Online Appendix

Bayesian Multifactor Spatio-Temporal Model for Estimating Time-Varying Network Interdependence

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Contents

A.1. Model Estimation and Factor Selection

We report the detailed MCMC algorithm for estimating the proposed MLST-MF model. In the `R` package `bpNet` we developed, we offer several options to specify the state equation of ρ_t, but our discussion here focuses on the algorithm based on the AR(1) specification of ρ_t. When specifying ρ_t as a random walk, the model can be estimated using the same simulation algorithm by setting $\kappa = 1$. When ρ_t is specified as a varying slop coefficient, we simply treat ρ_t as an ordinary random coefficient in a truncated parameter space, and the algorithm specified below can be applied with minor modifications.

A.1.1. The Posterior. To offer more flexibilities, the MCMC algorithm in the package `bpNet` is based on a more general form of the model than the one in the main text:

$$y_{it} = \rho_t \mathbf{w}_{it}\mathbf{y}_t + \mathbf{w}_{it}\mathbf{X}_t\boldsymbol{\beta}_{1it} + \gamma y_{i,t-1} + \mathbf{x}_{it}\boldsymbol{\beta}_2 + \nu_i + \psi_t + \boldsymbol{\zeta}_i\mathbf{f}_t + \epsilon_{it}, \tag{A1}$$

$$\boldsymbol{\beta}_{1it} = \boldsymbol{\beta}_1 + \mathbf{b}_i + \mathbf{c}_t, \tag{A2}$$

$$\rho_t = \kappa\rho_{t-1} + \mathbf{Z}_t'\boldsymbol{\alpha} + \eta_t \tag{A3}$$

The major difference between this model specification and the one in the main text is that here we also allow the exogenous network effects to vary. The user can choose whether to turn on or off the options of $\mathbf{b}_i$ and $\mathbf{c}_t$ when using the package. When estimating parameters, we put the fixed-effects into the multifactor term and reparameterize the factor loadings for applying Bayesian shrinkage. We use the following reduced form of the model in a matrix expression:

$$\mathbf{A}(\rho_t)\mathbf{y}_t = \gamma\mathbf{y}_{t-1} + \mathbf{X}_{1t}\boldsymbol{\beta} + \mathbb{X}_{2t}\mathbf{b} + \mathbf{X}_{2t}\mathbf{c}_t + \mathfrak{Z}(\boldsymbol{\omega}\cdot\mathbf{f}_t) + \boldsymbol{\epsilon}_t \tag{A4}$$

where $\mathbf{A}(\rho_t) = \mathbf{I} - \rho_t \mathbf{W}_t$ is an $N \times N$ matrix, $\mathbf{X}_{1t} = \{\mathbf{W}_t \mathbf{X}_t, \mathbf{x}_t\}$ is a $N \times 2p$ matrix, $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')'$ is a $2p \times 1$ column vector, $\mathbf{X}_{2t} = \mathbf{w}_t \mathbf{x}_t$ is a $N \times p$ matrix, and $\mathbb{X}_{2t}$ is a $N \times Np$ matrix with

$$\mathbb{X}_{2t} = \begin{pmatrix} \mathbf{x}_{2t,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \mathbf{x}_{2t,N} \end{pmatrix}$$

where $\mathbf{x}_{2t,i}$ is the i^{th} row of $\mathbf{X}_{2t}$. $\mathbf{b} = (\mathbf{b}_1', \cdots, \mathbf{b}_i', \cdots, \mathbf{b}_N')'$ is a $Np \times 1$ column vector, and $\mathbf{c}_t$ is a $p \times 1$ column vector. ω is an $r \times 1$ weights vector, f_t is an $r \times 1$ vector of common factor at time t, and $\mathfrak{Z}$ is an $N \times r$ matrix of factor loadings after reparameterization. "$\cdot$" represents point-wise product. Assume that $\mathrm{E}(\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t') = \sigma_e^2 \mathcal{I}_N$, $\forall t$, and that $\boldsymbol{\epsilon}_t$ and η_t are not correlated. The covariate matrices $\mathbf{X}_{1t}$ and $\mathbf{X}_{2t}$ can both include a constant. When there is a constant in $\mathbf{X}_{2t}$, no fixed effects should be included in the multifactor term.

The priors of the parameters to be estimated are specified as the following:

$$\boldsymbol{\beta} \sim \mathcal{N}(\boldsymbol{\beta}_0, \mathbf{B}_0),\ \boldsymbol{\alpha} \sim \mathcal{N}(\alpha_0, \mathbf{A}_0),\ \boldsymbol{\gamma} \sim \mathcal{U}_{S_\gamma},\ \rho_t \sim \mathcal{U}_{S_{\rho_t}},\ \kappa \sim \mathcal{U}_{S_\kappa}$$

$$\mathbf{b}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{D}),\ \mathbf{c}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{E}),\ \mathbf{f}_t \sim \mathcal{N}(\mathbf{0}, \mathcal{I}_r),\ \boldsymbol{\zeta}_i \sim \mathcal{N}(\mathbf{0}, \mathcal{I}_r)$$

$$\mathbf{D}^{-1} \sim \mathcal{W}(d_0, \mathbf{D}_0),\ \mathbf{E}^{-1} \sim \mathcal{W}(e_0, \mathbf{E}_0),\ \sigma_e^{-2} \sim \mathcal{G}(g_1, g_2),\ \sigma_\eta^{-2} \sim \mathcal{G}(g_3, g_4)$$

$$\omega_j | \tau_j^2 \sim \mathcal{N}(0, \tau_j^2), \quad \tau_j^2 | \lambda^2 \sim Exp(\frac{\lambda^2}{2}), \quad \forall 1 \le j \le r, \quad \lambda^2 \sim \mathcal{G}(g_5, g_6),$$

where $\mathcal{N}$ denotes (multivariate) Normal distribution, $\mathcal{G}(a, b)$ denotes Gamma distribution with parameters a and b, $\mathcal{W}$ denote Wishart distribution, $Exp(\lambda)$ denotes Exponential distribution with parameter λ, and $\mathcal{U}_S$ represents Uniform distribution within the space of S. Except the shrinkage priors, we make the priors flat and let the likelihood dominate the posterior. Following the Bayes Rule, the posterior is as the following:

$$
\begin{aligned}
\pi(\boldsymbol{\Theta}|\mathbf{Y}) \propto & \sigma_e^{-NT} \prod_{t=1}^{T} |\mathbf{A}(\rho_t)| \\
& \exp[-\frac{1}{2\sigma_e^2}\{\mathbf{A}(\rho_t)\mathbf{y}_t - \gamma\mathbf{y}_{t-1} - \mathbf{X}_{1t}\boldsymbol{\beta} - \mathbb{X}_{2t}\mathbf{b} - \mathbf{X}_{2t}\mathbf{c}_t - \mathfrak{Z}(\omega \cdot f_t)\}' \\
& \{\mathbf{A}(\rho_t)\mathbf{y}_t - \gamma\mathbf{y}_{t-1} - \mathbf{X}_{1t}\boldsymbol{\beta} - \mathbb{X}_{2t}\mathbf{b} - \mathbf{X}_{2t}\mathbf{c}_t - \mathfrak{Z}(\omega \cdot \mathbf{f}_t)\}] \\
& \sigma_\eta^{-T} \exp\left(-\frac{(\rho_1 - \mathbf{Z}_1'\boldsymbol{\alpha})^2}{2\sigma_\eta^2}\right) \prod_{t=2}^{T} \exp\left(-\frac{(\rho_t - \kappa\rho_{t-1} - \mathbf{Z}_t'\boldsymbol{\alpha})^2}{2\sigma_\eta^2}\right) \\
& \prod_{t=1}^{T} \pi_0(\rho_t) \prod_{i=1}^{N} \pi_0(\mathbf{b}_i|\mathbf{D}) \prod_{t=1}^{T} \pi_0(\mathbf{c}_t|\mathbf{E}) \prod_{j=1}^{r} \pi_0(\omega_j|\tau_j^2)\pi_0(\tau_j^2|\lambda^2) \\
& \pi_0(\gamma)\pi_0(\kappa)\pi_0(\boldsymbol{\beta})\pi_0(\boldsymbol{\alpha})\pi_0(\mathbf{D})\pi_0(\mathbf{E})\pi_0(\mathfrak{Z})\pi_0(\mathbf{F})\pi_0(\lambda^2)\pi_0(\sigma_e^{-2})\pi_0(\sigma_\eta^{-2})
\end{aligned}
\tag{A5}
$$

A.1.2. The Stationarity Space. When spatio-temporal modeling techniques are applied to network analysis, time-spatial stationarity is normally not a concern since the social space is not imagined to be infinite and the time-dimensional dynamic can be treated as a local trend. Nonetheless, the proposed model may also be applied to investigate interdependence in a geographical space, and then stationarity is relevant. Below we develop a joint stationarity space of the dynamic parameters for situations when asymptotic stability of the system is a concern.

When the response variable takes a spatial-temporal autoregressive process, stationarity is based on the concept of separability (Anselin, Gallo, and Jayet 2008). Separability requires that the space-time variance-covariance matrix be decomposed into two parts of the time and space covariance matrices, linked by a Kronecker product (Anselin, Gallo, and Jayet 2008). Therefore, the stationarity conditions of the time process depend on those of the spatial process, and *vice versa* (Debarsy, Ertur, and LeSage 2012).

Denote $S_\kappa,\ S_\gamma,\ S_{\rho_t}$, as the stationarity spaces of κ, γ, ρ_t respectively, we develop the multi-dimensional stationarity space following Elhorst 2012:

$$S_{\rho_t} : 1/\mu^t_{min} \leq \rho_t \leq 1/\mu^t_{max},\ S_\kappa : -1 \leq \kappa \leq 1 \tag{A6}$$

$$S_\gamma : \text{if } \rho_t > 0,\ -1 + \max(\mu^t_{max}\rho_t :\ t = 1, ..., T) < \gamma < 1 - \max(\mu^t_{max}\rho_t :\ t = 1, ..., T) \tag{A7}$$

$$\text{if } \rho_t \leq 0\ \ -1 + \max(\mu^t_{min}\rho_t :\ t = 1, ..., T) < \gamma < 1 - \max(\mu^t_{min}\rho_t :\ t = 1, ..., T) \tag{A8}$$

where μ^t_{max} and μ^t_{min} are the largest and smallest eigenvalues of the matrix $\mathbf{W}_t$. When stationarity is not a concern, we only need the condition of S_{ρ_t} to ensure the matrix $\mathbf{I} - \rho_t \mathbf{W}_t$ is invertible.

Note that in the MCMC algorithm, we do not restrict the dynamic parameters in the stationarity space. If the researcher thinks that stationarity is relevant in a particular research, she can check whether the conditions stated in Equation (A6) to Equation (A8) are met based on the MCMC output and decide whether to re-specify the model to achieve stationarity.

A.1.3. The MCMC Algorithm. The parameters can be sequentially updated by randomly sampling from their conditional posterior distributions, and the updating process is iterative until the Markov chain converges in the ergodic space of the joint posterior distribution. The sampling scheme can be briefly summarized as the following:

1. Choose starting values of the parameters

2. Sequentially sample the parameters from the following conditional posterior distributions with most updated values of the parameters:

(a) Update γ by sampling from the conditional posterior, $\gamma \sim \mathcal{TN}_{S_\gamma}(\bar{\gamma}, \sigma_\gamma^2)$, where :

$$\sigma_\gamma^2 = (\sigma_e^{-2} \sum_{t=2}^{T} \mathbf{y}_{t-1}' \mathbf{y}_{t-1})^{-1} \tag{A9}$$

$$\bar{\gamma} = \sigma_\gamma^2 \sigma_e^{-2} \sum_{t=2}^{T} \mathbf{y}_{t-1}' \left[\mathbf{A}(\rho_t)\mathbf{y}_t - \mathbf{X}_{1t}\beta - \mathbb{X}_{2t}\mathbf{b} - \mathbf{X}_{2t}\mathbf{c}_t - \mathfrak{Z}(\omega \cdot \mathbf{f}_t)\right] \tag{A10}$$

(b) Jointly update $\boldsymbol{\beta}$ and $\boldsymbol{\omega}$: denote $\tilde{X}_{1,it} = (X_{1,it}', \zeta_i' \cdot \mathbf{f}_t')'$ a $(2p+r) \times 1$ vector, and $\tilde{\mathbf{X}}_{1t}$ the corresponding $(2p+r) \times N$ matrix. $\tilde{\mathbf{B}}_0 = \begin{pmatrix} \mathbf{B}_0 & \mathbf{0} \\ \mathbf{0} & \Sigma_\omega \end{pmatrix}$ with $\Sigma_\omega = \mathrm{Diag}(\tau_1^2, \ldots, \tau_r^2)$. And $\tilde{\boldsymbol{\beta}}_0 = (\boldsymbol{\beta}_0', \mathbf{0}')'$. The joint conditional posterior distribution of $\boldsymbol{\beta}$ and $\boldsymbol{\omega}$ is $(\boldsymbol{\beta}', \omega')' \sim \mathcal{N}(\bar{\boldsymbol{\beta}}, \mathbf{B}_1)$, where

$$\mathbf{B}_1 = \left(\tilde{\mathbf{B}}_0 + \sigma_e^{-2} \sum_{t=1}^{T} \tilde{\mathbf{X}}_{1t}' \tilde{\mathbf{X}}_{1t}\right)^{-1} \tag{A11}$$

$$\bar{\boldsymbol{\beta}} = \mathbf{B}_1 \left(\tilde{\mathbf{B}}_0^{-1} \tilde{\boldsymbol{\beta}}_0 + \sigma_e^{-2} \sum_{t=1}^{T} \tilde{\mathbf{X}}_{1t}' \left[\mathbf{A}(\rho_t)\mathbf{y}_t - \gamma \mathbf{y}_{t-1} - \mathbb{X}_{2t}\mathbf{b} - \mathbf{X}_{2t}\mathbf{c}_t\right]\right) \tag{A12}$$

(c) Jointly update $\mathbf{c}_t$ and $\mathbf{f}_t$: denote $\tilde{X}_{2,it} = (X_{2,it}', \omega' \cdot \zeta_i')'$ and $\tilde{\mathbf{X}}_{2t}$ the corresponding matrix. $\tilde{\mathbf{E}} = \begin{pmatrix} \mathbf{E}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$. The joint conditional posterior distribution of $\mathbf{c}_t$ and $\mathbf{f}_t$ is $(\mathbf{c}_t', \mathbf{f}_t')' \sim \mathcal{N}(\bar{\mathbf{c}}_t, \mathbf{E}_t)$, where

$$\mathbf{E}_t = (\tilde{\mathbf{E}}^{-1} + \sigma_e^{-2} \tilde{\mathbf{X}}_{2t}' \tilde{\mathbf{X}}_{2t})^{-1} \tag{A13}$$

$$\bar{\mathbf{c}}_t = \mathbf{E}_t \sigma_e^{-2} \tilde{\mathbf{X}}_{2t}' (\mathbf{A}(\rho_t)\mathbf{y}_t - \gamma \mathbf{y}_{t-1} - \mathbf{X}_{1t}\beta - \mathbb{X}_{2t}\mathbf{b}) \tag{A14}$$

(d) Jointly update $\mathbf{b}_i$ and ζ_i: denote $\tilde{X}_{2,it} = (X'_{2,it}, \omega' \cdot \mathbf{f}'_t)'$ and $\tilde{\mathbf{D}} = \begin{pmatrix} \mathbf{D}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$. The joint conditional posterior distribution of $\boldsymbol{\beta}_i$ and ζ_i is $(\boldsymbol{\beta}'_i, \zeta'_i)' \sim \mathcal{N}(\bar{\mathbf{b}}_i, \mathbf{D}_i)$, where

$$\mathbf{D}_i = (\tilde{\mathbf{D}}^{-1} + \sigma_e^{-2} \sum_{t=1}^{T} \tilde{X}_{2,it} \tilde{X}'_{2,it})^{-1} \tag{A15}$$

$$\bar{\mathbf{b}}_i = \mathbf{D}_i \sigma_e^{-2} \sum_{t=1}^{T} \tilde{X}_{2,it} (y_{it} - \rho_t \sum_{j=1}^{N} w_{ijt} y_{jt} - X'_{1,it}\beta - X'_{2,it} c_t) \tag{A16}$$

(e) Update ρ_t: because the conditional posterior distribution of ρ_t is not a standard distribution from which we can directly sample, we apply an MH algorithm to update ρ's. Re-arrange the posterior to develop the conditional distribution of the parameters to be updated: define $\varepsilon(\rho_t) = \mathbf{y}_t - \rho_t \mathbf{W}_t \mathbf{y}_t - \gamma \mathbf{y}_{t-1} - \mathbf{X}_{1t}\boldsymbol{\beta} - \mathbb{X}_{2t}\mathbf{b} - \mathbf{X}_{2t}\mathbf{c}_t - \mathfrak{Z}(\omega \cdot \mathbf{f}_t)$. For $t = 2, \ldots, T-1$:

$$\pi(\rho_t|\mathbf{Y}) \propto \sigma_e^{-N} |\mathbf{A}(\rho_t)| \exp\left[-\frac{1}{2\sigma_e^2}\varepsilon(\rho_t)'\varepsilon(\rho_t)\right] \exp[-\frac{1}{2\sigma_\eta^2}(\rho_t - \kappa\rho_{t-1} - \mathbf{Z}'_t\boldsymbol{\alpha})^2] \exp[-\frac{1}{2\sigma_\eta^2}(\rho_{t+1} - \kappa\rho_t - \mathbf{Z}'_{t+1}\boldsymbol{\alpha})^2] \tag{A17}$$

The proposal density is $\rho^* \sim \mathcal{TN}_{S_{\rho_t}}(\phi_1, \Phi_1)$ where

$$\Phi_1 = \left(\sigma_e^{-2}(\mathbf{W}_t\mathbf{y}_t)'(\mathbf{W}_t\mathbf{y}_t) + (1+\kappa^2)\sigma_\eta^{-2}\right)^{-1} \tag{A18}$$

$$\phi_1 = \Phi_1 \left(\sigma_e^{-2}(\mathbf{W}_t\mathbf{y}_t)'\varepsilon(\rho_t) + \sigma_\eta^{-2}[\kappa(\rho_{t-1} + \mathbf{Z}'_t\boldsymbol{\alpha}) + \kappa(\rho_{t+1} - \mathbf{Z}'_{t+1}\boldsymbol{\alpha})]\right) \tag{A19}$$

Draw a proposal from the proposal distribution, and calculate the posterior:

$$\zeta_t = \frac{|\mathbf{A}(\rho_t^*)|\exp[-\frac{1}{2\sigma_e^2}\varepsilon(\rho_t^*)'\varepsilon(\rho_t^*)]\exp[-\frac{1}{2\sigma_\eta^2}(\rho_t^* - \kappa\rho_{t-1} - \mathbf{Z}_t'\boldsymbol{\alpha})^2]\exp[-\frac{1}{2\sigma_\eta^2}(\rho_{t+1} - \kappa\rho_t^* - \mathbf{Z}_{t+1}\boldsymbol{\alpha})^2]}{|\mathbf{A}(\rho_t)|\exp[-\frac{1}{2\sigma_e^2}\varepsilon(\rho_t)'\varepsilon(\rho_t)]\exp[-\frac{1}{2\sigma_\eta^2}(\rho_t - \kappa\rho_{t-1} - \mathbf{Z}_t'\boldsymbol{\alpha})^2]\exp[-\frac{1}{2\sigma_\eta^2}(\rho_{t+1} - \kappa\rho_t - \mathbf{Z}_{t+1}'\boldsymbol{\alpha})^2]} \tag{A20}$$

The determinant can be calculated as $|\mathbf{A}(\rho_t)| = \prod_{i=1}^{m}(1-\rho_t\lambda_i)$ where λ_i is an eigenvalue of $\mathbf{W}_t$ (citations). Accept ρ_t^* as the updated value with probability $\min\{\zeta_t, 1\}$.

(f) Update κ by sampling from its conditional posterior, $\kappa|\boldsymbol{\rho}, \boldsymbol{\alpha}, \sigma_\eta^2 \propto \exp[-\frac{1}{2\sigma_\eta^2}\sum_{t=2}^{T}(\rho_t - \kappa\rho_{t-1} - \mathbf{Z}_t'\boldsymbol{\alpha})^2]$, where

$$\kappa \sim \mathcal{N}_{S_\kappa}(\bar{\kappa}, \sigma_\kappa^2),\ \sigma_\kappa^2 = (\sigma_\eta^{-2}\sum_{t=2}^{T}\rho_{t-1}^2)^{-1},\ \bar{\kappa} = \sigma_\kappa^2\sigma_\eta^{-2}\sum_{t=2}^{T}(\rho_t - \mathbf{Z}_t'\boldsymbol{\alpha})\rho_{t-1} \tag{A21}$$

(g) Update $\boldsymbol{\alpha}$ by sampling from its conditional posterior, $\boldsymbol{\alpha} \sim \mathcal{N}(\bar{\boldsymbol{\alpha}}, \mathbf{A}_1)$, where

$$\mathbf{A}_1 = \left(\mathbf{A}_0 + \sigma_\eta^{-2}\sum_{t=1}^{T}\mathbf{Z}_t\mathbf{Z}_t'\right)^{-1} \tag{A22}$$

$$\bar{\boldsymbol{\alpha}} = \mathbf{A}_1\left(\mathbf{A}_0^{-1}\boldsymbol{\alpha}_0 + \sigma_\eta^{-2}\sum_{t=1}^{T}\mathbf{Z}_t\,(\rho_t - \kappa\rho_{t-1})\right) \tag{A23}$$

(h) Update τ_j^2 by sampling from its conditional posterior, $\tau_j^{-2} \sim \mathcal{IG}(\sqrt{\frac{\lambda^2}{\omega_j^2}}, \lambda^2), \quad \forall 1 \leq j \leq r$

(i) Update $\mathbf{D}$ by sampling from its conditional posterior, $\mathbf{D} \sim \mathcal{W}(d_1, \mathbf{D}_1)$, where $d_1 = d_0 + N$ and $\mathbf{D}_1 = \mathbf{D}_0 + \sum_{i=1}^{N} b_i b_i'$

(j) Update $\mathbf{E}$ by sampling from its conditional posterior, $\mathbf{E} \sim \mathcal{W}(e_1, \mathbf{E}_1)$, where

$e_1 = e_0 + T$, and $\mathbf{E}_1 = \mathbf{E}_0 + \sum_{t=1}^{T} c_t c_t'$

(k) Update σ_η^{-2} by sampling from its conditional posterior, $\sigma_\eta^{-2} \sim \mathcal{G}(\tilde{g}_3, \tilde{g}_4)$, where $\tilde{g}_3 = g_3 + T$, and $\tilde{g}_4 = g_4 + (\rho_1 - \mathbf{Z}_1'\boldsymbol{\alpha})^2 + \sum_{t=2}^{T}(\rho_t - \kappa\rho_{t-1} - \mathbf{Z}_t'\boldsymbol{\alpha})^2$

(l) Update σ_e^{-2}: define $\varepsilon(\rho_t) = \mathbf{y}_t - \rho_t \mathbf{W}_t \mathbf{y}_t - \gamma \mathbf{y}_{t-1} - \mathbf{X}_{1t}\beta - \mathbb{X}_{2t}\mathbf{b} - \mathbf{X}_{2t}\mathbf{c}_t - \mathfrak{Z}(\boldsymbol{\omega} \cdot \mathbf{f}_t)$, and sample from the conditional posterior $\sigma_e^{-2} \sim \mathcal{G}(\tilde{g}_1, \tilde{g}_2)$, where $\tilde{g}_1 = g_1 + NT$, and $\tilde{g}_2 = g_2 + \sum_{t=1}^{T} \varepsilon(\rho_t)'\varepsilon(\rho_t)$

(m) Update λ^2 by sampling from its conditional posterior, $\lambda^2 \sim \mathcal{G}(g_5 + r, g_6 + \frac{1}{2}\sum_{j=1}^{r} \tau_j^2)$

(n) Following Fruhwirth-Schnatter and Wagner (2010), we make a permutation on $\boldsymbol{\omega}$, $\mathfrak{Z}$ and $\mathbf{F}$: $\forall 1 \leq j \leq r$, since $\omega_j \cdot \boldsymbol{\zeta}_{ij} \cdot \mathbf{f}_{tj} = \omega_j \cdot (-\boldsymbol{\zeta}_{ij}) \cdot (-\mathbf{f}_{tj}) = (-\omega_j) \cdot (-\boldsymbol{\zeta}_{ij}) \cdot \mathbf{f}_{tj} = (-\omega_j) \cdot \boldsymbol{\zeta}_{ij} \cdot (-\mathbf{f}_{tj})$, we generate a random number v_j from $\mathcal{U}(0, 1)$. If $\frac{1}{4} < v_j \leq \frac{1}{2}$, let $\boldsymbol{\zeta}_{ij} = -\boldsymbol{\zeta}_{ij}$ and $f_{t,j} = -f_{t,j}$. If $\frac{1}{2} < v_j \leq \frac{3}{4}$, let $\omega_j = -\omega_j$ and $\boldsymbol{\zeta}_{ij} = -\boldsymbol{\zeta}_{ij}$. And if $\frac{3}{4} < v_j \leq 1$, let $\omega_j = -\omega_j$ and $f_{t,j} = -f_{t,j}$. $\quad 1 \leq i \leq N, \quad 1 \leq t \leq T, \quad 1 \leq j \leq r.$

3. repeat Step 2 until the chain converges

Based on our tests, the MH algorithm works smoothly and the acceptance rate is between 20% and 70%. When the proposal distribution is changed from a tailored one to a random walk, mixing does not change much and the acceptance rate range is almost the same.

A.2. Monte Carlo Studies

We generate datasets in the simulation studies reported in the main text following the DGP as below:

$$
\begin{gathered}
y_{it} = \rho_t \mathbf{w}_{it}\mathbf{y}_t + \beta_0 + \beta_1 x_{1,it} + \beta_2 x_{2,it} + \\
(\beta_3 + b_{1,i} + c_{1,t})\mathbf{w}_{it}\mathbf{x}_{1t} + (\beta_4 + b_{2,i} + c_{2,t})\mathbf{w}_{it}\mathbf{x}_{2t} + \\
\nu_i + \psi_t + \boldsymbol{\zeta}_i \mathbf{f}_t + \varepsilon_{it}, \\
\boldsymbol{b}_i = (b_{1,i}, b_{2,i}) \overset{iid}{\sim} \mathcal{N}_2(0, I_2), \quad \boldsymbol{c}_t = (c_{1,t}, c_{2,t}) \overset{iid}{\sim} \mathcal{N}_2(0, I_2), \quad \varepsilon_{it} \overset{iid}{\sim} \mathcal{N}(0,1).
\end{gathered}
\tag{A24}
$$

The true values of the coefficients are $\beta_0 = 5,\ \beta_1 = 1,\ \beta_2 = 3,\ \beta_3 = -2,\ \beta_4 = 2$. We do not include the lagged outcome variable in the model simply because it does not cause any further methodological complications.

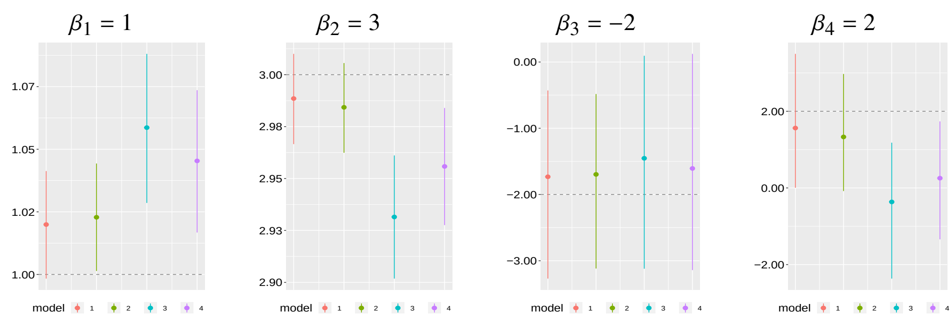
A.2.1. Study I. The adjacency matrix expression of the network is as follows:

$$
\mathbf{W} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{w}_2 & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{w}_{20} \end{bmatrix}_{200\times 200}, \text{ where } \mathbf{w}_1 = \cdots = \mathbf{w}_{20} = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 0 \end{bmatrix}_{10\times 10},
$$

and $\mathbf{0}$ denotes a 10×10 matrix with all elements equal to zero. We row-standardize the network when fitting the models.

We run 12 models in this study and each model has a large number of parameters. Due to the space limitations, here we only report the results based on data generated by setting $\kappa_w = 0.9$ (strong confounding). Figure A1 shows the estimated $\boldsymbol{\beta}$ based on the four model specifications. `M1`, the ρ_t-MLST-MF model with 2 factors performs the best, and `M2` with

10 initial factors and Bayesian shrinkage also performs well and almost as well as `M1`. The ρ_t-MLST-FE performs the worst, even worse than the ρ-MLST-MF model, which shows how important the multifactor term is for bias-correction. Figure A2 reports the posteriors of the group-level parameters in the state-equation of ρ_t, and the two multifactor models nicely recover the group-level parameters with 95% credibility intervals covering the true values. But the fixed-effect model overestimates some of the parameters. Figure A3 shows the posteriors of the variance of the error term ϵ. The two ρ_t-MLST-MF models correctly estimate the error distribution, but the constant-ρ model and the fixed-effect model leave the varying part of the spatial term or part of the latent confounders in the error term, resulting in over-estimation of its variance.

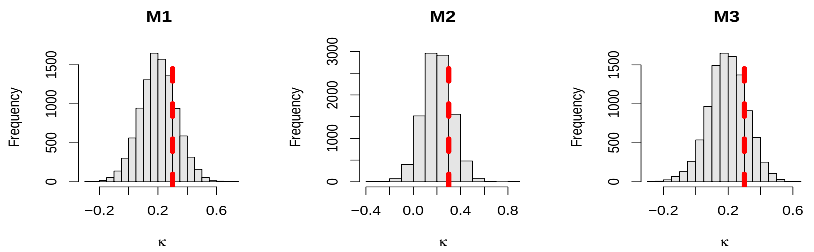


(a) (b) (c) (d)

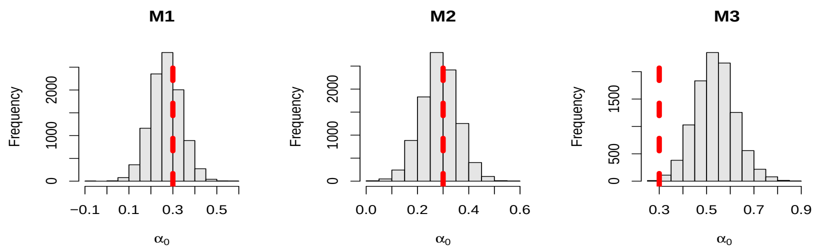
Figure A1. Coefficients $\boldsymbol{\beta}$ (Study I)

Note: the dashed line in each figure indicates the true value of the coefficient, and the vertical segments are 95% credibility intervals based on various model specifications.

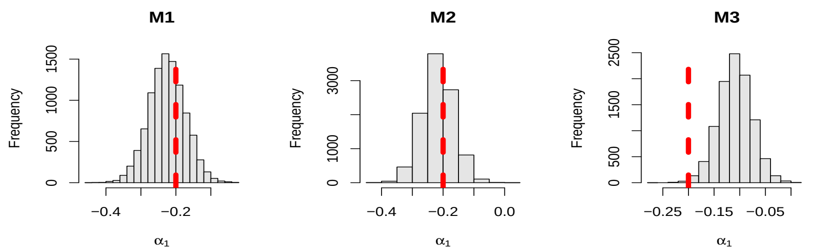
A.2.2. Study II. In this study, the true network interdependence is constant and equal to zero. Figure A4 reports the posteriors of the coefficients (β_1 and β_2) and the exogenous network effects (β_3 and β_4). All the multifactor models, `M1`, `M2`, `M4`, recover these parameters very well, but the fixed-effect model, `M3` mis-estimates three of the four coefficients.



(a) Autoregressive Coefficient κ



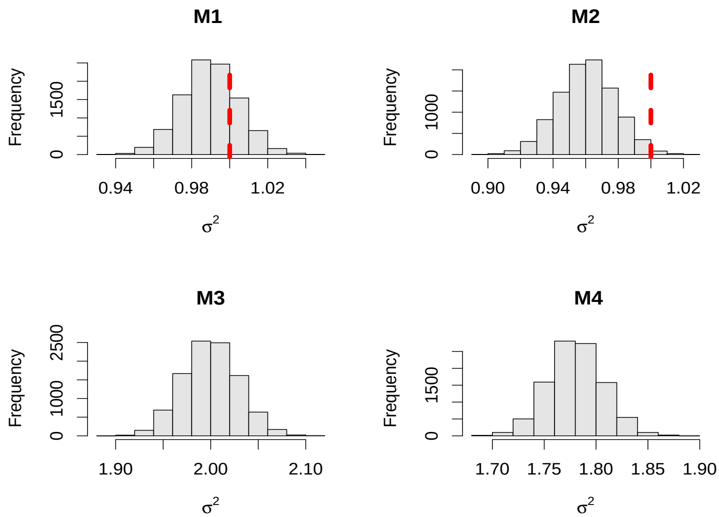
(b) Group-level Intercept α_0



(c) Group-level Coefficient α_1

Figure A2. Group-level Parameter α_1 (Study I)
Note: The vertical dashed line in each graph indicates the true value of the parameter.

Figure A5 shows the estimates of group-level parameters. The multifactor varying-ρ_t models encounter a convergence problem when estimating the parameters in the state equation. This is because `M1` and `M2` detect that interdependence barely varies over time and therefore the group-level parameters cannot be identified due to the lack of variance



Individual-Level Error σ_{ε}^{2}

Figure A3. Variance of Error Term ϵ (Study 1)
Note: The vertical dashed line in each graph indicates the true value of the parameter.

of the upper-level parameters. But the fixed-effect model produces the trajectory of ρ_t with variation so large that the group-level parameters do not have convergence problems. However, the well-shaped posteriors are biased. Figure A6 displays the estimated variance of the error term. Again, the multifactor models recover the true value, whereas the ρ_t-MLST-FE over-estimates the variance.

A.2.3. Study III. In this study, we use the ICEWS network that reflects “the undirected dyadic relations between the 50 most active countries...during a 112-month period from 2006 to 2015" (He and Hoff 2019). The network tie is defined as a degree of cooperation/conflict between two countries estimated using the event data. Because the network is too densely connected, we re-define the network by keeping the ties with top 3

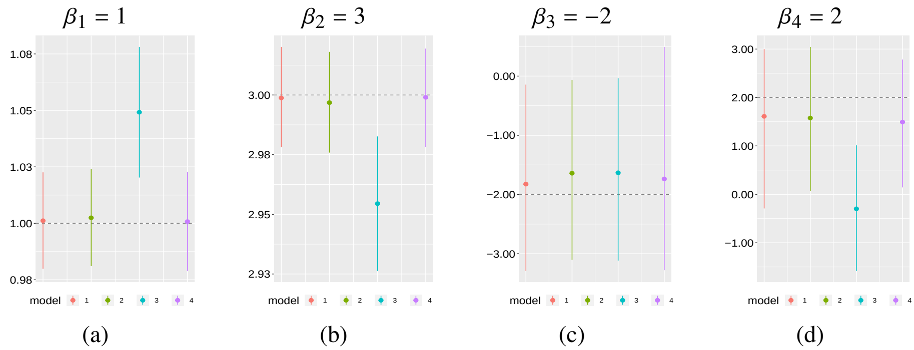


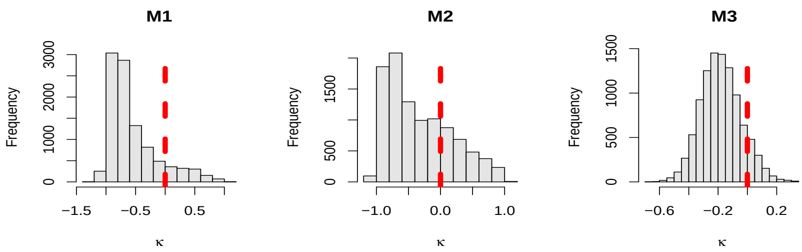
Figure A4. *Coefficients $\boldsymbol{\beta}$ (Study II)*

Note: the dashed line in each figure indicates the true value of the coefficient, and the vertical segments are 95% credibility intervals based on various model specifications.

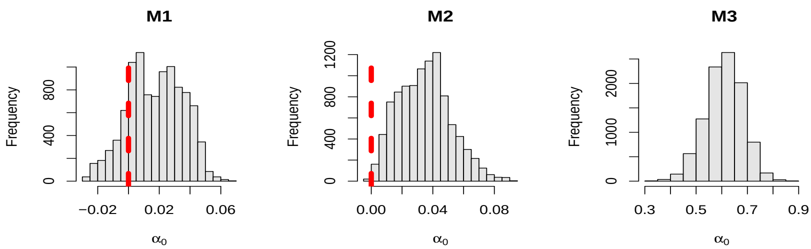
countries in terms of their strength of connections. Also, we take out three countries (the United States, China, and Russia) from the original network because their relationships with other countries are so strong that make all other ties virtually zero. Due to space limitations, here we only report the results of factor selection based on `M2` in Figure A7. The patterns of the results of other parameters are similar to those in Study I, and details of those posteriors are available upon request.

A.3. Empirical Applications

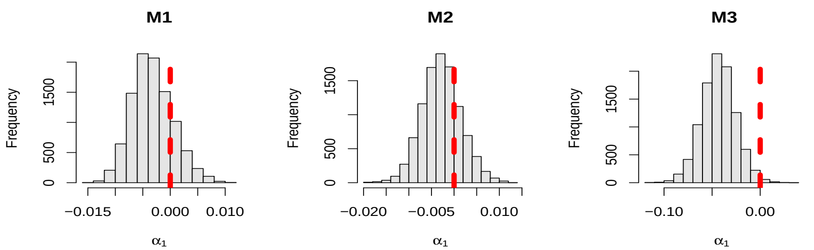
A.3.1. Migration and Terrorism. Table A1 reports descriptive statistics of the variables used in the re-analysis of network interdependence of terrorist attacks. For the definitions of these variables, refer to the original article of B&B. Figure A8 reports the results of factor selection and six latent factors (Factors 4-5, 7, 8-10) clearly escape Bayesian shrinkage. In Figure A9, we report the estimated posteriors of individual- and group-level coefficients. The two multifactor models produce similar posteriors, whereas estimates of



(a) Autoregressive Coefficient κ



(b) Group-level Intercept α_0



(c) Group-level Coefficient α_1

Figure A5. Group-level Parameter α_1 (Study II)

Note: The vertical dashed line in each graph indicates the true value of the parameter.

some coefficients based on the fixed-effect model are noticeably different from the other two models.

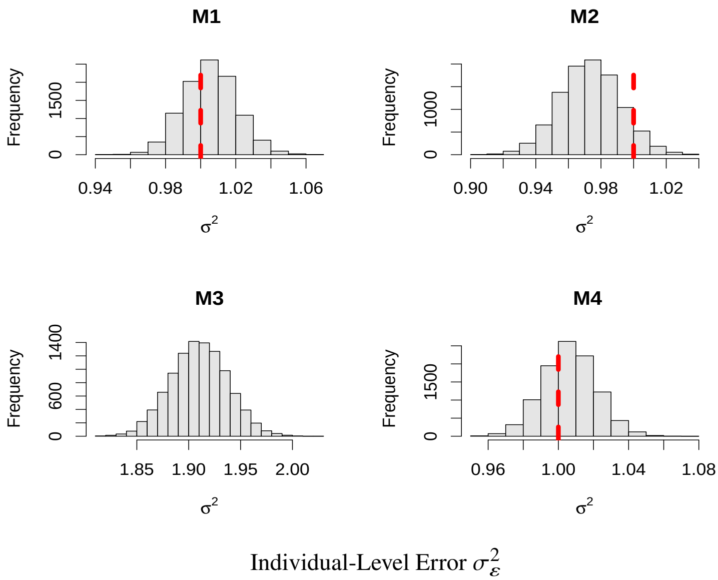


Figure A6. Variance of Error Term ϵ (Study II)

Note: The vertical dashed line in each graph indicates the true value of the parameter.

A.3.2. GATT/WTO and Free Trade. Table A2 and Table A3 report the definitions of variables and their descriptive statistics. For more details about the definitions, measures and data sources, refer to Chaudoin, Milner, and Pang (2015). Figure A10 compares the estimated ρ_t based on CMP and our initial analysis, when using the original scale of the dependent variable, tariff rates. The trajectories are more similar to each other than the results based on models with the dependent variable on a logarithm scale. Figure A11 shows that when not taking a logarithm of tariff rates, the number of factors is as large as 30. But with logged tariff rates, the number of factors is more reasonable, as shown in Figures A12 and A13. Figure A14 displays the posteriors of coefficients based on different model specifications. The two multifactor models mostly agree with each, and the fixed-effect model is more different.

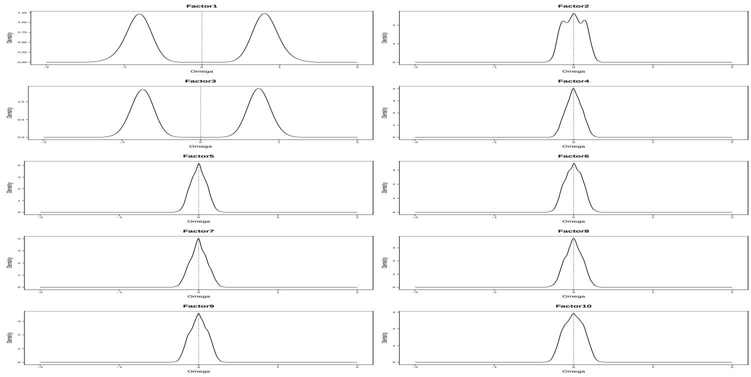


Figure A7. Factor Selection (Study III)
Note: The bimodal posteriors indicate that their associated factors should be included, while the unimodal ones indicate that their associated factors are virtually excluded from the model.

TABLE A1 *Descriptive Statistics: Migration and Terrorism*

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Pctl(75)	Max
Terrorist attacks (ln)	2,914	1.103	1.502	0.000	0.000	1.946	6.658
Personalist regime	2,914	0.120	0.325	0	0	0	1
Military regime	2,914	0.079	0.270	0	0	0	1
Single-party regime	2,914	0.225	0.418	0	0	0	1
Monarchy	2,914	0.065	0.246	0	0	0	1
Hybrid regime	2,914	0.044	0.206	0	0	0	1
GNI pc (ln)	2,914	7.344	1.560	3.919	6.061	8.553	10.752
Population (ln)	2,914	2.199	1.572	−1.609	1.225	3.119	7.146
Area (ln)	2,914	12.264	2.069	5.756	11.320	13.690	16.048
Inequality (GINI)	2,914	43.994	8.699	17.800	36.700	50.800	72.000
Durable regime	2,914	26.643	30.109	0	7	34	191
Failed state	2,914	0.496	1.474	0	0	0	14
Cold War	2,914	0.677	0.468	0	0	1	1
Interstate conflict	2,914	0.100	0.517	0	0	0	3
ucdp_type3	2,914	0.306	0.790	0	0	0	3
Domestic conflict	2,914	11.925	1.675	7.691	10.702	13.178	17.200
Lagged DV	2,914	1.086	1.494	0.000	0.000	1.792	6.658

A.4. Comparison of MLST-MF and SAOM

As discussed in the main text, the proposed MLST-MF addresses different methodological issues (*i.e.*, network interdependence and latent confounders) from what the SAOM model

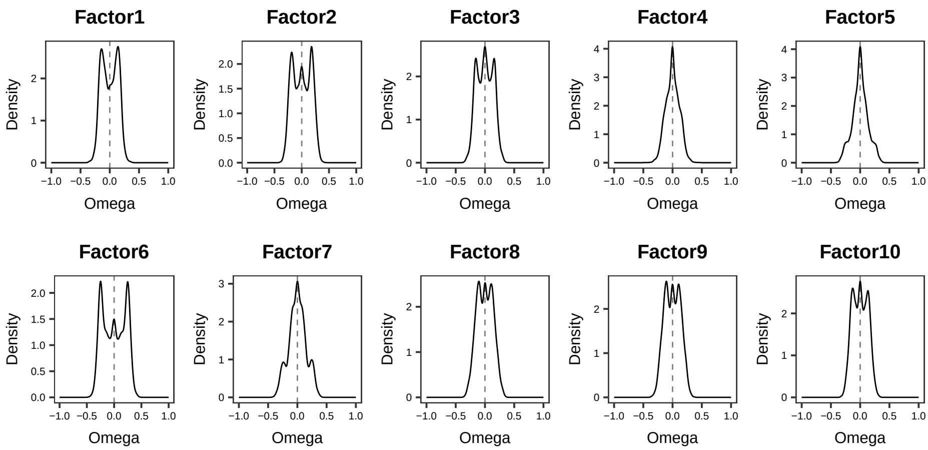


Figure A8. Factor Selection (Global Migration Network)

does. Nonetheless, like the SAOM model, MLST-MF also concerns the selection process of network evolution, though it treats the process as a nuisance. We use two examples to illustrate the comparison summarized in Table 2 in the main text.

A.4.1. A Simulated Example. We use a simulated example to show that MLST-MF can be used as a simpler alternative to SAOM when the researcher is only interested in network influence. We simulate a longitudinal network based on SAOM and generate nodal outcomes with the network and homophily. Then we fit a MLST-MF model to estimate network interdependence. For the nodal attributes that drives the network evolution, we leave them into the error term of MLST-MF and rely on the multifactor approach to correct for bias. This simulation study investigates whether the MLST-MF model is able to reduce bias without specifying the selection equation.

As in the simulation studies in the main text, we generate a sample of 200 units in 50 periods. The network starts in Period 0 as the exact network in Study I, consisting

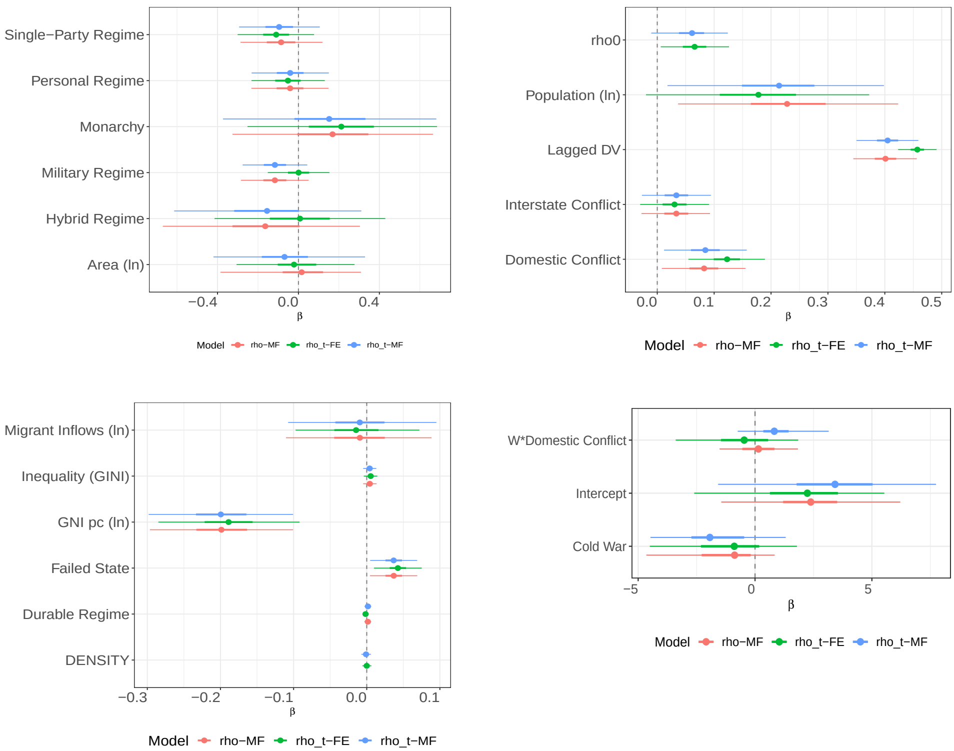


Figure A9. Posteriors of Coefficients (Migration & Terrorism)

Note: in each graph, the dots are the posterior means and the segments are 95% credibility intervals. In the legend, “Constant" indicates the ρ-MLST-MF model, “Fixed-Effects" indicates the ρ_t-MLST-FE model , and “Multifactor" indicates the ρ_t-MLST-MF model.

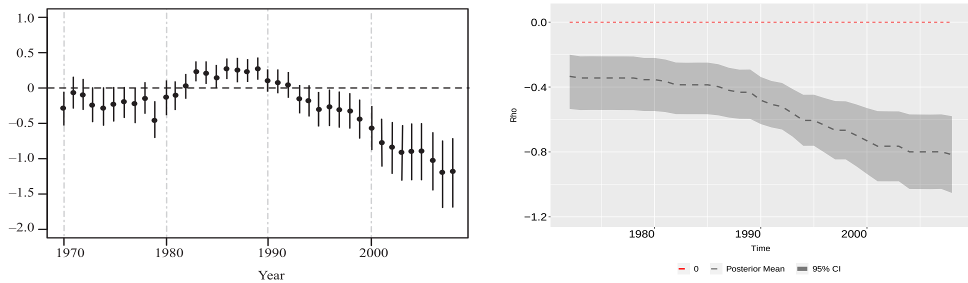
of 20 seperate groups each with 10 members. From Period 1 to Period 50, some units are randomly selected in each period to optimize their utilities by taking actor-oriented actions. Each of the selected units can choose to form a new link with a unit outside its group or drop an existing link with a unit within its group. The SAOM model specifies a tension function, and the selected units choose how to act by minimizing this function. We make the tension function partially determined by homophily between units. That is,

TABLE A2 *Covariates: GATT/WTO and Free Trade*

Covariate	Definition	Covariate	Definition
Ego's Own Conditions		Common Environment	
(possibly observed homophily)		`AV TARIFF`	Average Sample Tariff Level (t)
`REGIME`	Polity IV Score	`US HEGEMONY`	The degree of U.S. hegemony in the international market
`LN POP`	Log Population	`8 OPEN`	the openness of capital markets in the 8 largest countries in the world
`GDP PC`	GDP per capita	`GLOBAL WTO TRADE`	The percent of world trade that takes place in mutual-GATT/WTO member dyads
`EC CRISIS`	Economic Crisis	Network	
`BP CRISIS`	Balance of Payment Crisis	$\mathbf{W}_t$	Network formed by mutual WTO membership
`OFFICE`	Number of Years in Office	`DENSITY`	WTO Network Density
`GATT`	GATT/WTO Member State		

TABLE A3 *Descriptive Statistics: GATT/WTO and Free Trade*

Variable	N	Mean	St. Dev.	Min	Pctl(25)	Pctl(75)	Max
TARIFF	3,230	17.601	12.712	0.000	10.000	21.000	106.500
REGIME	3,230	0.117	6.944	−10	−7	7	10
LN POP	3,230	16.127	1.542	13.007	15.006	17.098	20.999
GDP PC	3,230	6.838	1.119	4.627	5.838	7.737	9.626
BP CRISIS	3,230	0.523	0.500	0	0	1	1
EC CRISIS	3,230	0.047	0.211	0	0	0	1
OFFICE	3,230	7.574	7.999	0	2	10	46
US HEGEMONY	3,230	0.139	0.011	0.117	0.130	0.146	0.161
AV TARIFF	3,230	17.516	3.517	10.450	15.424	19.949	23.825
8 OPEN	3,230	182.106	14.920	157.031	172.656	194.531	199.219
GLOBAL WTO TRADE	3,230	0.013	0.005	0.006	0.008	0.017	0.025
DENSITY	38	0.520	0.225	0.265	0.306	0.726	0.885



(a) CMP (b) ρ_t-MLST-MF

Figure A10. ρ_t: *Comparison with CMP2015*

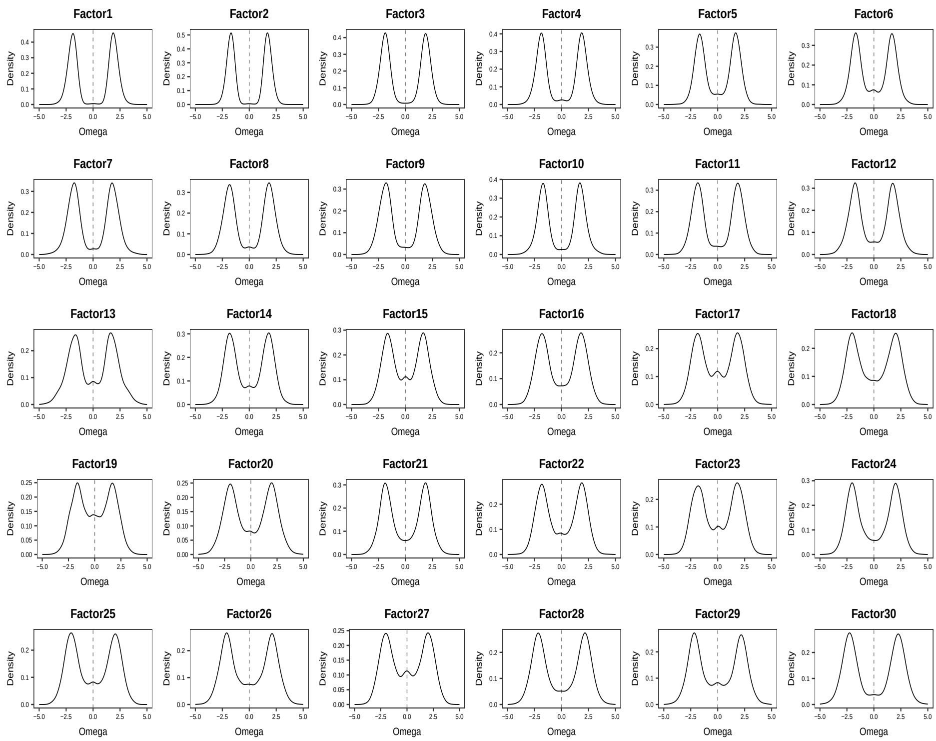


Figure A11. *Factor Selection (WTO Network and Varying-Slope* ρ_t*)*

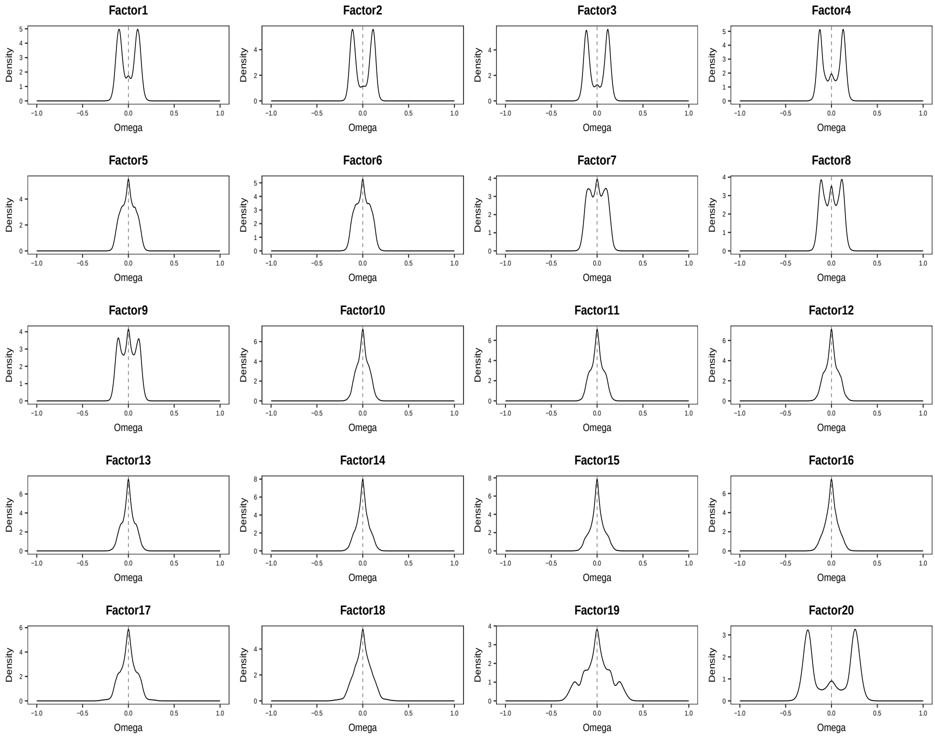


Figure A12. Factor Selection: ρ_t-MLST-MF

forming a tie with unit j changes unit i's utility as follows:

$$-a|(\xi_i - \xi_j)' f_t| + \varepsilon_{ij}. \tag{A25}$$

Here a is a structural parameter to be estimated in SAOM, and ξ_i and ξ_j are both two-dimensional nodal attributes. We assume that $a = 1$ in the DGP and sample ε_{ij} from $i.i.d.$ type I extreme value distribution with mean 0 and scale parameter 1. If unit i is to form a new tie with a unit outside its group at time t, the probability that it forms the tie with a

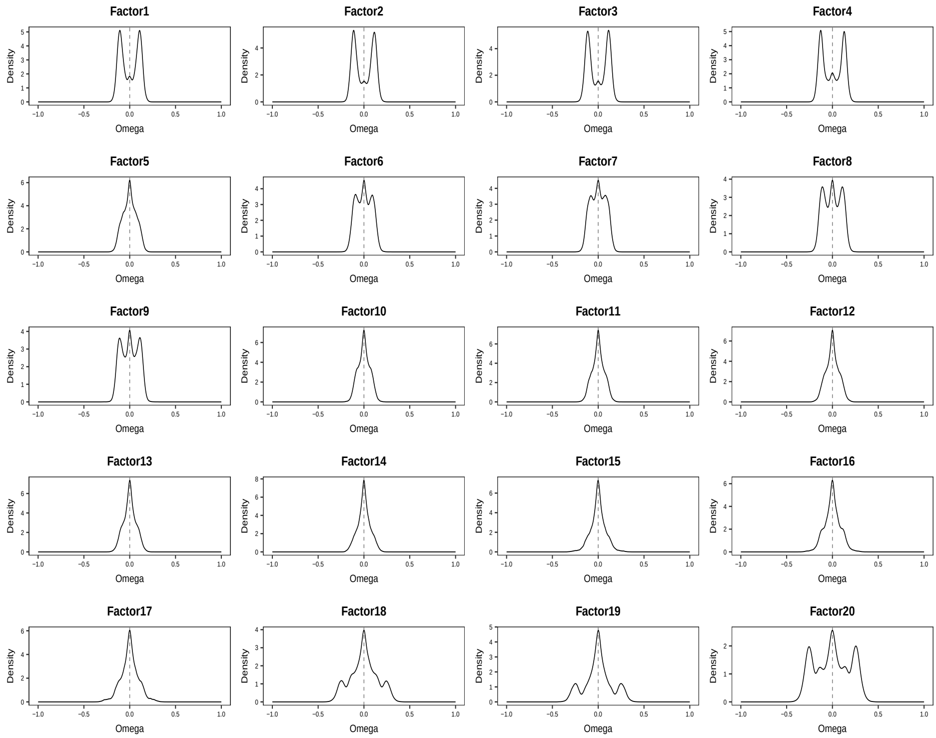


Figure A13. *Factor Selection: : ρ-MLST-MF*

specific unit j is:

$$\frac{\exp(-|(\xi_i - \xi_j)' f_t|)}{\sum_{k \notin G_i} \exp(-|(\xi_i - \xi_k)' f_t|)} \tag{A26}$$

where G_i is the set of units that are in the same group as unit i. Likewise, if unit i is to drop an existing tie with a unit within its group at time t, the probability that it drops the tie with a certain unit k is:

$$\frac{\exp(|(\xi_i - \xi_j)' f_t|)}{\sum_{k \in G_i} \exp(|(\xi_i - \xi_k)' f_t|)} \tag{A27}$$

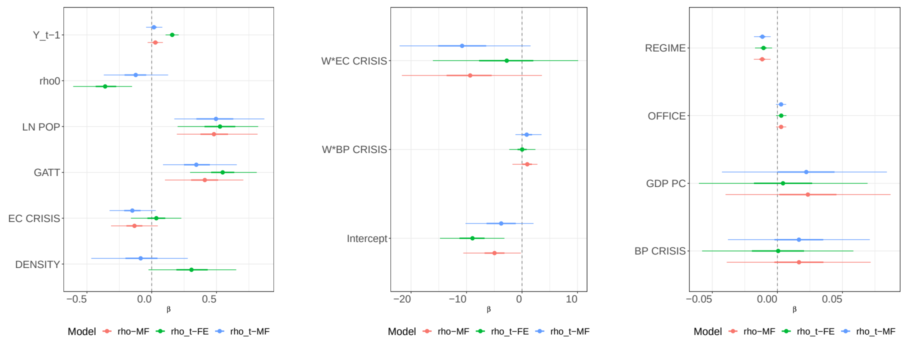


Figure A14. Posteriors of Invariant Parameters (GATT/WTO and Free Trade)

Note: in each graph, the dots are the posterior means and the segments are 95% credibility intervals.

Intuitively, a unit tends to form ties with similar units and drop ties with distinct ones.

The SAOM model sets network change to be a continuous time Markov process with structural parameters to be estimated in the rate of actions. In order to apply MLST-MF, we force selection to be a discrete process: In Period 1, we randomly select 4 units to either form a new link or drop an existing link. If the action is to form a new link, then the unit probabilistically forms a new link with a unit outside its group with probability stated in Equation (A26). If the action is to drop an existing link, then the units probabilistically drops a link with a unit within its group with probability specified in Equation (A27). We repeat the practice from Period 2 to Period 50 and generate a longitudinal network. This setup ensures that at expectation each unit only takes action once in the whole sample period so that the network does not change radically.

Nodal outcomes are generated following Equation (A24). We make the impact of homophily on outcomes vary over time; that is, the factor term in Equation (A24) is $\boldsymbol{\zeta}_i \mathbf{f}_t = \xi_{1i} f_{1t} + \xi_{2i} f_{2t}$ and ξ_{1i} and ξ_{2i} are the nodal attributes and f_{1t} and f_{2t} are the time-varying effects of these attributes in Equations (A26) and (A27). We fit the MLST-MF

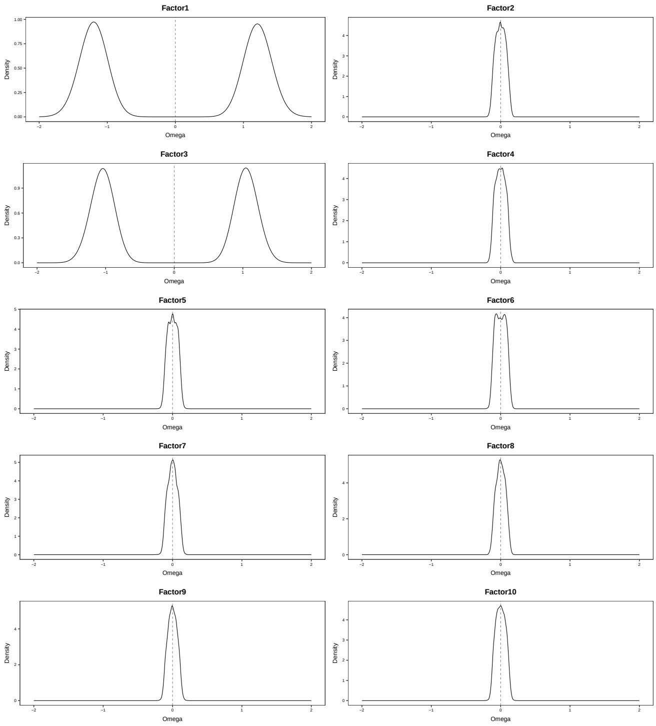


Figure A15. *Factor Selection for the Simulated Example*

model and apply the multifactor approach to approximate the impact of homophily rather than using the data of ξ_{1i} and ξ_{2i}. We include 10 initial factors and apply hierarchical shrinkage priors to the factor loadings to determine the number of factors. Bayesian shrinkage correctly detects 2 factors to be included in the regression model.

The true and estimated time-varying endogenous network influence is displayed in Figure A16. The 95% credibility interval of ρ_t covers the true value most of the time, and only occasionally it misses the true value. Note that this network evolves slowly and has binary ties, and identifying the effect of such a network is not easy. The MLST-MF model performs well in this simulation study, which demonstrates that the multifactor term is

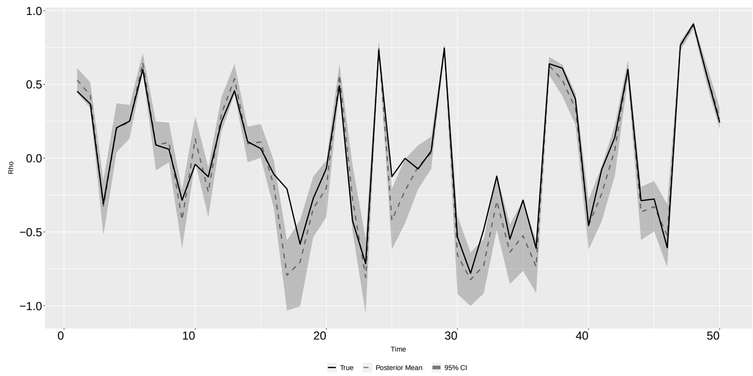


Figure A16. Estimated and True Network Influence of a Time-Varying Network

able to largely correct biases from the network selection process without modeling the network process.

A.4.2. An Empirical Example: Human Rights and Trade Network

Chyzh (2016) applies a co-evolution SAOM model to test two interactive hypotheses. Hypothesis 1 is “the level of human rights protections is positively (negatively) related to the probability of forming direct (indirect) trade-links." And Hypothesis 2 is “a state’s reliance on indirect trade-links is inversely related to its respect for human rights." Her empirical analysis draws on data of 126 countries in the years from 1987 to 1999. The tie of the global trade network is defined as the following: $w_{ijt} = 1$ if country i exports any goods to country j in year t; $w_{ijt} = 0$, otherwise. Under this definition, the network is a 126×126 directed, asymmetric, and binary network. Note that the network ties are dichotomized in the original study to meet the restrictions required by SAOMs. The trade network equation tests Hypothesis 1, and the dependent variable is a trade link from country i to country j at time t. Several nodal attributes of country i and country

j are included to explain the formation or dissolution of a trade link. The coefficient associated to human rights protections is primarily interesting. The behavior equation tests Hypothesis 2, and the outcome variable is human rights protections measured as a 9-scale ordinal variable with “0 as no respect for human rights" and “8 as full respect for human rights". The hypothesis is tested with the estimated impact of the ratio of indirect trade relationship (indirect degree), a nodal feature of network connections. Indirect degree is measured as the number of unique second-order links a node has. Standard control variables measuring national attributes are also included in the human rights equation. The empirical results based on the co-evolution model support both hypotheses.

We apply MLST-MF to estimate the interdependence of states' human rights protections in the global trade network as well as the effect of indirect degree on this nodal behavior. MLST-MF applies to continuous outcomes, and we treat the 9-scaled oridinal variable of human rights protections approximately as a continuous one. Because MLST-MF only focuses on explaining nodal behavior, it cannot test Hypothesis 1 on network evolution. The MLST-MF model is also different from the behavior equation in the SAOM model: network-effect terms are defining components for MLST-MF and they are interactive terms consisting of the network $\mathbf{W}_t$ and alters' outcomes or attributes. In other words, network effects in MLST-MF refer to *intervening* effects of the network. Therefore, the coefficient associated to a country's indirect degree is not regarded as a “network" effect but a direct effect of a nodal attribute (in this case, it is nodal's network position). Because of the substantive focus and the SAOM setup, the original study does not specify a network-effect term. We add a spatial lag term $\rho_t \mathbf{W}_t \mathbf{y}_t$ in the human rights equation to specify an MLST-MF model, where $\mathbf{W}_t$ is the global trade network and $\mathbf{y}_t$ is the vector of human rights practices of all the 126 sample countries at time t. There are reasons to suspect that human rights practices of trade partners are correlated with each other. For instance, Greenhill (2010) finds that inter-governmental organizations promote diffusion of human rights norms and practices among member states. Trade links also form channels for information flows, norm diffusion, and building of shared identities. Therefore, we expect network interdependence of human rights practices via trade links. Adding the spatial lag term turns the behavior equation into simultaneous equations.

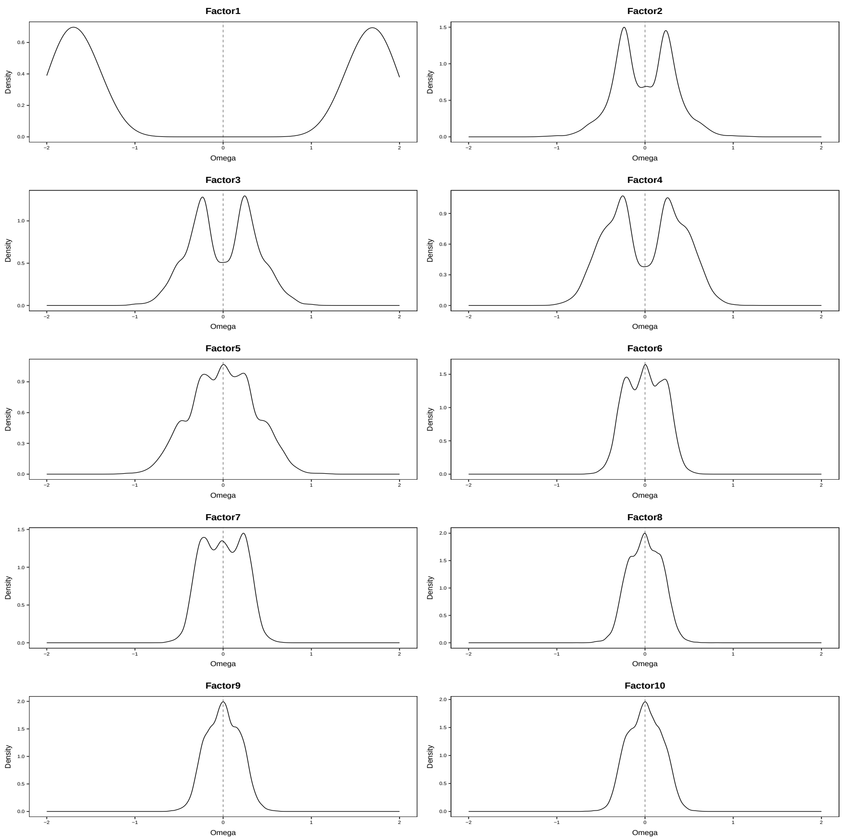


Figure A17. Results of Factor Selection in the Empirical Example

The spatial autoregressive coefficient is the parameter of primary interest in the MLST-MF model, whereas the original study focuses on the effect of indirect degree. From the perspective of the original study, MLST-MF is to correct for network interdependence, an omission of which may cause a biased estimate of the effect of indirect degree. Besides the spatial lag term, we include indirect degree as the only covariate in MLST-MF. We purposely omit all other covariates of nodal attributes in the human rights equation in the original study. Instead, we use two-way fixed effects and multiple factors to correct for bias caused by omitted attributes.[A1] The ρ_t process is simply specified as $\rho_t = \rho_0 + \varepsilon_t$. We include 10 initial factors, six of which (Factors 5-10) escape Bayesian shrinkage, as shown in Figure A17.

We report the estimated time series of ρ_t in Figure A18. MLST-MF finds that network influence is negative and barely varies over time. The coefficient associated to the ratio of indirect degree is -0.505, with the 95 % credible interval as $(-1.139,\ 0.156)$. The co-evolution model in the original study estimates this coefficient as -1.25 with the standard error as 0.23. The estimate based on MLST-MF is much smaller than SAOM, probably because the behavior equation of the SAOM model does not separate the negative impact of indirect degree from the negative network interdependence. The length of the 95% CI based on our model is also larger than that in SAOM, likely due to the difference between Bayesian and frequent inferences. In generally, our analysis provides much weaker evidence than the original study to support Hypothesis 2.

The negative network interdependence is not expected and is worthy of further theoretical explorations. Here we leave the task to future substantive research. As a methodological illustration, this example demonstrates that MLST-MF complements co-evolution models if network interdependence is the research interest. Even when

[A1]We fill in the missing values with observed mean value across 13 years for each country.

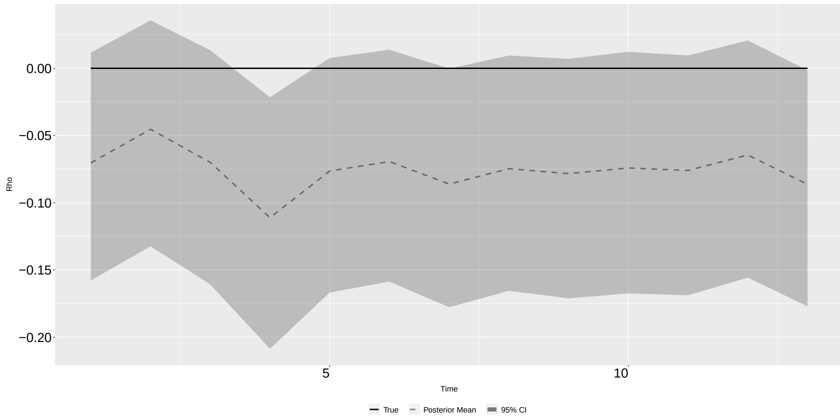


Figure A18. *Time-Varying Network Influence on Human Rights Protection*

network interdependence is not the quantity of primary interest, the proposed model could also serve as an alternative to co-evolution models because it estimates the coefficient of primarily interest by correcting for bias caused not only by the network selection but also by networked (interdependent) outcomes.

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