	60.56%		
Being stripped	27	15%		
Natural death	22	12.22%		
Overthrown by others	22	12.22%		
Monarchs	140	100%		
one crown prince	105	75%		
two crown princes	31	22.14%		
more than two	4	2.85%		
Age of enthronizing			31.35	13.95
Length of reign			18.24	13.48
Overthrown by crown princes	5			
Total conflicts	32	17.68%		

Notes: "Age gap with monarchs" only includes 171 observations, the birth year of 9 crown prince is missing. The number of "Total conflicts" is the sum of the number of the crown prince who were stripped by the monarchs and the number of monarchs who were overthrown by the crown princes, its percentage is the proportion of the conflict among all 180 monarch-crown prince relationship.