Appendices

A. SAVAR details

In this section we provide more details on the dynamical and statistical properties of SAVAR models. We start by showing the need of adding some constraint on how W is defined as well as more details on the noise. Then we describe the model with a forcing term (used for the experiment with a non-stationary trend). The next subsections describe SAVAR as a vector autoregressive process of order 1, which is the form used in the formal proofs of propositions 3.1 and 3.2 as well as in the last part of this appendix, where the autocovariance function of the SAVAR model is derived.

A.1. Rescaling of weight term

$(W)_{i\ell}$ defines the contribution of the ℓ-th grid point to the i-th mode. Scaling W by a scalar μ does not have any effect on the model, due to the inverse mapping of W^+. Suppose that $W' = \mu W$, then

$$\begin{aligned} \mathbf{y}_t &= \frac{W^+}{\mu} \sum_{\tau=1}^{\tau_{\max}} \Phi(\tau) \mu W \mathbf{y}_{t-\tau} + \boldsymbol{\varepsilon}_t, \\ \mathbf{y}_t &= W^+ \sum_{\tau=1}^{\tau_{\max}} \Phi(\tau) W \mathbf{y}_{t-\tau} + \boldsymbol{\varepsilon}_t, \end{aligned} \tag{18}$$

which is equivalent to equation (16).

A.2. Noise term

The SAVAR model's grid level noise term takes the form $\Sigma_{\mathbf{y}} = \lambda W^+ D_{\mathbf{x}} W^{+^{\mathrm{T}}} + D_{\mathbf{y}}$. To avoid overparametrization our definition requires some constraints on the weights W. While rescaling W

does not have an effect at the grid-level dynamics, as shown above, it does effect the noise's strength ε in the mode-level process. We, therefore, add some constraints to W. Let again $W' = \mu W$, then

$$cov(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}) = \frac{\lambda}{\mu^2} W^+ D_{\mathbf{x}} W^{+^{\mathrm{T}}}. \tag{19}$$

Consequently, by constraining the values of W, for example $\| \mathbf{w}^i \|_2 = 1$, the only free parameter is λ.

A.3. SAVAR with a forcing term

One can add a forcing term to the model yielding

$$y_t^\ell = \sum_{i=1}^{N} u^{\ell i} \sum_{j=1}^{N} \sum_{\tau=1}^{\tau_{\max}} \phi^{ij}(\tau) \sum_{\ell}^{L} w^{j\ell} y_{t-\tau}^\ell + b_t^\ell + \varepsilon_t^\ell, \tag{20}$$

where b_t^ℓ denotes the influence of the forcing term in the ℓ -th point at time t. The matrix form is given by

$$\mathbf{y}_t = W^+ \sum_{\tau=1}^{\tau_{\max}} \Phi(\tau) W \mathbf{y}_{t-\tau} + \mathbf{b}_t + \boldsymbol{\varepsilon}_t. \tag{21}$$

For the non-stationarity experiment we used $\mathbf{b}_t = W^+ \Psi(t, \sigma_o)$ as described in Section 4.1.1.

A.4. SAVAR as a process of order 1

SAVAR definitions from eqs. (16) and (17) can be rewritten into a process of order 1. To this end let $\tilde{\mathbf{y}}_t \in \mathbb{R}^{L\cdot\tau_{\max}}$ be the vector obtained by concatenating $\mathbf{y}_t$, ..., $\mathbf{y}_{t-\tau_{\max}}$ and $\tilde{\mathbf{x}}_t \in \mathbb{R}^{N\cdot\tau_{\max}}$ the vector obtained by concatenating $\mathbf{x}_t$, ..., $\mathbf{x}_{t-\tau_{\max}}$. In terms of these extended variables the processes read

$$\begin{aligned} \tilde{\mathbf{y}}_t &= A_y \tilde{\mathbf{y}}_{t-1} + \tilde{\varepsilon}_t, \\ \tilde{\mathbf{x}}_t &= A_x \tilde{\mathbf{x}}_{t-1} + \tilde{\varepsilon}_t, \end{aligned} \tag{22}$$

where $\tilde{\varepsilon}_t$ is obtained by concatenating ε_t with $L\cdot(\tau_{\max}-1)$ zeros, $\tilde{\varepsilon}_t$ by concatenating ε_t with $N\cdot(\tau_{\max}-1)$ zeros, and A_y and A_x are the block matrices

$$A_y = \begin{pmatrix} W^+\Phi(1)W & W^+\Phi(2)W & \cdots & W^+\Phi(\tau_{\max}-1)W & W^+\Phi(\tau_{\max})W \\ \mathbf{I}_L & 0 & \ddots & 0 & 0 \\ 0 & \mathbf{I}_L & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{I}_L & 0 \end{pmatrix},$$

$$A_x = \begin{pmatrix} \Phi(1) & \Phi(2) & \cdots & \Phi(\tau_{\max}-1) & \Phi(\tau_{\max}) \\ \mathbf{I}_N & 0 & \ddots & 0 & 0 \\ 0 & \mathbf{I}_N & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{I}_N & 0 \end{pmatrix}.$$

A.5. Autocovariance function of SAVAR

For simplicity we assume a stationary SAVAR process with mean zero. From equation (22) the covariance of $\tilde{\mathbf{y}}_t$ at lag 0, denoted by $\Omega(0)$, is given by

$$\begin{aligned} & \Omega(0) = \mathbb{E}[\tilde{\mathbf{y}}_t \tilde{\mathbf{y}}_t^{\intercal}] \\ \Leftrightarrow \quad & \Omega(0) = \mathbb{E}[(A_y \tilde{\mathbf{y}}_{t-1} + \tilde{\varepsilon}_t)(A_y \tilde{\mathbf{y}}_{t-1} + \tilde{\varepsilon}_t)^{\intercal}] \\ \Leftrightarrow \quad & \Omega(0) = \mathbb{E}[(A_y \tilde{\mathbf{y}}_{t-1})(A_y \tilde{\mathbf{y}}_{t-1})^{\intercal}] + 2\mathbb{E}(A_y \tilde{\mathbf{y}}_{t-1} \tilde{\varepsilon}_t) + \mathbb{E}(\tilde{\varepsilon}_t \tilde{\varepsilon}_t^{\intercal}). \end{aligned} \tag{23}$$

By assuming stationarity we have that

$$
\begin{aligned}
& \Omega(0) = \mathbb{E}[(A_y\tilde{\mathbf{y}}_{t-1})(A_y\tilde{\mathbf{y}}_{t-1})^{\top}] + 2\mathbb{E}(A_y\tilde{\mathbf{y}}_{t-1}\tilde{\varepsilon}_t) + \mathbb{E}(\tilde{\varepsilon}_t\tilde{\varepsilon}_t^{\top}) \\
\Leftrightarrow \quad & \Omega(0) = \mathbb{E}[(A_y\tilde{\mathbf{y}}_t)(A_y\tilde{\mathbf{y}}_t)^{\top}] + 2\mathbb{E}(A_y\tilde{\mathbf{y}}_t\tilde{\varepsilon}_t) + \mathbb{E}(\tilde{\varepsilon}_t\tilde{\varepsilon}_t^{\top}) \\
\Leftrightarrow \quad & \Omega(0) = A_y\Omega(0)A_y^{\top} + \Sigma_{\mathbf{y}}.
\end{aligned} \tag{24}
$$

Using the vectorization operator $(vec(ABC) = (C^{\top} \otimes A)vec(B))$ it is possible to express a close-term solution in terms of $vec(\Omega(0))$ where $\otimes$ denotes the Kronecker product. Then, from Eq. (24),

$$
\begin{aligned}
& vec(\Omega(0)) = (A_y \otimes A_y)vec(\Omega(0)) + vec(\Sigma_{\mathbf{y}}) \\
\Leftrightarrow \quad & vec(\Omega(0)) = (\mathbf{I}_L - (A_y \otimes A_y))^{-1}vec(\Sigma_{\mathbf{y}}),
\end{aligned} \tag{25}
$$

where $\mathbf{I}_n$ is the identity matrix.

The autocovariance of $\tilde{\mathbf{y}}_t$ at time lag τ, denoted $\Omega(\tau)$ for $\mu = 0$, is

$$
\Omega(\tau) = A_y^{\tau}\Omega(0). \tag{26}
$$

This can be shown by finding $\Omega(1)$ from multiplying $\tilde{\mathbf{y}}_t$ and $\tilde{\mathbf{y}}_{t-1}$ and taking the expectation.

$$
\begin{aligned}
& \mathbb{E}(\tilde{\mathbf{y}}_t\tilde{\mathbf{y}}_{t-1}^{\top}) = \mathbb{E}((A_y\tilde{\mathbf{y}}_{t-1} + \tilde{\varepsilon}_t)\tilde{\mathbf{y}}_{t-1}^{\top}) \\
\Leftrightarrow \quad & \Omega(1) = A_y\mathbb{E}(\tilde{\mathbf{y}}_{t-1}\tilde{\mathbf{y}}_{t-1}^{\top}) + \mathbb{E}(\tilde{\varepsilon}_t\tilde{\mathbf{y}}_{t-1}^{\top}) \\
\Leftrightarrow \quad & \Omega(1) = A_y\Omega(0).
\end{aligned}
$$

Then (26) follows by induction.

B. Proofs

In this section we prove Propositions 3.1 and 3.2 stated in the main text. We start with a detailed discussion that further clarifies the connection between the processes at grid- and mode-level, see eqs. (16) and (17) respectively.

Recall that $W \in \mathbb{R}^{N\times L}$ maps from $\mathcal{X} = \mathbb{R}^{L}$ to $\mathrm{im}(W) \subset \mathcal{Y} = \mathbb{R}^{N}$, where $\mathcal{X}$ and $\mathcal{Y}$ respectively are the spaces in which the grid-level variables $\mathbf{y}_t$ and mode-level variables $\mathbf{x}_t$ take their values. As long as $N < L$, i.e., as long as there are fewer modes than grid points (which we assume throughout) the matrix W necessarily has a nonzero kernel and can therefore not be inverted. We therefore work with its Moore-Penrose pseudo-inverse $W^{+} \in \mathbb{R}^{L\times N}$, that always exists and that maps from $\mathcal{Y}$ to $\mathrm{im}(W^{+}) \subset \mathcal{X}$. As stated below Eq. (16), the rows of W are linearly independent by assumption. This implies $WW^{+} = \mathbf{I}_N$ and $\mathrm{rank}(W) = \mathrm{rank}(W^{+}) = N$. Note that $W^{+}W \neq \mathbf{I}_L$. Further, $\mathcal{X} = \mathrm{im}(W)$ and $\mathcal{Y}$ decomposes as $\mathcal{Y} = \mathrm{im}(W^{+}) \oplus \ker(W)$. This can be made manifest by means of the matrices

$$P_1 = W^{+}W, \quad P_2 = \mathbf{I}_L - W^{+}W$$

with the properties $P_1 + P_2 = \mathbf{I}_L$, $P_1^2 = P_1$, $P_2^2 = P_2$, and $P_1P_2 = P_2P_1 = 0$. These matrices respectively are projectors to the subspaces $\mathrm{im}(W^{+})$ and $\ker(W)$ that can be used to uniquely decompose any $v \in \mathcal{Y}$ into $v = v_1 + v_2$ with $v_1 = vP_1 \in \mathrm{im}(W^{+})$ and $v_2 = vP_2 \in \ker(W)$.

Further, this decomposition of $\mathcal{Y}$ is consistent with dynamics described by the grid-level process (16). That is to say, the process $\mathbf{y}_t$ be can accordingly decomposed as $\mathbf{y}_t = \mathbf{y}_t^{(1)} + \mathbf{y}_t^{(2)}$ into two individual processes that do not mix over time. These are $\mathbf{y}_t^{(1)} = P_1\mathbf{y}_t \in \mathrm{im}(W^{+})$ and $\mathbf{y}_t^{(2)} = P_2\mathbf{y}_t \in \ker(W)$ with the dynamics

$$\begin{aligned}
\mathbf{y}_t^{(1)} &= W^+ \sum_{\tau=1}^{\tau_{\max}} \Phi(\tau) W \mathbf{y}_{t-\tau}^{(1)} + \varepsilon_t^{(1)} \quad \text{with} \quad \varepsilon_t^{(1)} \sim \mathcal{N}(P_1\mu_{\mathbf{y}}, \lambda W^+ D_{\mathbf{x}} W^{+\top} + P_1 D_{\mathbf{y}} P_1^\top),\\
\mathbf{y}_t^{(2)} &= \varepsilon_t^{(2)} \quad \text{with} \quad \varepsilon_t^{(2)} \sim \mathcal{N}(P_2\mu_{\mathbf{y}}, P_2 D_{\mathbf{y}} P_2^\top).
\end{aligned}$$

This follows from Eq. (16) by using $WP_1 = W$, $WP_2 = 0$, $P_1W^+ = W^+$, and $P_2W^+ = 0$. Note that $\mathbf{y}_t^{(2)}$ is a mere white noise process.

The connection between the grid-level process $\mathbf{y}_t$ and the mode-level process $\mathbf{x}_t$ can now be understood in the following way: First, $\mathbf{x_t}$ has the same distribution as $W\mathbf{y}_t = W\mathbf{y}_t^{(1)}$. Second, $W^+\mathbf{x_t}$ has the same distribution as $\mathbf{y}_t^{(1)}$. The processes $\mathbf{y}_t$ and $\mathbf{x}_t$ are therefore in a one-to-one correspondence up to the additive independent white noise process $\mathbf{y}_t^{(2)}$ at grid-level. While this already gives strong intuitive indication in favour of the propositions, their formal proofs now follow.

The **proposition 3.1** states that *a SAVAR process as in Eq.* (16) *is stable if and only if the corresponding VAR process* (17) *is stable. In particular, the choice of W does not influence stability.*

Proof of Proposition 3.1:

We begin by rewriting both processes into processes of order 1 as shown in Appendix A Section A.4. Stability of the SAVAR process in Eq. (16) (the VAR process in Eq. (17)) then means that all eigenvalues of A_y (of A_x) have modulus strictly smaller than one. The proposition can therefore be proven by showing that A_y and A_x have the same nonzero eigenvalues.

One direction of this equivalence, namely that all nonzero eigenvalues of A_x are also nonzero eigenvalues of A_y, follows readily: Let v_x be a nonzero eigenvector of A_x with eigenvalue $\delta \neq 0$, i.e., $A_x v_x = \delta v_x \neq 0$. A calculation then shows

$$A_y(W^+ \otimes \mathbf{I}_{\tau_{\max}})v_x = (W^+ \otimes \mathbf{I}_{\tau_{\max}})A_x v_x = \delta(W^+ \otimes \mathbf{I}_{\tau_{\max}})v_x.$$

Here, $\otimes$ denotes the Kronecker product such that $(W^+ \otimes \mathbf{I}_{\tau_{\max}})$ is a block diagonal matrix with $\tau_{\max}$ copies of W^+ on its diagonal. This identifies $(W^+ \otimes \mathbf{I}_{\tau_{\max}})v_x$ as eigenvector of A_y with eigenvalue δ.

The other direction of the equivalence, namely that all nonzero eigenvalues of A_y are also nonzero eigenvalues of A_x, is more complicated. For this purpose we first split up A_y as $A_y = A_y^{(1)} + A_y^{(2)}$, where $A_y^{(1)}$ and $A_y^{(2)}$ respectively correspond to the processes $\mathbf{y}_t^{(1)}$ and $\mathbf{y}_t^{(2)}$ introduced above. This can be achieved by extending the projection matrices P_1 and P_2 to the enlarged space $\mathcal{X}^{\tau_{\max}} = \mathbb{R}^{L\cdot\tau_{\max}}$, namely

$$\tilde{P}_1 = P_1 \otimes \mathbf{I}_{\tau_{\max}}, \quad \tilde{P}_2 = P_2 \otimes \mathbf{I}_{\tau_{\max}}.$$

These are block diagonal matrices (with $\tau_{\max}$ copies of, respectively, P_1 and P_2 on their diagonal) that still obey the projector properties $\tilde{P}_1 + \tilde{P}_2 = \mathbf{I}_{L\cdot\tau_{\max}}$, $\tilde{P}_1^2 = \tilde{P}_1$, $\tilde{P}_2^2 = \tilde{P}_2$, and $\tilde{P}_1\tilde{P}_2 = \tilde{P}_2\tilde{P}_1 = 0$. Then

$$A_y = (\tilde{P}_1 + \tilde{P}_2)A_y(\tilde{P}_1 + \tilde{P}_2) = \tilde{P}_1 A_y \tilde{P}_1 + \tilde{P}_2 A_y \tilde{P}_2,$$

where $\tilde{P}_1 A_y \tilde{P}_2 = \tilde{P}_2 A_y \tilde{P}_1 = 0$ follows from $\tilde{P}_1\tilde{P}_2 = \tilde{P}_2\tilde{P}_1 = 0$ together with $P_2 W^+ = W P_2 = 0$. This shows that the desired decomposition indeed exists with $A_y^{(1)} = \tilde{P}_1 A_y \tilde{P}_1$ and $A_y^{(2)} = \tilde{P}_2 A_y \tilde{P}_2$. By inspection $A_y^{(2)}$ is found to be lower triagonal with only zeros on its diagonal, hence all its eigenvalues are zero.

Now let v_y be a nonzero eigenvector of A_y with eigenvalue $\delta \neq 0$, i.e., $A_y v_y = \delta v_y \neq 0$. Using $v_y = \tilde{P}_1 v_y + \tilde{P}_2 v_y$, $A_y = A_y^{(1)} + A_y^{(2)}$, and left-multiplying $A_y v_y = \delta v_y \neq 0$ with $\tilde{P}_1$ and $\tilde{P}_2$ we then find

$$\tilde{P}_1 A_y v_y = A_y^{(1)} \tilde{P}_1 v_y = \delta \tilde{P}_1 v_y,$$
$$\tilde{P}_2 A_y v_y = A_y^{(2)} \tilde{P}_2 v_y = \delta \tilde{P}_2 v_y.$$

This shows that $ia)$ $\tilde{P}_1 v_y = 0$ or $ib)$ $\tilde{P}_1 v_y$ is an eigenvector of $A_y^{(1)}$ with eigenvalue δ and $iia)$ $\tilde{P}_2 v_y = 0$ or $iib)$ $\tilde{P}_2 v_y$ is an eigenvector of $A_y^{(2)}$ with eigenvalue δ. Since all eigenvalues of $A_y^{(2)}$ are zero $iib)$ cannot hold and thus $iia)$ must be the true, i.e., $\tilde{P}_2 v_y = 0$. But then $\tilde{P}_1 v_y \neq 0$ so $ia)$ cannot hold and $ib)$ must be true, i.e., $v_y = \tilde{P}_1 v_y$ is an eigenvector of $A_y^{(1)}$ with eigenvalue δ. A calculation now shows

$$\begin{aligned} A_x (W \otimes \mathbf{I}_{\tau_{\max}}) v_y &= \underbrace{(W \otimes \mathbf{I}_{\tau_{\max}})(W^+ \otimes \mathbf{I}_{\tau_{\max}})}_{\mathbf{I}_{N \cdot \tau_{\max}}} \underbrace{(W \otimes \mathbf{I}_{\tau_{\max}}) A_y (W^+ \otimes \mathbf{I}_{\tau_{\max}})}_{A_x} (W \otimes \mathbf{I}_{\tau_{\max}}) v_y \\ &= (W \otimes \mathbf{I}_{\tau_{\max}}) \tilde{P}_1 A_y \tilde{P}_1 v_y \\ &= (W \otimes \mathbf{I}_{\tau_{\max}}) A_y^{(1)} v_y \\ &= \delta (W \otimes \mathbf{I}_{\tau_{\max}}) v_y, \end{aligned}$$

which identifies $(W \otimes \mathbf{I}_{\tau_{\max}}) v_y$ as eigenvector of A_x with eigenvalue δ. □

Proposition 3.2 states that *given A_y it is possible to identify A_x up to similarity. Similarly, given A_x it is possible to identify A_x up to similarity.*

Proof of Proposition 3.2:

Given A_y we find $\Phi_y(\tau) \equiv W^+ \Phi(\tau) W$ for all $1 \leq \tau \leq \tau_{\max}$ unambiguously. Similarly, given A_x we find $\Phi_x(\tau) \equiv \Phi(\tau)$ for all $1 \leq \tau \leq \tau_{\max}$ unambiguously. It is therefore sufficient to show that given $\Phi_y(\tau)$ it is possibile to identify $\Phi_x(\tau)$ up to similarity, and that given $\Phi_x(\tau)$ it is possibile to identify $\Phi_y(\tau)$ up to similarity.

Recall that $\mathcal{Y}=\mathbb{R}^L$ decomposes as $\mathcal{Y}=\mathrm{im}(W^+)\oplus\ker(W)$ and that P_1 projects to $\mathrm{im}(W^+)$. The equality $\Phi_y(\tau)=P_1\Phi_y(\tau)P_1$ thus shows that $\Phi_y(\tau)$ maps $\mathrm{im}(W^+)$ to a subspace of itself while it maps $\ker(W)$ to zero. Now choose a basis $e_1,\ldots,e_N$ of $\mathrm{im}(W^+)$ as well as a basis $f_1,\ldots,f_{L-N}$ of $\ker(W)$, and let $S\in\mathrm{GL}(L,\mathbb{R})$ be the matrix that changes the basis of $\mathcal{Y}$ to the basis $e_1,\ldots,e_N,f_1,\ldots,f_{L-N}$. Then

$$S\Phi_y(\tau)S^{-1}=\begin{pmatrix}\Phi'_y(\tau) & 0_{N,L-N}\\ 0_{L-N,N} & 0_{L-N,L-N}\end{pmatrix}$$

for some $\Phi'_y(\tau)\in\mathbb{R}^{N\times N}$ and where $0_{n,m}$ denotes the $n\times m$-dimensional matrix with all zero entries. Further

$$WS^{-1}=\begin{pmatrix}W' & 0_{L-N,N}\end{pmatrix},\quad SW^+=\begin{pmatrix}(W')^{-1}\\ 0_{N,L-N}\end{pmatrix},$$

with $W'\in\mathrm{GL}(N,\mathbb{R})$. Note that W' is invertible because of $\mathrm{rank}(W)=N$.

To show that $\Phi_x(\tau)$ can be determined up to similarity from $\Phi_y(\tau)$ we first observe

$$\Phi_x(\tau)=W\Phi_y(\tau)W^+=WS^{-1}S\Phi_y(\tau)S^{-1}SW^+=W'\Phi'_y(\tau)(W')^{-1}.$$

This shows that $\Phi_x(\tau)$ is similar to $\Phi'_y(\tau)$, so we have to prove that $\Phi'_y(\tau)$ can be determined up to similarity from $\Phi_y(\tau)$. For this purpose let $T'\in\mathrm{GL}(N,\mathbb{C})$ be such that $T'\Phi'_y(\tau)(T')^{-1}$ is in Jordan normal form and let T be the block diagonal matrix $T=\mathrm{diag}(T',\mathbf{I}_{L-N,L-N})$. Then

$$(TS)\Phi_y(\tau)(TS)^{-1}=\begin{pmatrix}T'\Phi'_y(\tau)(T')^{-1} & 0_{N,L-N}\\ 0_{L-N,N} & 0_{L-N,L-N}\end{pmatrix}\tag{27}$$

is in Jordan normal form too. Now let $U \in \mathrm{GL}(L,\mathbb{C})$ be such that $U\Phi_y(\tau)U^{-1}$ is in Jordan normal form with the individual Jordan blocks being ordered such that a block J_1 with eigenvalue δ_1 is before (i.e., to the upper left of) a block J_2 with eigenvalue δ_2 if any of the following three conditions holds: $i)$ $|\delta_1|>|\delta_2|$. $ii)$ $|\delta_1|=|\delta_2|$ and $\delta_1=-\delta_2>0$. $iii)$ $\delta_1=\delta_2$ and J_1 is larger than J_2. These conditions determine $\Phi_y^{(J)}(\tau)\equiv U\Phi_y(\tau)U^{-1}$ uniquely, which is why $\Phi_y^{(J)}(\tau)$ can be found from $\Phi_y(\tau)$ by bringing the latter to Jordan normal from and then reordering the blocks appropriately. From Eq. (27) we further know that $\Phi_y(\tau)$ has at least $L-N$ one-dimensional Jordan blocks with eigenvalue 0. Hence, by definition of the imposed ordering of Jordan blocks, $\Phi_y^{(J)}(\tau)$ necessarily is of the block diagonal from

$$\Phi_y^{(J)}(\tau)=\begin{pmatrix}\Phi_y^{(J)}(\tau)' & 0_{N,L-N}\\ 0_{L-N,N} & 0_{L-N,L-N}\end{pmatrix}.$$

Now recall that any two Jordan normal forms of the same matrix can be obtained from each other by reordering the Jordan blocks, i.e., any two Jordan normal forms of the same matrix are similar. This means there is $V \in \mathrm{GL}(L,\mathbb{R})$ such that

$$(VTS)\Phi_y(\tau)(VTS)^{-1}=V(TS)\Phi_y(\tau)(TS)^{-1}V^{-1}=\Phi_y^{(J)}(\tau).$$

Due to the block diagonal form of both $\Phi_y^{(J)}(\tau)$ and $(TS)\Phi_y(\tau)(TS)^{-1}$, the matrix V too can be chosen to be block diagonal, namely $V=\mathrm{diag}(V',\mathbf{I}_{L-N})$. The previous equation then implies

$$V'T'\Phi_y'(\tau)(V'T')^{-1}=\Phi_y^{(J)}(\tau)'.$$

This shows that $\Phi_y^{(J)}(\tau)'$ is similar to $\Phi'_y(\tau)$ and hence also similar to $\Phi_x(\tau)$. Since $\Phi_y^{(J)}(\tau)'$ can be determined uniquely from $\Phi_y(\tau)$ as explained above, $\Phi_x(\tau)$ can be determined from $\Phi_y(\tau)$ up to similarity.

The opposite direction of the proposition now follows readily. A calculation gives

$$\begin{aligned}
\begin{pmatrix} \Phi_x(\tau) & 0_{N,L-N} \\ 0_{L-N,N} & 0_{L-N,L-N} \end{pmatrix} &= \begin{pmatrix} W'\Phi'_y(\tau)(W')^{-1} & 0_{N,L-N} \\ 0_{L-N,N} & 0_{L-N,L-N} \end{pmatrix} \\
&= \begin{pmatrix} W' & 0_{N,L-N} \\ 0_{L-N,N} & \mathbf{I}_{L-N} \end{pmatrix} \begin{pmatrix} \Phi'_y(\tau) & 0_{N,L-N} \\ 0_{L-N,N} & 0_{L-N,L-N} \end{pmatrix} \begin{pmatrix} (W')^{-1} & 0_{N,L-N} \\ 0_{L-N,N} & \mathbf{I}_{L-N} \end{pmatrix} \\
&= \begin{pmatrix} W' & 0_{N,L-N} \\ 0_{L-N,N} & \mathbf{I}_{L-N} \end{pmatrix} T^{-1}\Phi_y(\tau)T \begin{pmatrix} (W')^{-1} & 0_{N,L-N} \\ 0_{L-N,N} & \mathbf{I}_{L-N} \end{pmatrix} \\
&= \left[\begin{pmatrix} W' & 0_{N,L-N} \\ 0_{L-N,N} & \mathbf{I}_{L-N} \end{pmatrix} T^{-1}\right] \Phi_y(\tau) \left[\begin{pmatrix} W' & 0_{N,L-N} \\ 0_{L-N,N} & \mathbf{I}_{L-N} \end{pmatrix} T^{-1}\right]^{-1}.
\end{aligned}$$

This shows that $\Phi_y(\tau)$ is similar to $diag(\Phi_x(\tau), 0_{L-N,L-N})$, which is known uniquely once $\Phi_x(\tau)$ is known. □

C. Algorithms

C.1. Mapped-PCMCI

Algorithm 1 describes an implementation of Mapped-PCMCI using a modified version of Varimax (12) and multivariate linear regression, MLR (9). Note that since all entries of $\hat{W}$ are nonzero, when using it to map $\hat{\Phi}$ to the grid space, many grid points will be connected to each other. To address this problem, we suggest a slight modification of the Varimax algorithm (Varimax$^+$) that determines whether each loading's value statistically differs from 0 (see algorithm 3).

For a more detailed explanation of the method see Section 2.3.

Algorithm 1 Mapped-PCMCI

```
procedure ($Y \rightarrow (\hat{\Phi}_{Y,h}, \hat{\mathcal{G}}_Y)$ )
    $\hat{W} =$ Varimax$^{+}(Y)$
    $\hat{X} = \hat{W}Y$
    $(\hat{\mathcal{G}}, \hat{\mathcal{P}}) \leftarrow$ PCMCI $(X)$
    $\hat{\Phi} \leftarrow$ MLR $(\hat{X}, \hat{\mathcal{P}})$
    $\hat{\Phi}_Y = W^{+}\hat{\Phi}W$
    $\hat{\mathcal{G}}_Y \leftarrow \hat{\Phi}_Y$
    **return:** $(\hat{\mathcal{G}}_Y, \hat{\Phi}_Y)$
```

C.2. *Matching permutation*

Algorithm 2 offers an approach for the problem given by the fact that the rows of $\hat{W}$ can be permuted in relation to W. Therefore we need to look for a permutation to find a suitable order that allows us to compare W and $\hat{W}$. Here we reorder the rows of $\hat{W}$ to increase the Pearson correlation coefficient between the rows of W and $\hat{W}$. This algorithm outputs one of the many solutions to the problem. Let $\mathcal{I}$ be the true order of the weights, $\mathcal{I} = (1, \ldots, I)$, the algorithm reads

Algorithm 2 Finding a matching permutation

```
procedure $(W, \hat{W}, \mathcal{I}) \rightarrow \hat{\mathcal{I}}$
    $\hat{\mathcal{I}} = ()$
    $\mathcal{U} = \{\}$
    for $i \in \mathcal{I}$ do
        $j^* = \operatorname{argmax}_{j \notin \mathcal{U}} |\rho(\mathbf{w}^i, \hat{\mathbf{w}}^j)|$
        $\hat{\mathcal{I}}^i = j^*$
        $\mathcal{U} = \mathcal{U} \cup \{j^*\}$
    return: $\hat{\mathcal{I}}$
```

C.3. Varimax$^+$

The Varimax rotation is an algorithm that rotates loadings resulting from PCA to simplify its interpretation. It maximizes the sum of the variances of the squared loadings. As a result most of the values of those loadings are pushed towards zero. More information of PCA and Varimax can be found in Sec. 2.2.1. Varimax$^+$ aims to assess which values of the resulting loadings are statistically different from 0 and which are not.

Starting from the original dataset (Y), $\mathrm{n\_rep}$ datasets of $\mathrm{n\_sam}$ samples (Y^i) are generated using random sampling with replacement ($\mathrm{Bootstrap}$). The Varimax algorithm is applied to each of those sets, and then the resulting loadings $\tilde{W}^i$ are ordered to match the order of the estimated loadings of Y, $\hat{W}$. This reordering is done with Algorithm 2 ($\mathrm{Find\_Permutation}$). Finally, for each variable (grid-point) $\ell \in \mathcal{L}$ in each of the $k \in \{1, \ldots, N\}$ permuted Varimax loading distributions, we test whether the value 0 lies inside that permuted distribution at a given alpha level α. If that hypothesis cannot be rejected, then that grid point in the estimated weights ($\hat{W}_k^{\ell}$) for that Varimax component is set to 0.

Algorithm 3 Varimax^{+}

```
procedure ($Y, \mathrm{n\_rep}, \mathrm{n\_sam} \rightarrow \hat{W}$)
    $\tilde{\mathcal{W}} = \{\}$                    ▷ Set of estimated $\tilde{W}$ from bootstrap data
    $\hat{W} \leftarrow \mathrm{Varimax}(Y)$
    while $i \le \mathrm{n\_rep}$ do
        $\tilde{Y}^i \leftarrow \mathrm{Bootstrap}(Y, \mathrm{n\_sam})$
        $\tilde{W}^i \leftarrow \mathrm{Varimax}(\tilde{Y}^i)$
        $\tilde{W}^i_{\mathrm{per}} \leftarrow \mathrm{Find\_Permutation}(\tilde{W}^i, \hat{W})$
        $\tilde{\mathcal{W}} \leftarrow \tilde{\mathcal{W}} \cup \tilde{W}^i_{\mathrm{per}}$
```

```
            $i = i + 1$
    for $\ell \in \mathcal{L}$ and $k \in \{1, \dots, N\}$ do    ▷ for every grid location among all estimated $\tilde{W}_{\text{per}}$
        $\text{p\_val}_k^{\ell} \leftarrow \text{p} - \text{value of 0 wrt. } \tilde{\mathcal{W}}_k$
        if $\text{p\_val}_k^{\ell} > \alpha$ then
            $\hat{W}_k^{\ell} = 0$
    return: $\hat{W}$
```