Resilience of Single-Layer and Multiplex Networks Following Sudden Changes to Tie Costs
SI APPENDIX

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I. EXPLORATION OF ISOLATED EFFECTS

A. Isolated effects: Triangle benefits only

We first examine single-layer networks (for which no spillover is possible) in which there are additional benefits to closed triangles. Network statistics are presented in Fig. 1. Under low costs, many triangles form, and the average degree of the network increases as triangles are incentivized more (Fig. 1A). This is because closed triangles scaffold the creation of addition triangles by providing affordances (e.g., new two-stars), forming a cascade. Such a cascade does not go on indefinitely, however. The costs of ties can set a practical limit, especially when each new edge must yield an increase in utility. The HL condition closely tracks the LL condition, because both conditions result from dynamics under low tie costs.

For values of d below a critical threshold, the LH condition tracks the HH condition. This is because the shock in which tie costs increase causes agents to drop ties, and triangles cannot be maintained. Past the critical threshold ($d = 0.8$ in our runs), some amount of resilience occurs. Some nodes are dropped, but the network is denser than networks that began with high tie costs. This first threshold is when the benefit of a closed triangle can offset the higher tie cost, so that a node in a closed triangle need not drop any ties. Past a second threshold ($d = 1.2$ in our runs), when the benefits to closed triangles are high enough, networks in the LH condition are indistinguishable from networks in the LL condition. See below for a derivation of these thresholds. Examining the average clustering of the network mirrors this finding (Fig. 1B). When triangles are incentivized and tie costs permit their closure, clustering maximizes fairly rapidly. Fig. 2 illustrates the types of network structures that emerge under each shock condition and varying benefits to closed triangles. Examining the average node utility at equilibrium, we also observe that resilience allows post-shock LH nodes to maintain higher utility even after costs increase than they would if costs had always been high.

B. Isolated effects: Spillover benefits only

We next consider a two-layer multiplex (and will do so for all subsequently presented results). Like incentives for closed triangles in a single-layer network, incentives for spillover ties provide a minimal model of structural entrenchment in a multiplex network. In general, we see a similar pattern of resilience for spillover as we did for triangles (Fig. 3). Unlike with triangles, however, the average degree under low tie costs does not continue to increase with the benefit to spillover ties (Fig. 3A). Rather, it plateaus. This is because spillover ties do not scaffold the creation of additional spillover ties, as closing triangles does. In other words, the existence of a spillover tie does not provide new opportunities for additional spillover ties. The critical threshold for some resilience in the LH condition is the same for spillover as for triangles, which is unsurprising when the benefit of a spillover tie can prevent a node from needing to drop a tie due to increased costs. Unlike with triangles, the network is not fully resilient until a much higher spillover benefit has been reached as compared with the triangle case ($e = 2$ in our runs). This is because each tie can only confer one unit of spillover benefit, whereas a single tie can be part of many triangles. See below for derivation of critical thresholds.

Past the first critical threshold, the average degree of the HL condition is slightly lower than for the LL condition. This is because all ties will be spillover ties under high costs and large e. This ends up making it more difficult for some nodes to find partners who would accept their offer to form a tie. The reason is that fewer would-be partners stand to increase their utility from adding a tie. Interestingly, the average utility received by a node at equilibrium in the LH condition is not any higher than that of a node in the HH condition. That is, nodes who end up in a high-cost environment experience no benefits nor costs, in the short run, on the basis of whether tie

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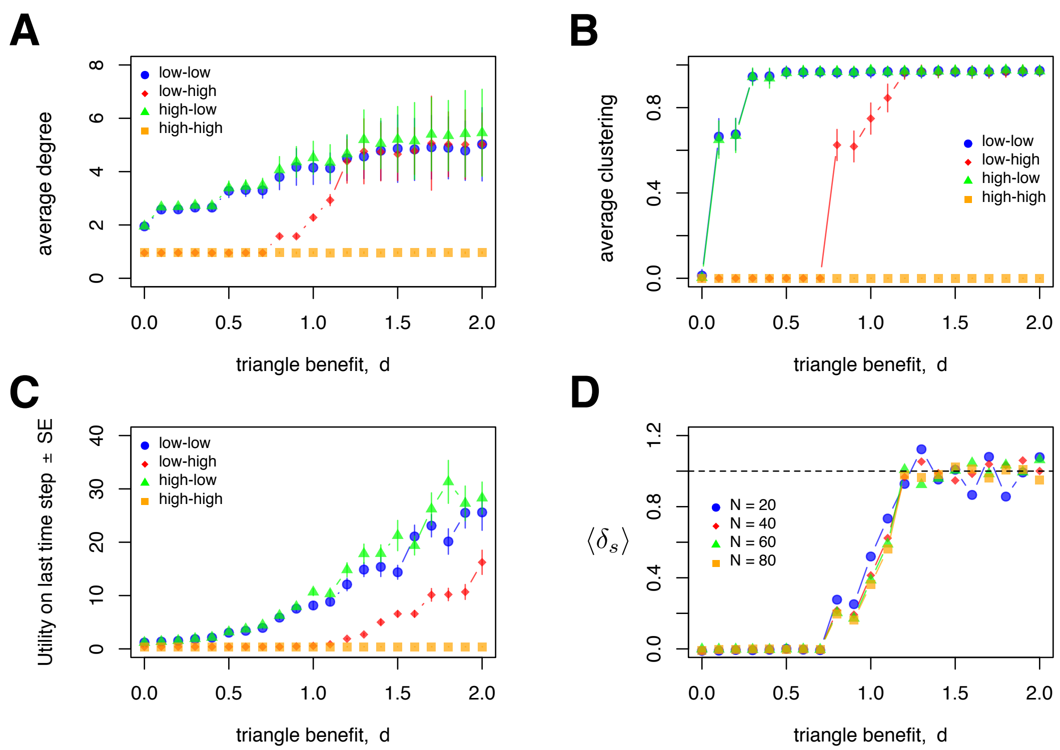


FIG. 1. Isolated effects: triangle benefits only ($e = 0$). (A–C) Average results for each of four shock conditions on a 40-node network. (A) Average node degree ± SD, (B) Average node clustering ± SD, (C) Average node utility at equilibrium ± SE, (D) Average resilience for LH condition, showing insensitivity to network size.

costs were initially high or low. This contrasts the case where we examine variations in triangle benefits. The difference is due to the fact that the benefit of additional spillover ties is compensated by the higher costs of more ties.

The network structures that emerge from spillover incentives are quite different from those that emerge from triangle incentives (Fig. 4). Under low tie costs, incentives for triangles created several tightly clustered but completely discrete communities. Incentives for spillover, on the other hand, tends to create fully connected graphs that exhibit low levels of triadic closure (Fig. 3B). This structure is not fully recovered in the LH shock condition. Rather, an intermediate structure emerges composed of several isolated chains or circles.

II. EXPLANATION OF TRANSITION POINTS

A. When resilience begins.

Here we derive conditions for when, on average, $k_{LH} > k_{HH}$. We consider only the isolated effects conditions for clarity. In addition, we focus on a minimal type of resilience observed in our simulations: when two or more edges are not possible under high tie costs but are present following a shock to high costs from initially low tie costs. Different types of resilience for different degree thresholds are also possible, as shown in later sections of this Appendix.

First, let us condition only triangle benefits ($e = 0$) under a single-layer network. Figure 5 indicates the threshold parameter values for additional ties. We see that under low tie costs, there is always the incentive to have at least two social ties, while under high costs, only one tie is incentivized. Under low tie costs, the utility for two and three ties is identical, and so adding a third tie only occurs if $d > 0$, which is what we observed (main text, Fig. 2). When tie costs increase from low to high, we see that the triangle benefits must be quite high to maintain three ties, unless the individual node already has three triangles. Thus, there is often a reduction from four or three ties to two. However, two ties can be stable as long as $d \geq 0.8$ and the agent is in a closed triangle, because only when it is below this threshold is there a strict increase in agent utility from dropping an edge. More generally, this minimal level of resilience between two and one network ties will be seen when the following two conditions are met: (1) a second edge will never be added *de novo* under high tie costs but will always be favored under low tie costs, and (2) the benefit to triangles ensures that, if a closed triangle exists, dropping an edge, and hence losing the triangle, will not be favored under either ties cost level.

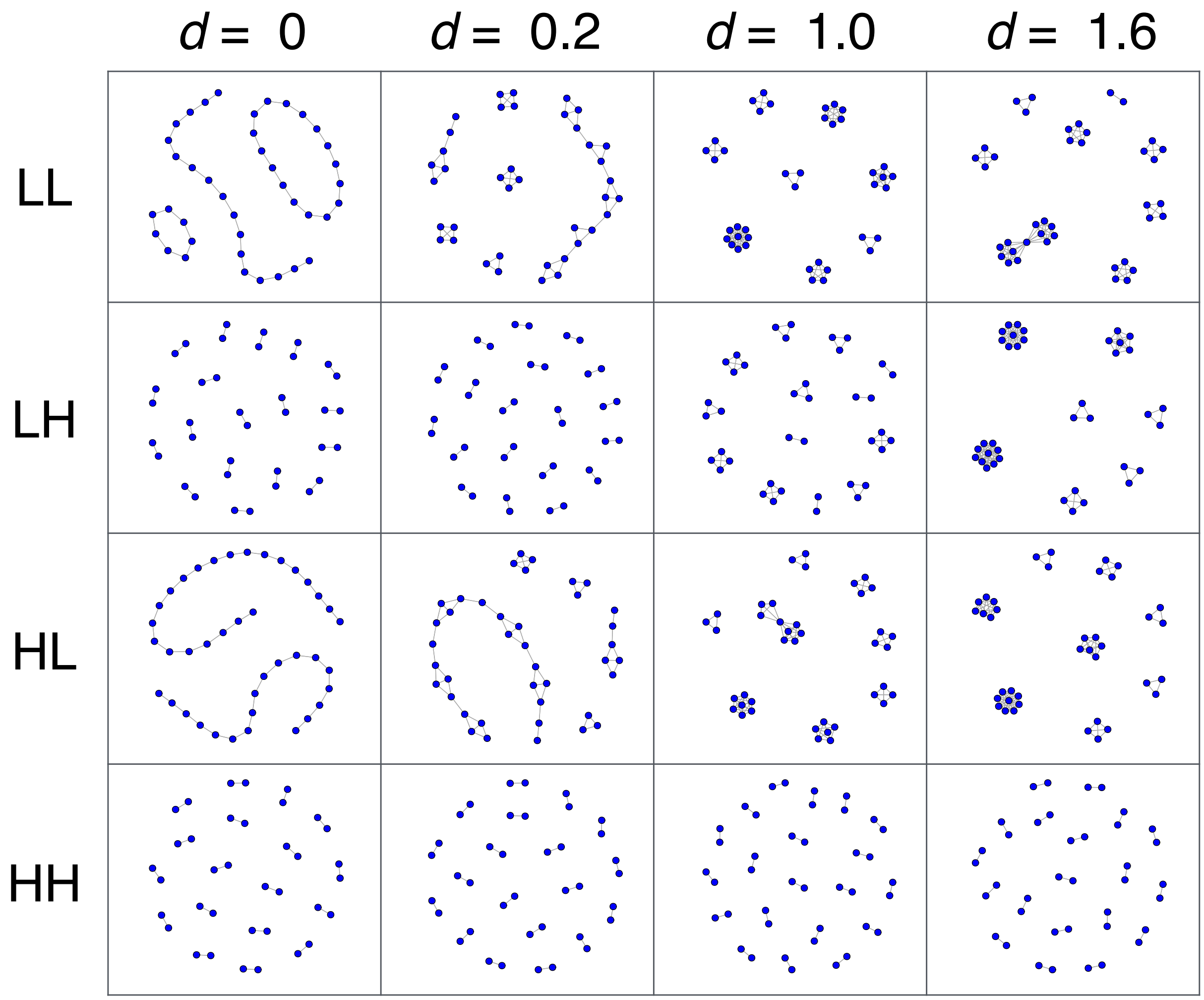


FIG. 2. Isolated effects: triangle benefits only ($e = 0$). Representative single-layer networks that emerge as a result of varying incentives for closed triangles, d. Unconnected nodes do occasionally occur but are not represented in these plots.

Condition 1 is met when

$$1 - c_{low} < 2 - 4c_{low},$$

or when $c_{low} < 1/3$, and, correspondingly, $c_{high} \geq 1/3$.

Condition 2 is met when

$$d \geq 3c_{high} - 1.$$

Under the value we used, c_{high}, the threshold value of d is 0.8, which is exactly what we observed in our simulations.

The logic of this analysis is easily extended to the case of spillover benefits only ($d = 0$, $e > 0$), although there are are a greater number of relevant ego networks, making the transition diagram quite complicated. See Figure 6. The presence of a spillover benefit for either of a node's ties allows a second tie to be maintained after a shock from low to high tie costs. This logic also explains the diagonal threshold line seen in Figure 4B (main text) in the LH shock condition. The resiliency effects of triangle and spillover benefits are additive, such that if an agent possesses *both* a spillover edge and a closed triangle, it is resilient to shocks as long as the total $d + e$ is greater than the threshold, which in this case is 0.8.

B. When resilience is perfect.

When does $k_{LH} = k_{LL}$? In our simulations, we observe that, when triangle or spillover benefits are sufficiently large, resilience is perfect, and the average degree of the network does not diminish when a network formed under low tie costs experiences a sudden increase to tie costs. What this means is that the incentives are such that a stable state reached under low tie costs will not become unstable when tie costs are suddenly made high.

This is most easily illustrated by considering the case of spillover benefits only ($d = 0$; see Figure 6). Under low tie costs, nodes will often reach degree 4 (Note that this is the average degree for Layer 1 only. When the layers have the same incentives, network statistics are

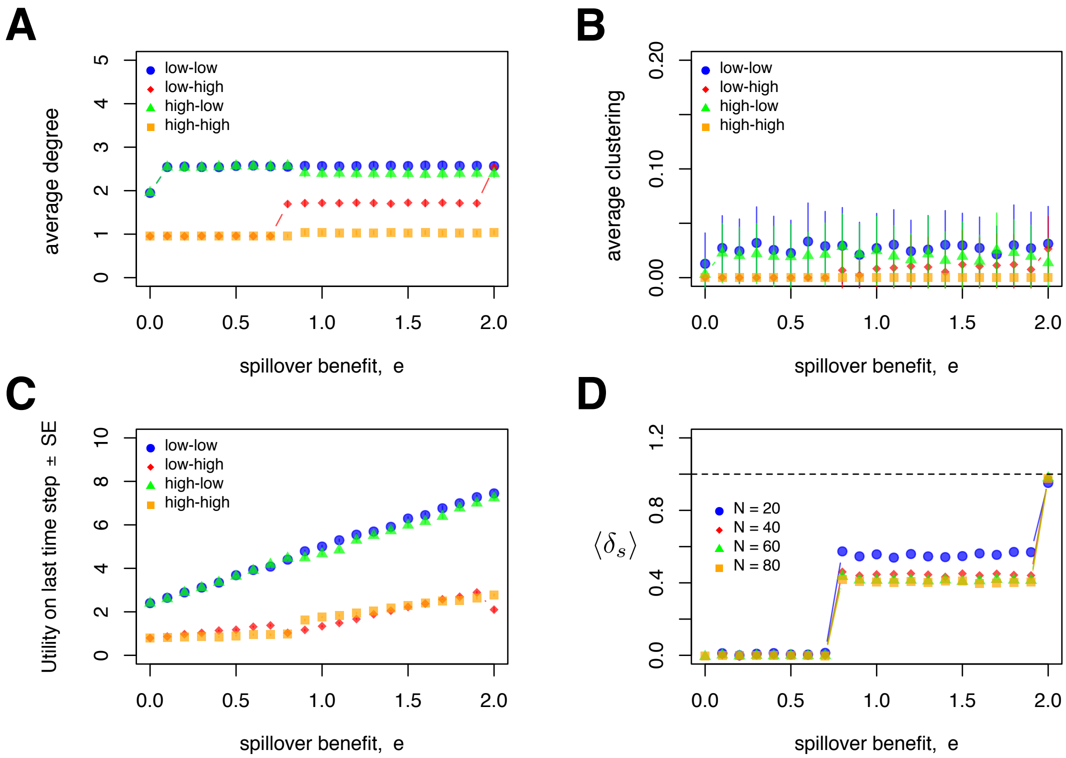


FIG. 3. Isolated effects: spillover benefits only ($d = 0$). (A–C) Average results for each of four shock conditions on a 40-node network. (A) Average node degree *pm* SD, (B) Average node clustering ± SD, (C) Average node utility at equilibrium ± SE, (D) Average resilience for LH condition, showing insensitivity to network size. All but (C) are from Layer 1 only.

the same for both layers.). However, such a high degree is unstable unless all edges are spillover edges. And because it is not always possible to increase the degree of a node and increase the number of spillover edges simultaneously, degree 3 (and even degree 2) is the common and stable network state for low tie costs (see main text Figures 4 and 5). Although degree 3 is stable under low tie costs, it is unstable after post-shock high tie costs unless $e \geq 2.0$, which is the threshold point for perfect resilience. If e is less than this, a node's degree will decrease to $k = 2$ if $e \geq 0.8$ (partial resilience), and $k = 1$ otherwise (no resilience). This argument can be extended for all shock-related network dynamics for any incentive parameter values.

C. When complete spillover occurs in HH condition.

Under constant high tie costs (HH condition), there is a threshold value of the spillover benefit, e, above which all ties are spillover ties (see Figure 8 in the main text). For our simulations, this value is $e > 0.8$. Such a state occurs when the cost of adding a new tie that completes a spillover edge is favored, but subsequently dropping any non-spillover ties is also favored. This is illustrated in Figure 7.

III. PROBABILITY OF SPILLOVER PAIRS FROM RANDOM PAIRING

Earlier in this Appendix, we calculated the threshold transition parameter for when all ties will be spillover ties. Before that, our simulations indicate that a smaller number ties are spillover ties, except with $e = 0$, for which spillover ties are rare. In such a case, spillover ties are rare due to the fact that they will only occur by chance, and the number occurring may be less than expected in a purely random model due to high numbers of isolated clusters that form when only triangle benefits are present. In other cases, spillover ties follow a pattern in which they are weakly incentivized, and therefore the proportion corresponds to numbers higher than should be expected by chance. What is this number?

We can calculate this for the special case in which each node has degree of 1, which occurs under high tie costs. Under random pairing, each node chooses the name node as its neighbor in each layer with probability $1/(N-1)$, and this is equal to the expected proportion of edges that will co-occur in both layers, i.e., the proportion of spillover edges. For a 40-node network, as was used in most simulations presented in the main text, this approximately equal to 0.026.

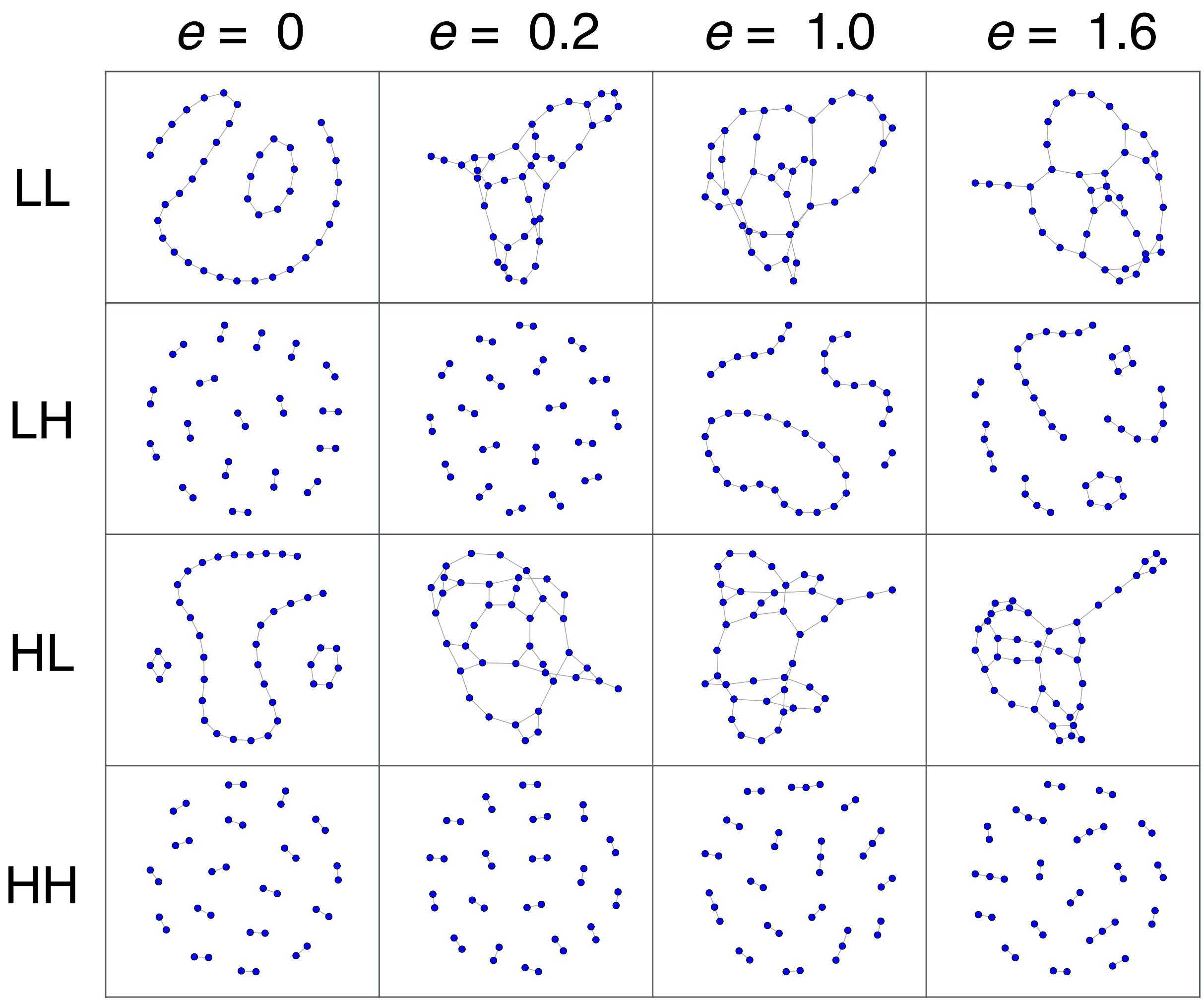


FIG. 4. Isolated effects: spillover benefits only ($d = 0$). Representative networks (Layer 1 only) that emerge as a result of varying incentives for spillover ties, e. Unconnected nodes do occasionally occur but are not represented in these plots.

IV. EXPLANATION OF FIG. 4B IN THE MAIN TEXT

Figure 4B, bottom row, in the main text shows a curious result: the resilience of under HL shocks is lower when structural benefits are significantly high—specifically when $d, e > 0.8$. To explain this we note that under such high structural incentives, many nodes will form triangles even under high tie costs. Thus, when tie costs are lowered, there are fewer new connections that will be incentivized. Figure 8 provides an illustration of how this can occur.

V. SUPPLEMENTAL SIMULATION RESULTS

A. Sensitivity to noise

When network dynamics exhibit resilience, post-shock equilibria are metastable in LH conditions, befitting the path-dependent nature of the equilibria (i.e., the network states cannot be obtained from an empty network). As such, random events—adding or dropping edges at random—will eventually eliminate resilience, causing the system to settle into a state resembling those obtained under high initial tie costs. The key word here is *eventually.* To investigate the time scale of these dynamics, we ran simulations in which adds and drops occurred with probability ν (see details in main text). We found that after shocks from low to high tie costs, the system moved from the metastable (LH) higher-degree state to the stable (HH) low-degree state at a timescale that was approximately $t \sim 1/\nu$. This was confirmed for $\nu \in [10^{-4}, 10^{-1}]$. Our results therefore hold as long as most events are strictly utility-increasing, relative to the characteristic timescale of dynamics.

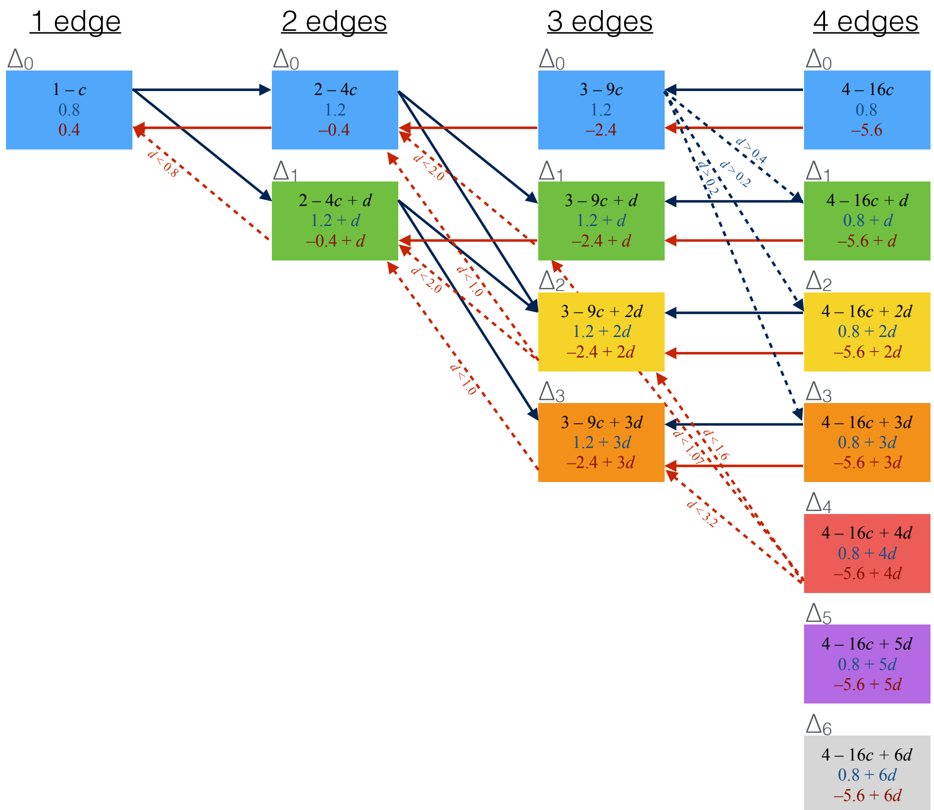


FIG. 5. Individual utilities for triangle benefits only ($e = 0$), for different number of ties and triangles (Δ), and utility-increasing state transitions under LH shock conditions. Precise values are given for low tie costs, c_{low}, and high tie costs, c_{high}, in blue and red, respectively. Blue arrows indicate when adding (or dropping) an edge would result in a utility increase under low tie costs. Red arrows indicate where dropping an edge would be incentivized under a post-shock tie-cost increase. Solid lines indicate a move that is always favored (sometimes only when $d > 0$), dashed lines indicate moves dependent on the triangle benefit, d. For the transition between 3 and 4 ties, only a subset of transition lines are shown for clarity. The remaining transitions can be easily calculated with the values shown.

B. Sensitivity to population size

Our results were very robust to changes in population size. This is largely because the numerical values of individual incentives operated on the tie capacity of nodes, independent of the size of the network. Larger network created few shortages for social ties, and the so the average degree of agents tended to be slightly higher in larger networks than in smaller networks, but this affect was minimal (Figure 10). Average clustering was similarly robust (Figure 11). For triangle benefits only, clustering was slightly higher in very small networks, due to more triangles forming through random chance as a result of the small population. The same was true for the spillover benefits only case. In this case, incentives tended to push the network *away* from clustering. When network size was very small, some additional clustering happened as a result of change connections. This effect disappeared for larger networks.

C. Sensitivity to tie costs

The results shown in the main text used parameters chosen for maximal clarity. For example, when tie costs were always high (HH condition), the equilibrium degree was exactly one. However, the broader principle of our results—namely, resilience from structural entrenchment—should hold for a wide range of parameters. To demonstrate this, we rand simulations for which the “high” cost of social ties was sufficiently low to generate higher degree networks, and thus the possibility of triangles. Figure 12 illustrates that, although the resilience effects are less stark, there are similar patterns of resilience as seen with more extreme tie cost values.

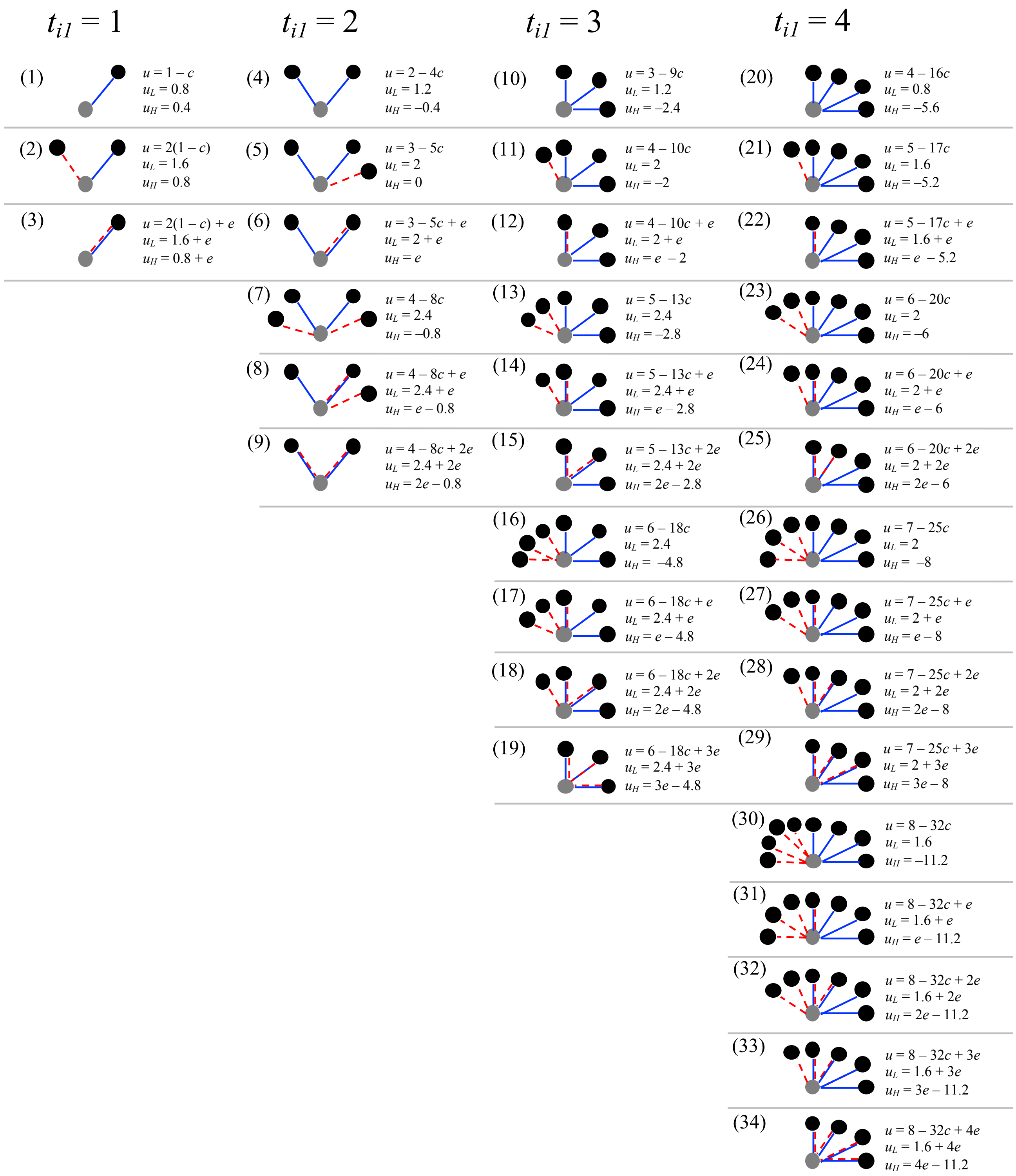
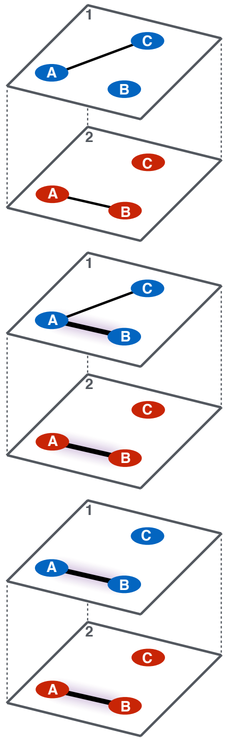


FIG. 6. All 34 possible ego networks (centered on the grey node) and corresponding utilities for spillover benefits only ($d = 0$). Ties in layer 1 are indicated by solid blue lines, ties in layer 2 are indicated by dashed red lines. Utility values are given for low tie costs (u_L) and high tie costs (u_H), based on the values for c_{low} and c_{high} used in the main text. A transition diagram for these networks under each cost and shock condition (LH, HL), similar to that shown in Figure 5, can be derived from these states and corresponding utilities, though it will be considerably more complicated.



$$u_A = 2(1 - c) \\ = 0.8$$

$$u_A = 2 - 4c + 1 - c + e \\ = 3 - 5c + e \\ = e$$

$$u_A = 2(1 - c) + e \\ = 0.8 + e$$

FIG. 7. The dynamics of spillover under high tie costs. Consider an agent A who has a tie with agent C in layer 1 and a tie with agent B in layer 2 (top row).The agent can add a tie in one of the two layers to complete a spillover edge; in this case with agent B in layer 1 (middle row). Such a move is favored if the utility gained from the new spillover tie is great enough to compensate the added costs of the second tie in layer 1; in our case, this occurs when $e > 0.8$. Once this new tie is formed, it is then always beneficial to agent A to drop its previously held tie with agent C in layer 1 (bottom row). Thus, under high tie costs, we observe a threshold value of e above which the average degree does not change (it remains $k = 1$), but for which the proportion of spillover edges increases to unity. Below this threshold, there are still more spillover ties than expected from random assortment, due to limited incentives to form spillover ties. Utilities calculated assume our simulation value of $c_{high} = 0.6$.

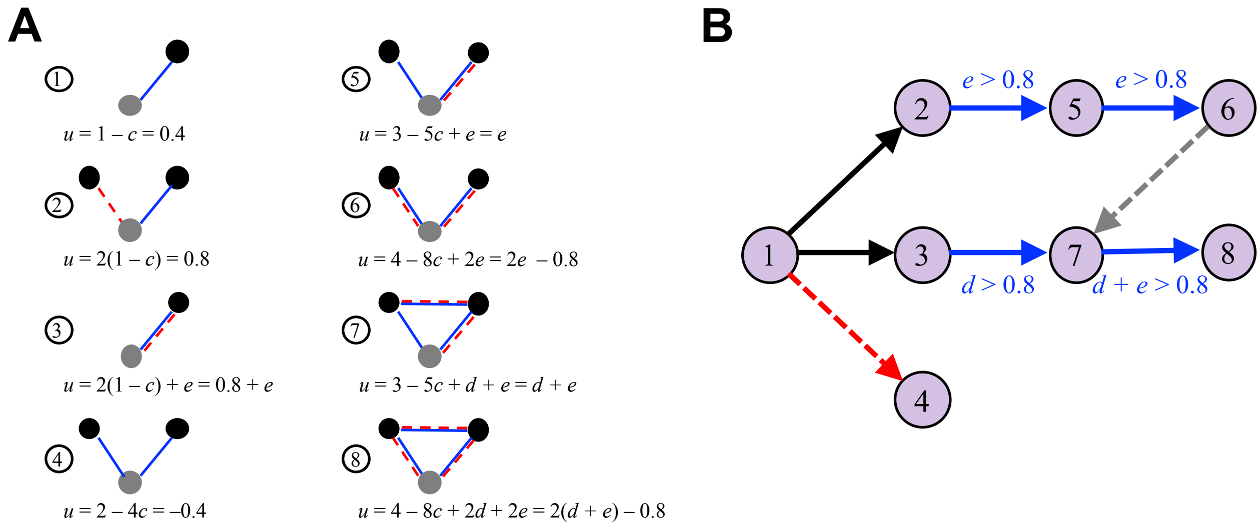


FIG. 8. (A) Eight distinct ego networks (for the grey node) under high tie costs ($c_{high} = 0.6$), with utilities indicated. Edges can exist both in layer 1 (blue) and layer 2 (red) of the multiplex. (B) Transitions between the states indicated in subfigure A. Black arrows indicate transitions that are always favored. Blue arrows indicate transitions that are sometimes favored; the required condition for each transition is shown in blue text. The grey dashed arrow between state 6 and state 7 indicates that the presence of state 6 centered on an adjacent node is required for ego to transition from state 3 to state 7. The red dashed line indicates that this transition is never favored under the indicated cost condition.

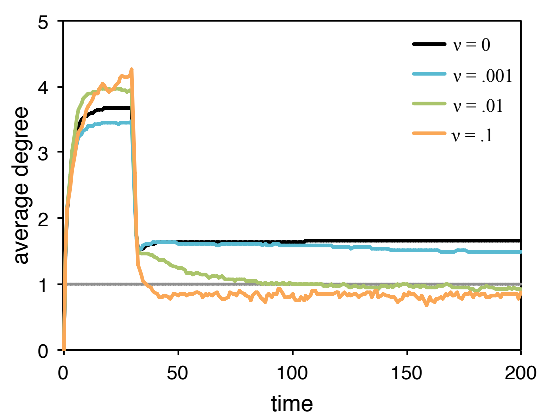


FIG. 9. Sensitivity to noise. Temporal dynamics of representative simulation runs. Under initially low tie costs, average degree increases to a dynamic equilibrium of about 3 or 4. A shock to high tie costs occurs at $t = 40$, after which we see a decrease in average degree. In the absence of noise, this stabilizes to an average degree of just under 2. The more noise is present, the more quickly the system goes from the metastable LH condition (black line) to the stable HH condition (grey line).

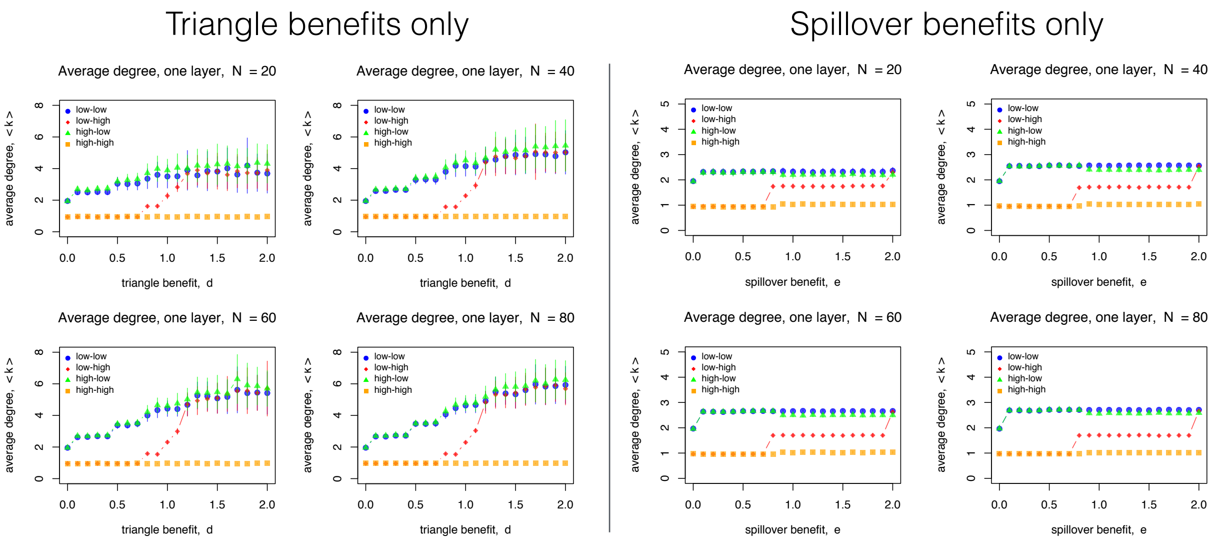


FIG. 10. Sensitivity to network size: average degree.

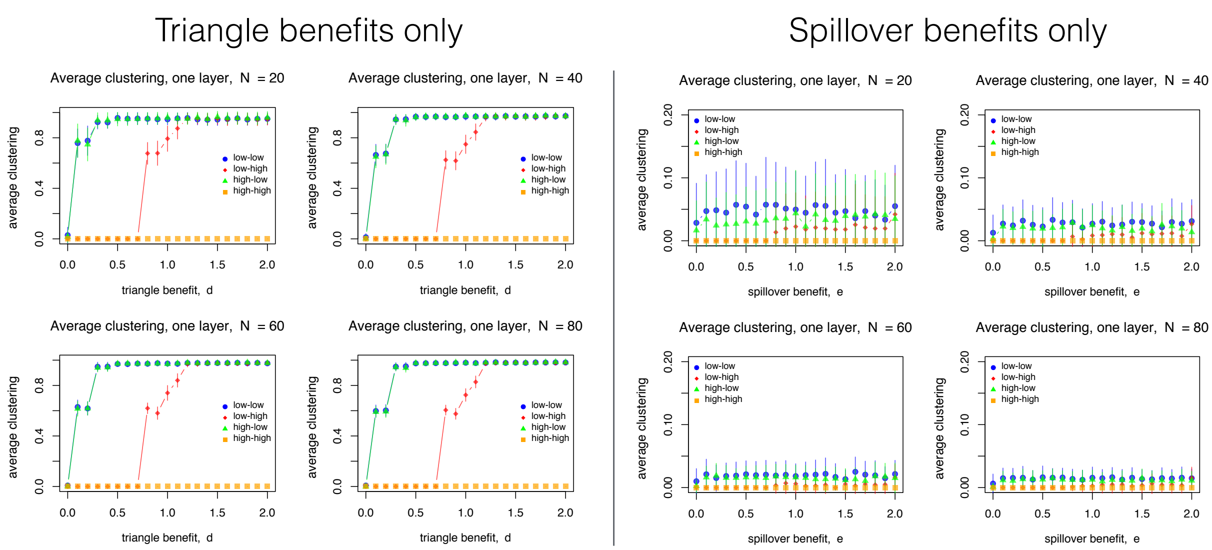


FIG. 11. Sensitivity to network size: average clustering.

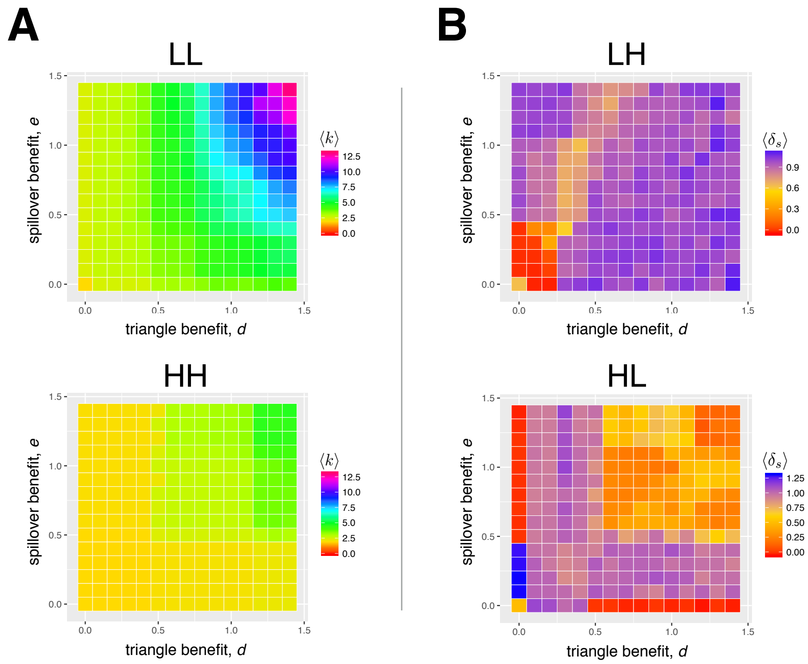


FIG. 12. Average degree for all four shock conditions when $c_{high} = 0.3$ and $c_{low} = 0.2$.